THE GENERALIZED FINITE ELEMENT METHOD APPLIED IN FREE VIBRATION ANALYSIS UNDER PLANE STRESS

Ricardo Custódio

Marcos Arndt

ricardo.cstd@gmail.com

arndt.marcos@gmail.com

Program of Numerical Methods in Engineering, Federal University of Paraná, Curitiba, Paraná, Brazil

Abstract. *The Generalized Finite Element Method can be defined as a generalization of the Finite Element Method. The main principle of this method consists in the enrichment of the finite element through construction of a subspace filled with shape functions, that not to be necessarily polynomial. Another characteristics of this method, it is the uses of concepts of the Partition of Unity, wherein the shape functions need to have that properties. However, the enrichment functions can often be defined through previously known boundary value problems. The aim of this paper is to investigate an enrichment with shape functions for a finite quadrilateral element and also to verify the response of its application in a problem of free vibration for a structure subjected to the plane stress, in order to check its convergence and precision when compared to conventional Finite Element Method.*

Keywords: *Generalized Finite Element Method; Partition of Unity; Free Vibration; Plane Stress.*

1 INTRODUCTION

Nowadays with advances that occur in engineering practice, new techniques arise for solving problems, that need to be solved to describe the behavior of structural systems problems.

The precise definition of eigenvalues is as important in engineering as in the field of mathematics, because many problems of dynamic analysis results generalized problems of eigenvalues and eigenvectors, such as, in structural problems, where the eigenvalues represent the natural frequencies of vibration. However, there are difficulties in the exact numerical approximation of the highest order eigenvalues, which can affect the prediction quality of the dynamic responses, as shown by Arndt (2009).

The Generalized Finite Element Method (GFEM) can be defined as the generalization of the conventional Finite Element Method (FEM). The main principle of the GFEM is the enrichment of the master element through the construction of a subspace of enriched shape functions, using the concepts of the Partition of Unit Method. The shape function need not be polynomial only, but can also assume properties previously known from solutions of boundary value problems. The application of the GFEM can take to more precise results for the higher order eigenvalues, because of the hierarchical characteristic obtained by incorporating new functions in the master element domain, taking to higher convergence rates when compared to the h refinement. In the present work it is investigated the dynamic response obtained by the use of sinusoidal trigonometric functions, which were established by Torii (2015).

The choice of the enrichment functions can be did arbitrarily or by set from a previously known analytical solutions, since they are in accordance with mathematical condition established by GFEM. Thus, it is possible to use the Partition of Unity to provide local approximation spaces in which functions may be reproduced in each element as piece of the overall approximation, Melenk & Babuska (1997), here the functions adopted have the property of assuming null values on boundary of the master element.

The numerical integrations often require a greater number of integration points, therefore, here, the stiffness and mass matrices are obtained through the numerical integration of Gauss-Legendre, with use of thirty integration points, that are considered enough to guarantee the accuracy of the components terms, however the numerical stability of equations system can also depends on the number of significant digits used.

There are cases of structural problems in the dynamic analysis, wherein the solution is independent of one of the reference coordinates, due to geometry, boundary conditions or due to external loads, these problems are defined as Plane Stress, the problem here analyzed is of this nature. Plane Stress in two dimensions is governed by partial differential equations system, whose analytical solution becomes difficult for complex geometries, that justifies the study of GFEM for these problems.

2 THE GENERALIZED FINITE ELEMENT METHOD

The GFEM formulation for a finite quadrilateral element was already presented by Torii (2012) and Shang (2017), wherein the base it is related to the Lagrangian shape functions as partition of unity, through of local enrichment promoted by appropriate functions and defined from the previous knowledge of the partial differential equation analytical solution.

