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FLUX LIMITERS COMPARISON FOR RESOLUTION OF EULER EQUATION BY TVD METHOD IN QUASI ONE-DIMENSIONAL NOZZLES WITH SUBSONIC/SUPERSONIC FLOW

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Abstract. *The resolution of a system of partial differential equations can result in infinite solutions, but only with the correct definition of boundary conditions will be possible to determine the desired approximate solution. Considering the analysis of an isentropic flow in a quasi one-dimensional nozzle, which is governed by a hyperbolic equation, the correct boundary condition specification is directly related to the method of characteristics and depends on the direction of information propagation within the studied domain, which is determined by the problem eigenvalues. However, when the flow is subsonic at the inlet section and supersonic at the outlet section, accordingly to the method of characteristics, one of the eigenvalues will not be defined neither at inlet nor outlet section, what could cause the non convergence of the numerical method to the correct physical solution. This work will investigate the effects of use different flux limiters for solve the subsonic/supersonic case of the quasi one-dimensional Euler equations using the TVD method.*

Keywords: *Euler Equation, TVD Method, Flux Limiter, Boundary Condition, Isentropic Flow*

1 INTRODUCTION

The solution of the Euler equation is completely dependent of the adequate boundary conditions definition, because of that, cases like the subsonic/supersonic quasi one-dimensional flow represents a challenge to be solved due to the non-definition of one of the eigenvalues, neither at the inlet nor outlet sections of the nozzle.

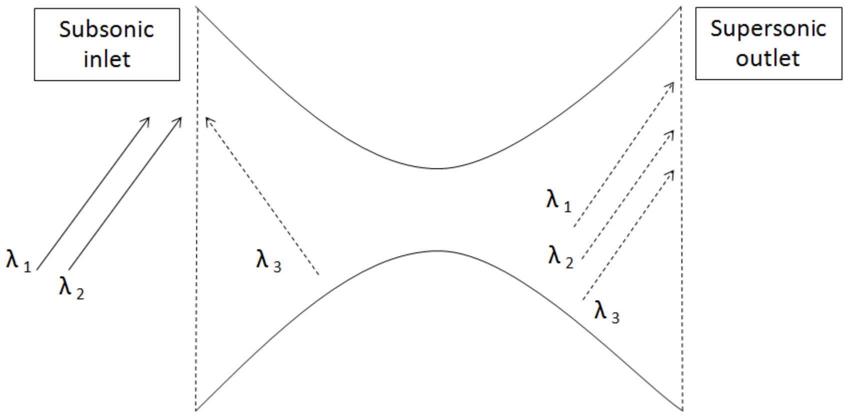


Figure 1: Boundary Conditions

At Fig. 1, $(\lambda_1, \lambda_2, \lambda_3)$ are the problem eigenvalues and the ones represented by dashed arrows are those which will be extrapolated from the domain inner part. The combination of the non-definition of all eigenvalues, flux limiter and geometry lead the numerical method to provide different results for the same case.

This work proposal is evaluate some kinds of flux limiters at different geometries and compare the results with the ones provided by the isentropic flow equations, thus determining which flux limiter is the most adequate for the subsonic/supersonic quasi one-dimensional flow.

2 METHODOLOGY

To produce and analyze the numerical results, the following steps will be executed.

1. Present the quasi one-dimensional geometries that will be studied
2. Define the Euler equation for the quasi one-dimensional case
3. Simplify the Euler equation and reduce it to isentropic flow equations
4. Apply the TVD method and present the flux limiters that will be investigated

2.1 Geometry

In this article, the behavior of the numerical method will be evaluated in two different geometries. Being the first symmetrical with respect to the y-axis and the second one asymmetrical regarding the y-axis.

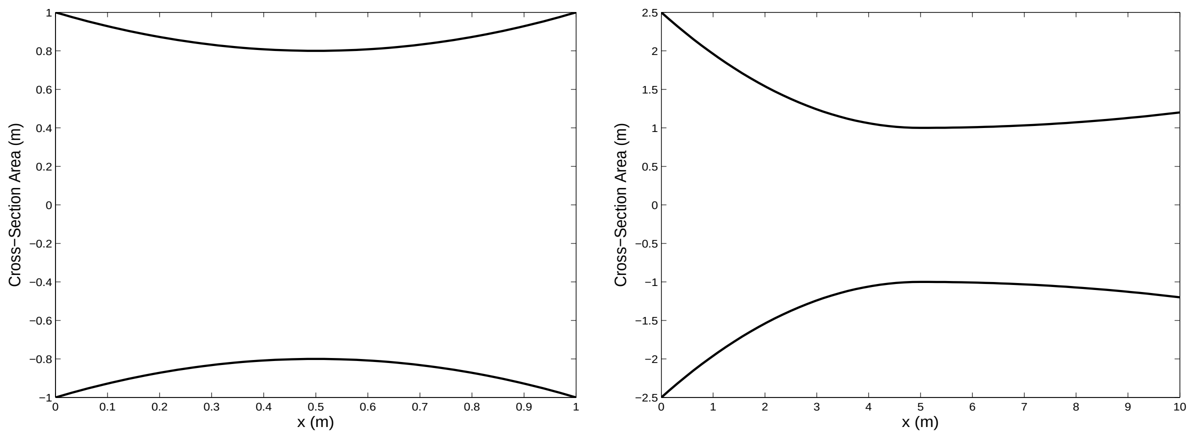
The symmetrical nozzle is defined by the following equation