Overall the enrichment process can be get by the Eq. (1):

$$\widetilde{u}(\bar{x}) = \sum_{i=1}^{N} \phi_i(\bar{x}) \left\{ u_i + \sum_{n=1}^{qj} {}_{jn}(\bar{x}) b_{jn} \right\} \tag{1}$$

$$\widetilde{u} = \widetilde{u}_{FEM} + \widetilde{u}_{ENRIQ} \tag{2}$$

Or, in matrix form:

$$\widetilde{u}(\bar{x}) = \Phi^T U \tag{3}$$

Where:

$$U^T = \begin{bmatrix} u_1 & b_{11} & ... & b_{1qj} & ... & u_N & b_{j1} & ... & b_{jqj} \end{bmatrix} \tag{4}$$

$$\Phi^T = \begin{bmatrix} \phi_1 & l_{11}\phi_1 & ... & {}_{1qj}\phi_1 & ... & \phi_N & l_{j1}\phi_N & ... & l_{jqj}\phi_N \end{bmatrix} \tag{5}$$

In Eq. (1), ϕ_i it is the PU functions, u_i the nodal displacements, b_{jqj} the field degrees of freedom, l_{qj} the enriched shape functions, qj the level of enrichment functions and N is the total number of nodes in the system. The functions used in this study are called bubble functions, because they have null values along the elementary boundary and non-null values within the domain. The Lagrangian shape functions are obtained from the following one-dimensional functions:

$$l_1(\xi) = \frac{1-\xi}{2} \tag{6}$$

$$l_2(\xi) = \frac{1+\xi}{2} \tag{7}$$

$$l_3(\xi) = \frac{1-\xi}{2}[\sin(\frac{\beta_j(\xi+1)}{2})] \tag{8}$$

$$l_4(\xi) = \frac{1+\xi}{2}[\sin(\frac{\beta_j(\xi-1)}{2})] \tag{9}$$

Another characteristic of enrichment process can be performed by incorporating new shape functions or by a fixed set of shape functions, in which only β_j parameter is changed. The use of bubble functions facilitates the imposition of boundary conditions at the moment of the solution global system of equations, because their values are null on nodal coordinates, Arndt (2009).

2.1 QUADRILATERAL FINITE ELEMENT FORMULATION

The procedure to definition of stiffness and mass matrices are presented by Shang (2016). In Fig. (1) it is shown the domain of quadrilateral finite element, composed by four nodes.

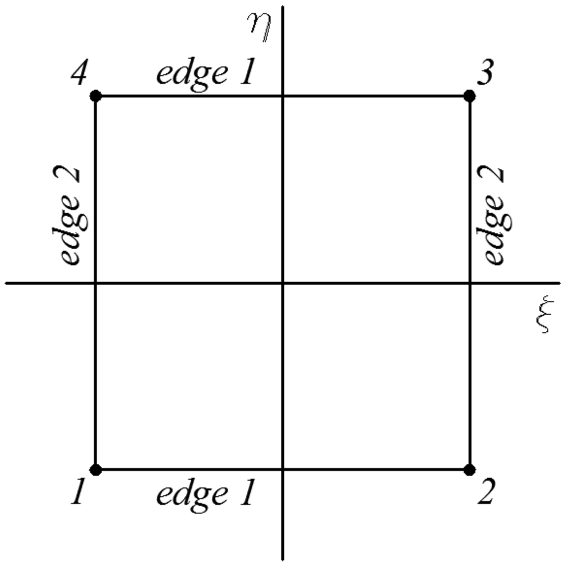

Figure 1: Element domain.

The shape functions related to PU, can be obtained through product combination between Eq.(6) and (7), where these combinations are graphically shown in Figs. (2) to (5).

$$\phi_1(\xi,\eta) = l_1(\xi)l_1(\eta) \tag{10}$$

$$\phi_2(\xi,\eta) = l_2(\xi)l_1(\eta) \tag{11}$$

$$\phi_3(\xi,\eta) = l_2(\xi)l_2(\eta) \tag{12}$$

$$\phi_4(\xi,\eta) = l_1(\xi)l_2(\eta) \tag{13}$$

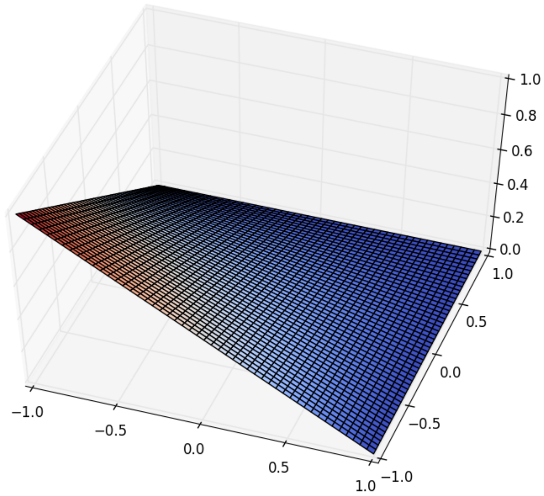

Figure 2: PU function given by product $l_1(\xi)l_1(\eta)$.

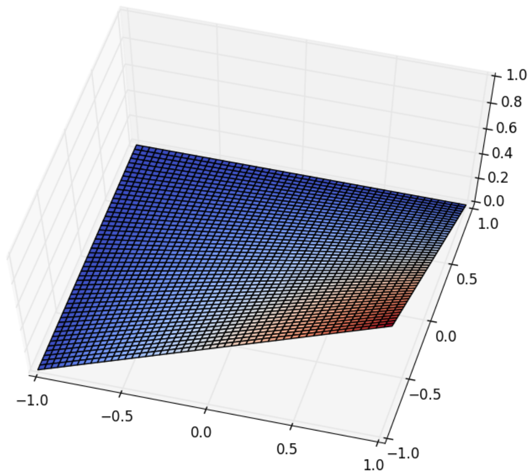

Figure 3: PU function given by product $l_1(\xi)l_2(\eta)$.

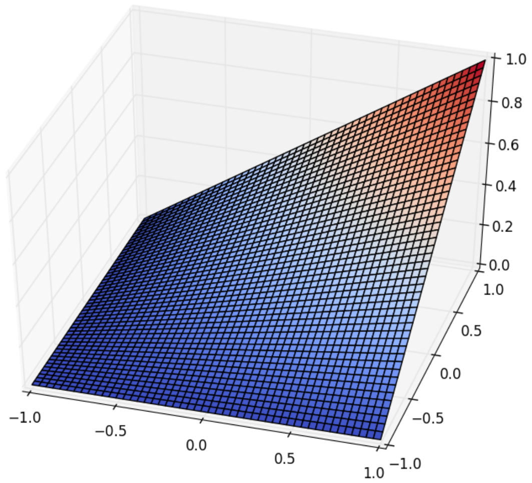

Figure 4: PU function given by product $l_2(\xi)l_2(\eta)$.

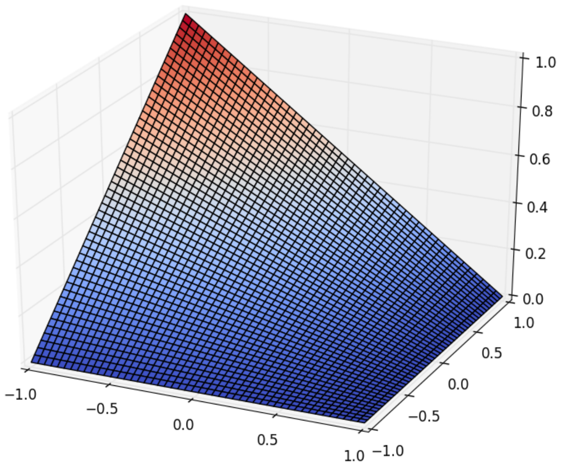

Figure 5: PU function given by product $l_2(\xi)l_1(\eta)$.

The enriched shape function can be generated by the combination of product between the Equations (8) and (9), both them are defined on direction ξ e η. The Figures (7) to (9) show graphically these products.

$$h_1^b(\xi,\eta) = l_3(\xi)l_3(\eta) \tag{14}$$

$$h_2^b(\xi,\eta) = l_3(\xi)l_4(\eta) \tag{15}$$

$$h_3^b(\xi,\eta) = l_4(\xi)l_3(\eta) \tag{16}$$

$$h_4^b(\xi,\eta) = l_4(\xi)l_4(\eta) \tag{17}$$

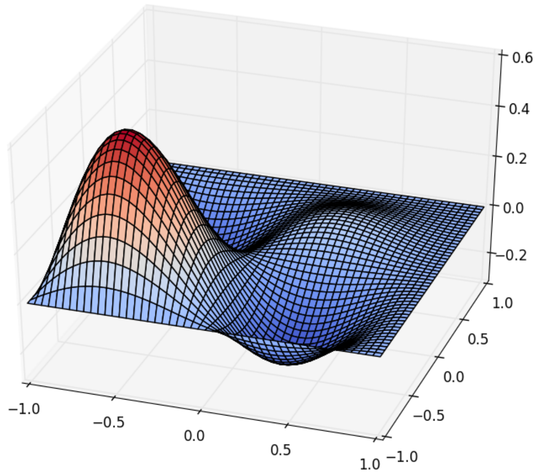

Figure 6: Enrichment function h_1^b, for $\beta_j = \pi$.