$$A\left(x\right) = 1 - 0,8 \cdot x \cdot \left(1 - x\right); \quad 0 \leq x \leq 1 \tag{1}$$

and the asymmetrical nozzle is described by

$$A\left(x\right) = \begin{cases} 1 + 1,5\left(1 - \dfrac{x}{5}\right)^2, & x \leq 5 \\ 1 + 0,008\left(x - 5\right)^2, & x > 5 \end{cases}; \quad 0 \leq x \leq 10 \tag{2}$$

where A is the nozzle cross-sectional area. The equations of both nozzles are given by (Lobão, 1992)



(a) Symmetrical nozzle

(b) Asymmetrical nozzle

Figure 2: Quasi one-dimensional nozzles

2.2 Euler equation

The Euler equation for the quasi one-dimensional flow is described by the following equation as shown by (Shubin et al., 1981)

$$\frac{\partial Q}{\partial t} + \frac{\partial F\left(Q\right)}{\partial x} + H\left(Q\right) = 0 \tag{3}$$

where

$$Q = A \begin{bmatrix} \rho \\ \rho u \\ e \end{bmatrix}, F = A \begin{bmatrix} \rho u \\ \rho u^2 + P \\ \left(e + P\right) u \end{bmatrix}, H = \frac{\partial A}{\partial x} \begin{bmatrix} 0 \\ -P \\ 0 \end{bmatrix} \tag{4}$$

which results in a system with four unknows and three equations, which turn out to be necessary use the state equation in order to complete the system of governing equations, and this equation is given by

$$P = (\gamma - 1)\left[e - \frac{\rho u^2}{2}\right] \tag{5}$$

where Q is the vector of conserved variable, F is the flux vector, H is the forcing term related to the nozzle area variation, ρ is the density, u is the velocity, e is the energy, P is the pressure; and γ is the ratio of specific heats at constant pressure and volume, equal to 1.4 to the atmospheric air.

2.3 Isentropic flow

The Euler equation, when there is no thermal exchange (adiabtic) and the viscous effects can be negleted, will remain with constant entropy, thus being able to be modeled by the following relations, for a perfect gas, as given by (Liepmann & Roshko, 2001).

$$\frac{P_0}{P} = \left[1 + \frac{\gamma - 1}{2} M^2\right]^{\frac{\gamma}{\gamma - 1}} \tag{6}$$

$$\frac{\rho_0}{\rho} = \left[1 + \frac{\gamma - 1}{2} M^2\right]^{\frac{1}{\gamma - 1}} \tag{7}$$

where P_0 is the stagnation pressure, ρ_0 is the stagnation density and M is the Mach number.

The Mach number will be determined through the Newton-Raphson method applied on the following equation

$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left[\frac{2}{\gamma + 1}\left(1 + \frac{\gamma - 1}{2} M^2\right)\right]^{\frac{\gamma + 1}{\gamma - 1}} \tag{8}$$

where A^* is the cross-sectional area at the nozzle throat.

2.4 TVD method

The main idea of the TVD method, presented by (Yee et al., 1983), is to ensure that the oscillation of the numerical solution does not grow between two consecutive iterations. All monotone schemes have the first-order TVD characteristic and with the use of flow limiters it is possible to develop second order schemes, where the formulation will be of second order in regions with smooth solution and of first order in regions with steep gradients, thus avoiding the formation of spurious oscillations.

The general formulation for the family of conservative schemes of one parameter and three points is given by

$$Q_j^{n+1} + \lambda\theta\left(\tilde{F}^{n+1}_{j+\frac{1}{2}} - \tilde{F}^{n+1}_{j-\frac{1}{2}}\right) = Q_j^n - \lambda\left(1-\theta\right)\left(\tilde{F}^{n}_{j+\frac{1}{2}} - \tilde{F}^{n}_{j-\frac{1}{2}}\right) \tag{9}$$

where θ is a parameter, and if $\theta = 0$, the scheme will be explicit, and if $\theta = 1$, the scheme will be implicit and $\lambda = \frac{\Delta t}{\Delta x}$.