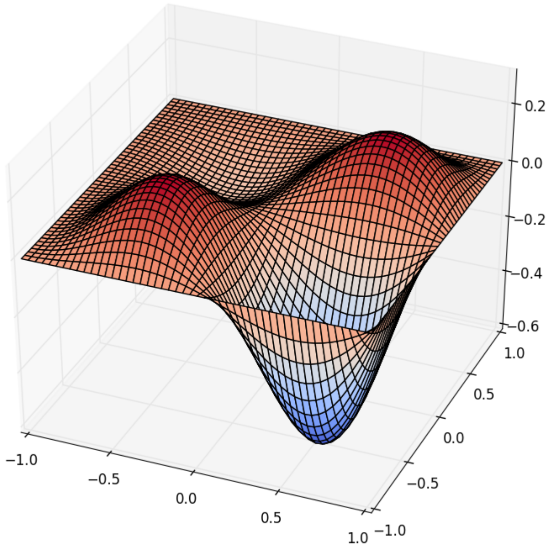

Figure 7: Enrichment function h_2^b, for $\beta_j = \pi$.

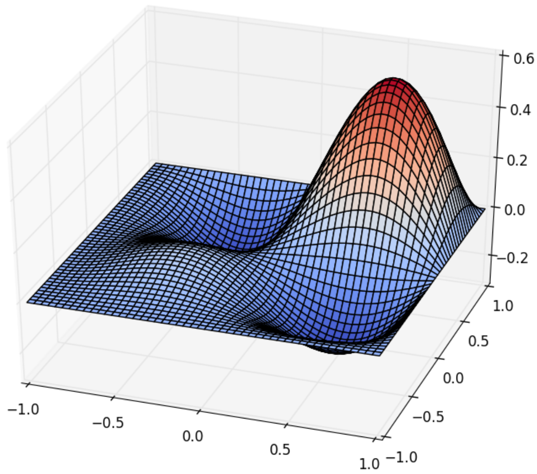

Figure 8: Enrichment function h_3^b, for $\beta_j = \pi$.

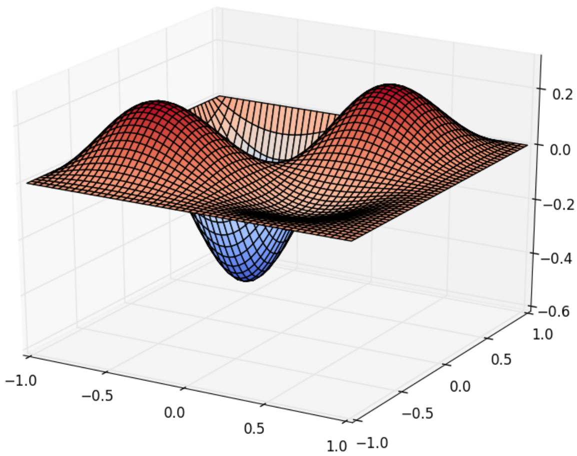

Figure 9: Enrichment function h_4^b, for $\beta_j = \pi$.

2.2 STIFFNESS AND MASS MATRICES

According to Houmat (2007) the potential energy and kinetic related to plane stress are given by equations (18) and (19), both them are defined in elementary domain.