The use of weak solutions can produce correct results from a mathematical point of view but will not be in accordance with the problem physics, which makes it necessary to use a function to "select" the physically correct solution, this function is the entropy correction. As not all TVD schemes automatically meet the entropy condition it is necessary to put the following equation on the scheme.

$$\Psi\left(z\right) = \begin{cases} \left|z\right|, & \left|z\right| \geq \delta \\ \dfrac{\left(z^2+\delta^2\right)}{2\delta}, & \left|z\right| < \delta \end{cases} \tag{10}$$

where $\Psi\left(z\right)$ é is a nonvanishing approximation, continously differentiable for $|z|$, which represents eigenvalues of the Euler equation set, and δ é is the entropy correction parameter, being its value contained within the range $0 \leq \delta \leq 0,125$, as shown by (Yee et al., 1983).

The scheme order control will be done by the flux limiter which is added at the numerical flux function that will combine functions of first and second order.

$$\tilde{F}^{n+1}_{j+\frac{1}{2}} = \frac{1}{2}\left(F_j^{n+1} + F_{j+1}^{n+1} + \left(R^l_{j+\frac{1}{2}}\Phi^l_{j+\frac{1}{2}}\right)^{n+1}\right) \tag{11}$$

where the flux limiter Φ is a vector with three components, $\Phi = \begin{bmatrix}\phi_1 & \phi_2 & \phi_3\end{bmatrix}^T$ and each one of its components is given by

$$\phi^l_{j+\frac{1}{2}} = \sum_{l=1}^{m}\left[\frac{1}{2}\Psi\left(a^l_{j+\frac{1}{2}}\right)\left(g_j^l + g_{j+1}^l\right) - \Psi\left(a^l_{j+\frac{1}{2}} + \gamma^l_{j+\frac{1}{2}}\right)\alpha^l_{j+\frac{1}{2}}\right] \tag{12}$$

where the variable g is the flux function limiter, and the following options shown by (Yee et al., 1983) and (Yee, 1989) will be compared.

- Minmod flux limiter

$$g_j^l = minmod\left(\alpha^l_{j-\frac{1}{2}}, \alpha^l_{j+\frac{1}{2}}, \alpha^l_{j+\frac{3}{2}}\right) \tag{13}$$

- Yee flux limiter

$$g_j^l = S\cdot max\left[0, min\left(\sigma^l_{j+\frac{1}{2}}\left|\alpha^l_{j+\frac{1}{2}}\right|\right), S\cdot\sigma^l_{j-\frac{1}{2}}\alpha^l_{j-\frac{1}{2}}\right] \tag{14}$$

- Van Leer flux limiter

$$g_j^l = \frac{\left\{\alpha_{j-\frac{1}{2}}^l\left[\left(\alpha_{j+\frac{1}{2}}^l\right)^2 + \delta_2\right] + \alpha_{j+\frac{1}{2}}^l\left[\left(\alpha_{j-\frac{1}{2}}^l\right)^2 + \delta_2\right]\right\}}{\left[\left(\alpha_{j+\frac{1}{2}}^l\right)^2 + \left(\alpha_{j-\frac{1}{2}}^l\right)^2 + 2\delta_2\right]} \tag{15}$$

- Superbee flux limiter

$$g_j^l = S \cdot max\left[0, min\left(2\left|\alpha_{j+\frac{1}{2}}^l\right|, S \cdot \alpha_{j-\frac{1}{2}}^l\right), min\left(\left|\alpha_{j+\frac{1}{2}}^l\right|, 2S \cdot \alpha_{j-\frac{1}{2}}^l\right)\right] \tag{16}$$

where

$$S = sign\left(\alpha_{j+\frac{1}{2}}^l\right) \tag{17}$$

$$\sigma_{j+\frac{1}{2}}^l = \frac{1}{2}\Psi\left(z\right) \tag{18}$$

and δ_2 is a small parameter to prevent division by zero and $10^{-7} \leq \delta_2 \leq 10^{-5}$ is a commonly used range.

The variable *a* corresponds to the eigenvalues, γ is the correction to increase the numerical flux order and α is the average of the local characteristics, being these variables given by

$$\left(a^1, a^2, a^3\right) = \left(u - c, c, u + c\right) \tag{19}$$

$$\gamma_{j+\frac{1}{2}} = \begin{cases} \dfrac{\left(g_{j+1}^l - g_j^l\right)}{\left(\alpha_{j+\frac{1}{2}}^l\right)}, & \alpha_{j+\frac{1}{2}}^l \neq 0 \\ 0, & \alpha_{j+\frac{1}{2}}^l = 0 \end{cases}; \qquad \alpha_{j+\frac{1}{2}}^l = R_{j+\frac{1}{2}}^{-1}\Delta_{j+\frac{1}{2}}U \tag{20}$$

and the eigenvector matrix and its inverse is given by

$$R = \begin{bmatrix} 1 & 1 & 1 \\ u - c & u & u + c \\ H - uc & \frac{1}{2}u^2 & H + uc \end{bmatrix}; \qquad R^{-1} = \begin{bmatrix} \frac{1}{2}\left(b_1 + \frac{u}{c}\right) & \frac{1}{2}\left(-b_2 u - \frac{1}{c}\right) & \frac{1}{2}b_2 \\ 1 - b_1 & b_2 u & -b_2 \\ \frac{1}{2}\left(b_1 - \frac{u}{c}\right) & \frac{1}{2}\left(-b_2 u + \frac{1}{c}\right) & \frac{1}{2}b_2 \end{bmatrix} \tag{21}$$

where the matrices terms are

$$b_1 = b_2\frac{u^2}{2}; \qquad b_2 = \frac{\gamma - 1}{c^2}; \qquad H = \frac{\gamma P}{\left(\gamma - 1\right)\rho} + \frac{1}{2}u^2 \tag{22}$$

and Roe's average, (Roe, 1997), is given by

$$\bar{D} = \sqrt{\frac{\rho_{j+1}}{\rho_j}} \tag{23}$$

$$u_{j+\frac{1}{2}} = \frac{\bar{D}u_{j+1} + u_j}{\bar{D} + 1} \tag{24}$$

$$H_{j+\frac{1}{2}} = \frac{\bar{D}H_{j+1} + H_j}{\bar{D} + 1} \tag{25}$$

$$c^2_{j+\frac{1}{2}} = (\gamma - 1)\left(H_{j+\frac{1}{2}} - \frac{1}{2}u^2_{j+\frac{1}{2}}\right) \tag{26}$$

Thus, using the implicit approach, it is possible to write in matrix form as following

$$E_1 D_{j-1} + E_2 D_j + E_3 D_{j+1} = -RHS \tag{27}$$

where the coefficients *D* are given as

$$D_j = Q_j^{n+1} - Q_j^n \tag{28}$$

$$E_1 = -\frac{\lambda}{2}\left(A_{j-1} + \sum_{l=1}^{m} \Omega^l_{j-\frac{1}{2}}\right) \tag{29}$$

$$E_2 = I + \Delta t \cdot B_j + \frac{\lambda}{2}\left(\sum_{l=1}^{m} \Omega^l_{j+\frac{1}{2}} + \sum_{l=1}^{m} \Omega^l_{j-\frac{1}{2}}\right) \tag{30}$$

$$E_3 = \frac{\lambda}{2}\left(A_{j+1} + \sum_{l=1}^{m} \Omega^l_{j+\frac{1}{2}}\right) \tag{31}$$

$$\begin{aligned} RHS = \frac{\lambda}{2}\Bigg(&\sum_{l=1}^{m}\left[R^l_{j+\frac{1}{2}} \cdot \left(-\Psi\left(a^l_{j+\frac{1}{2}}\right)\left[\alpha^l_{j+\frac{1}{2}} - g^l_{j+\frac{1}{2}}\right]\right)\right] \\ &- \sum_{l=1}^{m}\left[R^l_{j-\frac{1}{2}} \cdot \left(-\Psi\left(a^l_{j-\frac{1}{2}}\right)\left[\alpha^l_{j-\frac{1}{2}} - g^l_{j-\frac{1}{2}}\right]\right)\right] + F_{j+1} - F_{j-1}\Bigg) + \Delta t \cdot H_j^n \end{aligned} \tag{32}$$

where *A* and *B* are jacobian matrices for the vectors *F* and *H* from Eq. (4) and are given by

$$A = \frac{\partial F}{\partial Q} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{u^2}{2}(\gamma - 3) & u(3 - \gamma) & (\gamma - 1) \\ \frac{-\gamma u e}{\rho} + (\gamma - 1)u^3 & \frac{\gamma e}{\rho} - \frac{3}{2}u^2(\gamma - 1) & \gamma u \end{bmatrix} \tag{33}$$

$$B = \frac{\partial H}{\partial Q} = \frac{(\gamma - 1)}{S}\frac{dS}{dx}\begin{bmatrix} 0 & 0 & 0 \\ -\frac{u^2}{2} & u & -1 \\ 0 & 0 & 0 \end{bmatrix} \tag{34}$$

and the Ω terms may be written in a triple product form or as a aproximation used to save computational effort given by (Yee,1990), being

$$\Omega^l_{j+\frac{1}{2}} = R_{j+\frac{1}{2}} \cdot \Psi\left(a_{j+\frac{1}{2}} + \gamma_{j+\frac{1}{2}}\right) \cdot R_{j+\frac{1}{2}} \quad or \quad \Omega^l_{j\pm\frac{1}{2}} = M \cdot I \tag{35}$$

being *I* the identity matrix and *M*,

$$M = max^l \Psi\left(a^l + \gamma^l\right)_{j\pm\frac{1}{2}} \tag{36}$$

2.5 Boundary conditions

The boundary conditions will be defined as shown in the introduction, and considering the vector of primitive variables at Eq. (4), at the inlet section, as the flow is subsonic, the value of components U_1 and U_2 will be constant and the value of component U_3 will be extrapolated from the domain interior through the pressure value.