$$P = \frac{D_0}{2} t \int_A [(\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial y})^2 + 2\nu \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{(1-\nu)}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})^2]\mathrm{d}A \tag{18}$$

$$T = \frac{1}{2}\rho t \int_A (\dot{u}^2 + \dot{v}^2)\mathrm{d}A \tag{19}$$

Where D_0 is given by:

$$D_0 = \frac{E}{(1-\nu^2)} \tag{20}$$

The term D_0 relates the Elastic Module E with the Poisson coefficient ν. The term t represents de thickness of the finite element. It can be observed that in Eq. (18) and (19) that the terms $\dot{u}$ e $\dot{v}$ represents the first derivatives of the displacements in relation to time. By the formulation of the principle of the virtual works of the integral equations, the stiffness and mass matrices results in the following expressions:

$$\mathbf{K} = \int_V \mathbf{B}^T \mathbf{D} \mathbf{B} \mathrm{d}V \tag{21}$$

$$\mathbf{M} = \rho \int_V \mathbf{N}^T \mathbf{N} \mathrm{d}V \tag{22}$$

$$\mathbf{D} = D_0 \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \tag{23}$$

Where **B** represents the first derivative matrix shape functions, **D** the relative constitutive tensor, **N** the shape functions matrix, ρ and t, the density and thickness respectively. However the Eq. (21) and Eq.(22) are transformed from the real coordinates to the natural coordinates, Fig (1), since the shape functions are defined in this system. So **K** and **M** results:

$$\mathbf{K} = t \int_{-1}^{+1} \int_{-1}^{+1} \mathbf{B}^T \mathbf{D} \mathbf{B} \left| J \right| \mathrm{d}\xi \mathrm{d}\eta \tag{24}$$

$$\mathbf{M} = \rho t \int_{-1}^{+1} \int_{-1}^{+1} \mathbf{N}^T \mathbf{N} \left| J \right| \mathrm{d}\xi \mathrm{d}\eta \tag{25}$$

$$\mathbf{N} = \begin{bmatrix} \phi_1 & 0 & ... & \phi_4 & 0 & h_1^b & 0 & ... & h_4^b & 0 \\ 0 & \phi_1 & 0 & ... & \phi_4 & 0 & h_1^b & 0 & ... & h_4^b \end{bmatrix} \tag{26}$$

Here, the matrix **N** is the local where the shape functions are gathering.

3 NUMERICAL EXAMPLE

The model adopted to numerical application GFEM can be viewed in Fig. (10), which consists in a L-shaped piece, with density ρ equal to $7500kg/m^3$, Elastic Module $E = 210GPa$ and Poisson coefficient ν equal to 0.30. The eigenvalues are shown on Table 1, and they are compared with the results obtained by conventional FEM. The mesh generated has 3 e 12 finite elements, Figs. (11) e (12), for both methods.

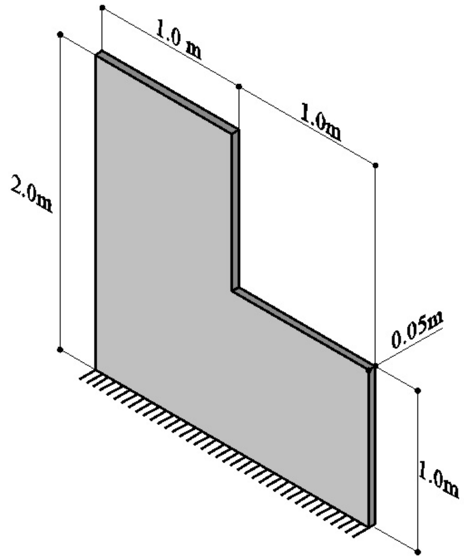

Figure 10: L-shaped used to application of the GFEM.

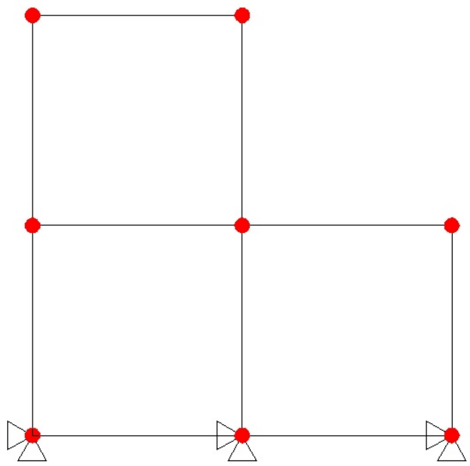

Figure 11: Mesh with 3 elements defined to modeling GFEM and conventional FEM.