At the outlet section, as the flow is supersonic, all components of the vector *U* will be extrapolated from the domain interior.

3 RESULTS

At this section the influence of flux limiters shown at Eq. (13), Eq. (14), Eq. (15) e Eq. (16) will be evaluated considering the symmetrical and asymmetrical nozzles. The numerical results for Mach number, speed, pressure and density will be compared through an overlap of the graphics considering all the flux limiters studied and also through the percentage difference between the numerical and exact solution.

3.1 Symmetrical nozzle

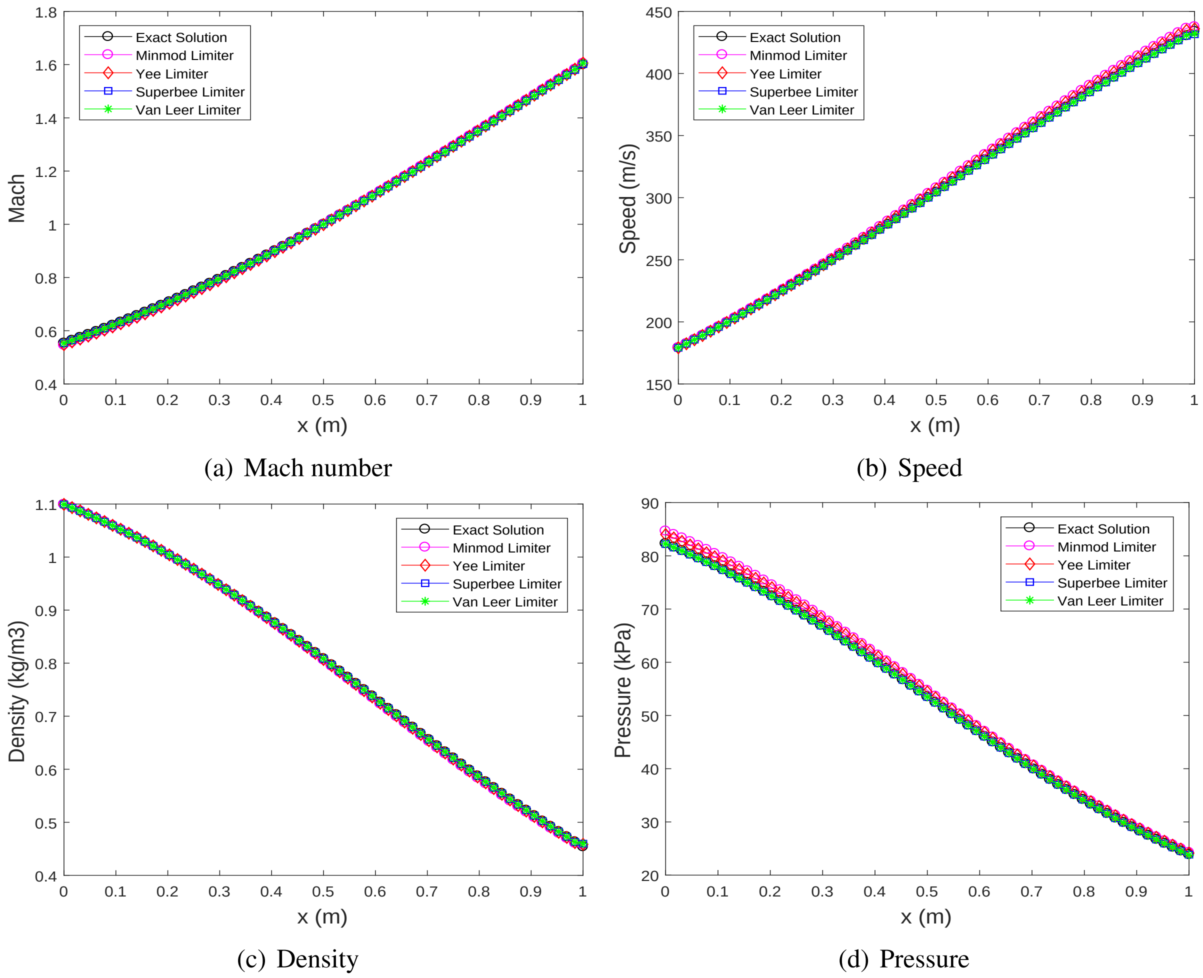
This nozzle is defined by Eq. (1) and is shown in Fig. 2. The primitive variables initial values the boundaries are given by

Table 1: Symmetrical nozzle - Boundaries initial values

Primitive variable	Inlet	Outlet
Speed	0,6355	1,5394
Density	0,8619	0,3558
Pressure	0,8122	0,2354

All the values are adimensionalized and were obtained from the isentropic flow equations considering the stagnation pressure and stagnation dsensity equal to 1.

The exact and numerical results considering this geometry are given the figure below.



(a) Mach number

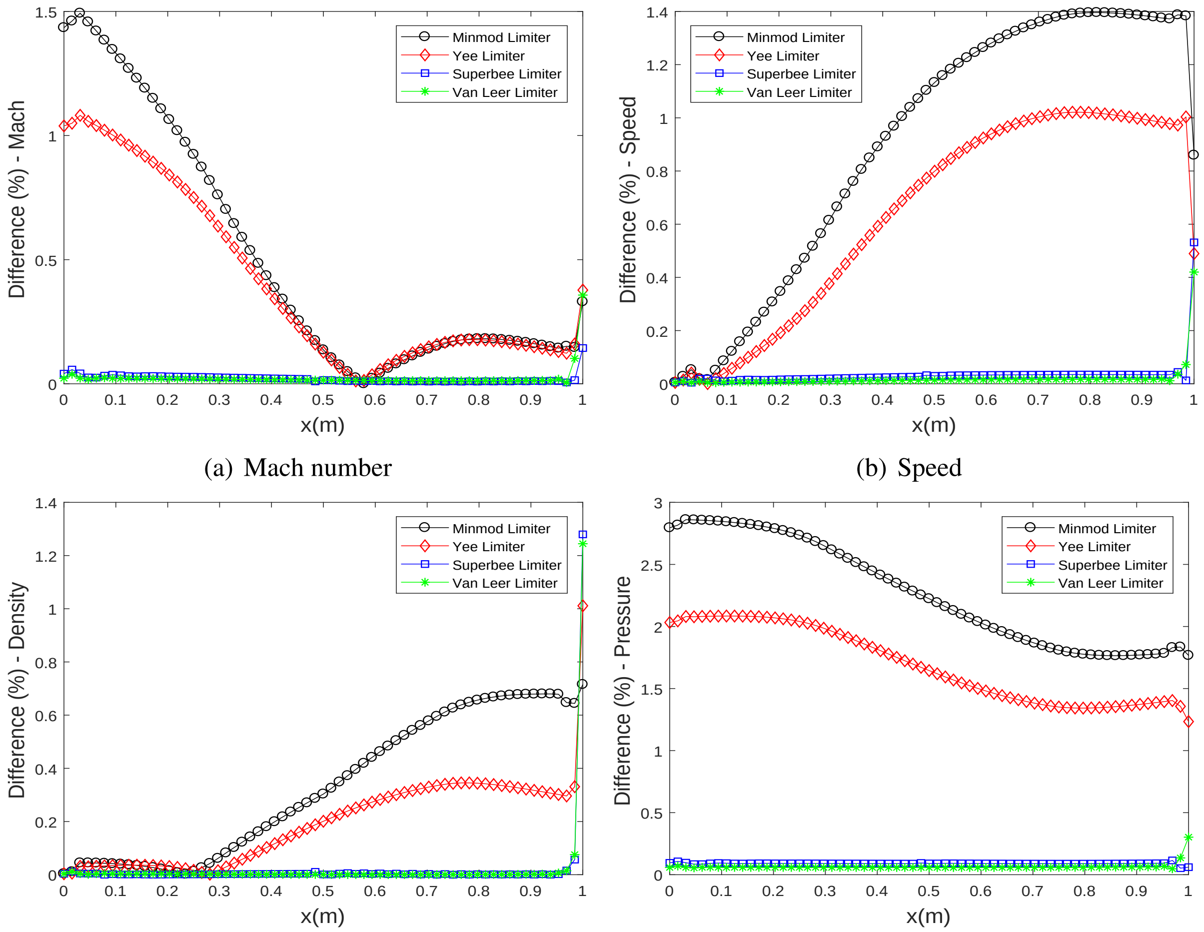
(b) Speed

(c) Density

(d) Pressure

Figure 3: Quasi one-dimensional symmetrical nozzle flow

Analyzing Fig. 3, it is possible to see that the numerical results are very close of the exact solution, thus, a statiscal analysis will be made to determine which flux limiter provide the best results for this geometry.