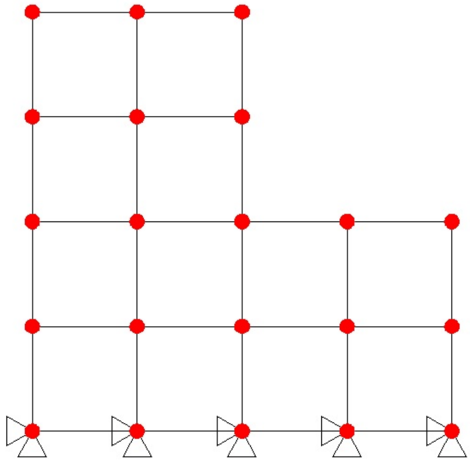

Figure 12: Mesh with 12 elements defined to modeling GFEM and conventional FEM

Table 1: Relation of the first six natural frequency vibration (rad/s).

N	FEM-3	FEM-12	GFEM-3	GFEM-12
1	2123.2312	1928.1409	2069.6163	1902.7049
2	4952.6227	4645.6203	4820.8666	4592.5767
3	5522.8128	4877.0504	5269.7351	4789.2342
4	9169.0052	8297.7950	8822.6321	8083.6877
5	12291.8361	9718.1258	10388.8427	9346.3894
6	14237.3742	12197.3462	12970.1481	11721.6665

In Table 1, can be observed that values of *N* from GFEM are lower than FEM, due to the finite element's enrichment by addition of 4 bubble functions in subspace of shape functions. The model with 3 elements, for GFEM-3, the system has 10 nodal degrees of freedom u_n, more 25 field degrees of freedom b_{j1}, from nodal degrees of freedom 6 were constrained, in order to configure the boundary condition. The application of the GFEM-12, with 12 elements, it has 128 degrees of freedom, whose 32 are nodal degrees of freedom and 96 are field degrees of freedom. In the Fig. (13) it is shown graphically the natural frequency distribution contained in Table 1.

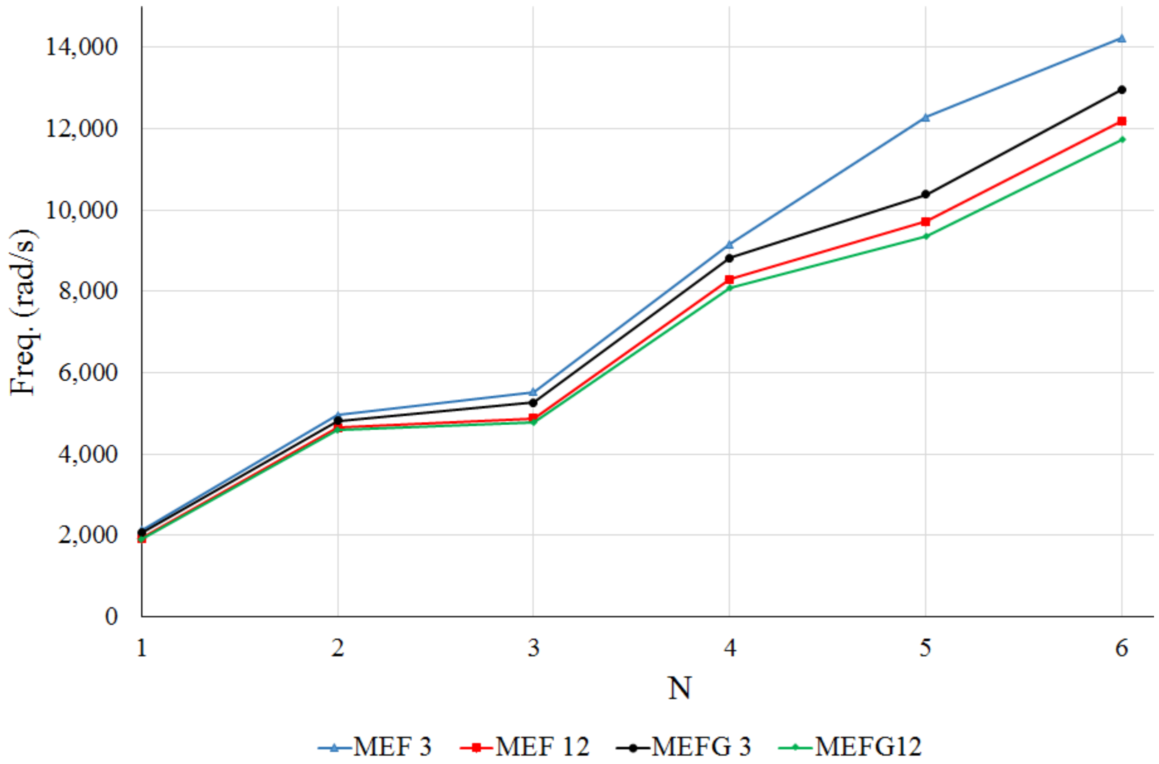

Figure 13: Relation of the frequencies given by Table 1.