(a) Mach number (b) Speed

(c) Density (d) Pressure

Figure 4: Symmetrical nozzle - Flux limiters percentual difference

The results shown of Fig. 4, indicate that the minmod is the flux limiter what is providing the worst results, followed by the Yee flux limiter, however, it is difficult to say which flux limiter is the best one, as the results provided by the superbee and Van Leer flux limiters are very close together. Therefore, the median and standard deviation of Mach number, speed, density and pressure will be calculated to determine which one is closer of the exact solution. To avoid outliers results at the inlet and outlet sections, the fist point and the two last points of the domain will be disregarded.

Table 2: Median

Flux limiter	σ Mach number	σ Speed	σ Density	σ Pressure
Minmod	0,1817	1,1180	0,2950	2,2388
Yee	0,1777	0,7863	0,1943	1,6578
Superbee	0,0149	0,0300	0,0019	0,0906
Van Leer	0,0151	0,0135	0,0012	0,0588

Table 3: Standard deviation

Flux limiter	σ Mach number	σ Speed	σ Density	σ Pressure
Minmod	0,4852	0,4913	0,2601	0,4200
Yee	0,3550	0,3755	0,1321	0,2967
Superbee	0,0105	0,0088	0,0028	0,0041
Van Leer	0,0047	0,0050	0,0025	0,0026

Therefore, according to Table 2 and Table 3, the Van Leer flux limiter has the best performance to this geometry, with a small advantage to the superbee flux limiter.

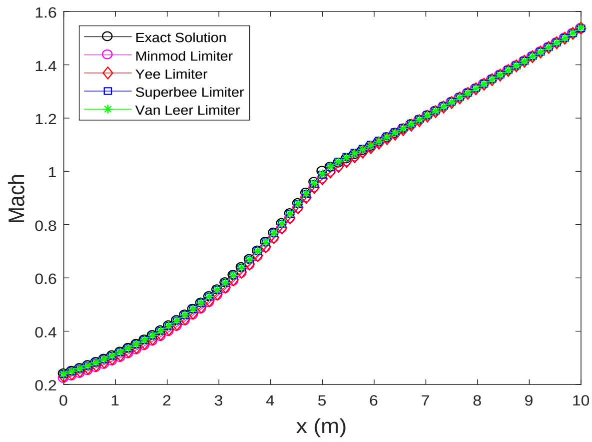
3.2 Asymmetrical nozzle

This nozzle is defined by Eq. (2) and is shown in Fig. 2. The primitive variables initial values the boundaries are given by

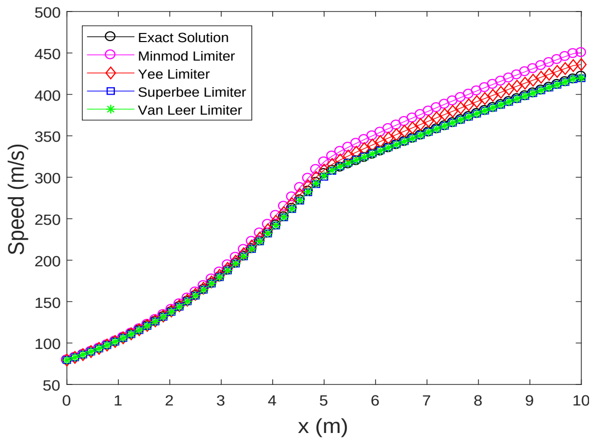
Table 4: Asymmetrical nozzle - Boundaries initial values

Primitive variable	Inlet	Outlet
Speed	0,2818	1,4968
Density	0,9719	0,3812
Pressure	0,0105	0,0088

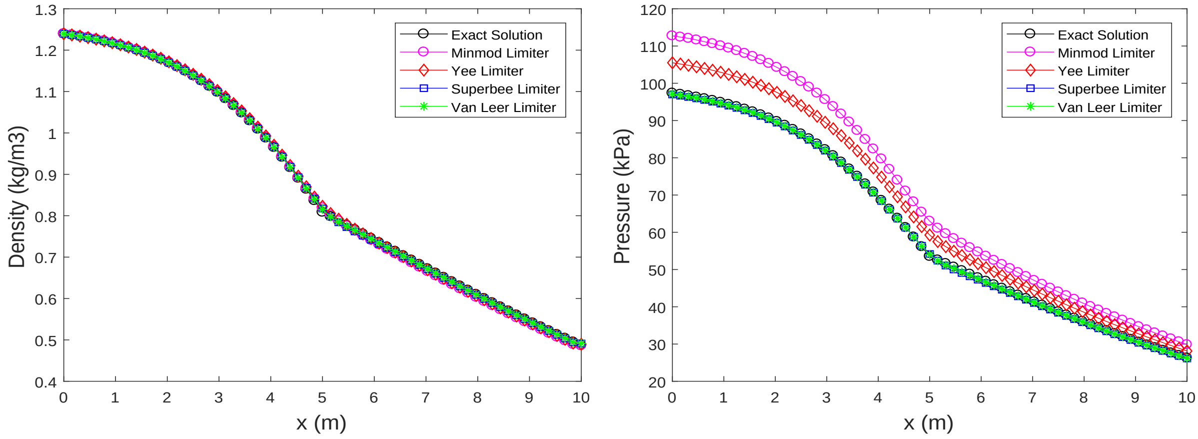
The exact and numerical results considering this geometry and the results statiscal analysis are given at the figures and tables below.