4 CONCLUSION

The present study had as objective the use of GFEM composed by enrichment with sinusoidal bubble functions, in order to investigate the natural vibration frequencies of a L-shaped piece under plane stress. Compared with conventional FEM, for mesh with same quantity of elements, the GFEM showed natural frequencies relatively low, tending to a convergence with higher rates than conventional FEM. Thus futures studies will be realized to establish the increase of new enriched shape functions, as well as the application for other complex geometries.

ACKNOWLEDGMENTS

The authors note in this preamble, the thanks to CAPES for the research funding, and to UFPR for the structure provided for the development of this work.

5 REFERENCES

Arndt, M., 2009., *O Método dos Elementos Finitos Generalizados Aplicado Á Análise de Vibrações Livres de Estruturas Reticuladas*, Tese de Doutorado, Programa de Pós-Graduação em Métodos Numéricos em Engenharia. Curitiba. Doutorado em Métodos Numéricos em Engenharia.

Babuska, I., & Melenk, J.M., 1997. *The partition of unity method*.Int. J. Numer. Methods Eng. 40, 727-758.

Duarte, C. A., Kim, D. J., 2008. *Analysis and application of a generalized finite element method with global-local enrichment funtions*. Comput. Methods Appl. Mech. Eng. 197, 487-504.

Houmat, A., 2007. *In-plane vibration of plates with curvilinear plan-forms by a trigonometrically enriched curved triangular p-element*. Thin-Walled Structures 46 (2008), pp 103-111

Leung, A.Y.T., Zhu, B., Zheng, J., Yang, H., 2004. *Analytical trapezoidal fourier p-element for vibrating plane problems*. J. Sound Vib. 271, 67-81. Comput. Methods Appl. Mech. Eng. 197, 487-504.

Petyt, M., 2010. *Introduction to finite element vibration analysis*. 2 ed. Cambridge University Press.

Rauen. M., Machado. R. & Arndt. M., 2016., *Isogeometric Analysis Applied to Free Vibration Analysis of Plane Stress and Plane Strain Structures*, XXXVI Iberian Latin American Congress on Computacional Method in Engineering, Brasília, DF, Brasil.

Segerlind, L., 1984. *Applied finite element analysis*. Vol I. John Wiley & Sons, Inc. Canada.

Shang, H. Y., Machado, Roberto D., Abdalla Filho, J. E., & Arndt, M., 2017. *Numerical analysis of plane stress free vibration in severely distorted mesh by Generalized Finite Element Method*. European Journal of Mechanics-A/Solids, 62, pp 50-66.

Shang, H. Y., Machado, R. D., Abdalla Filho, J. E., & Arndt, M., 2016. *Dynamic analysis of Euler-Bernoulli beam problems using the generalized finite element method*. Comput. Struct. 173, pp 109-122.

Solin, P., & Segeth, K.,& Dolezel, I., 2004. *Higher-order finite element methods*. Chapman & Hall/ CRC.

Szabó, B. & Babuska, I. 1991. *Finite element analysis*. Vol II. John Wiley & Sons, Inc. Canada.

Timoshenko, S. P., & Goodier, J. N., 1980. *Teoria da Elasticidade.* Guanabara Dois.

Torii, A.J., Machado, R. D. & Arndt, M., *GFEM for modal analysis of 2D wave equation*, Emerald Insight, 2015.

Torii, A.J., 2012., *Análise Dinâmica de Estruturas com o Método Dos Elementos Finitos Generalizado*, Tese de Doutorado, Programa de Pós-Graduação em Métodos Numéricos em Engenharia.Curitiba. Doutorado em Métodos Numéricos em Engenharia.