(a) Mach number



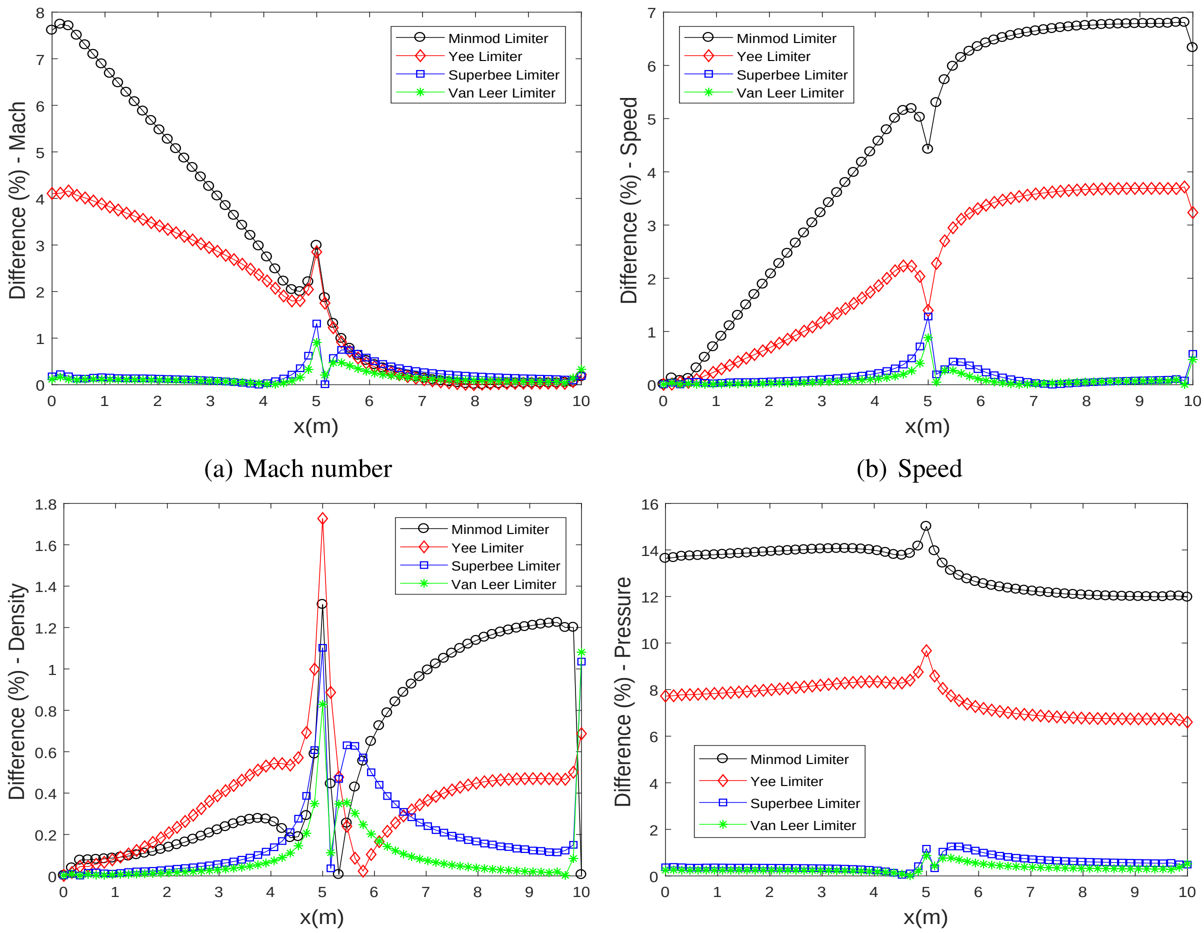
(b) Speed



(c) Density (d) Pressure

Figure 5: Quasi one-dimensional asymmetrical nozzle flow

Figure 5 shows that the results for Mach number and density are very close to the exact solution, but for speed and pressure they presents bigger differences between themselves.



(a) Mach number (b) Speed

(c) Density (d) Pressure

Figure 6: Asymmetrical nozzle - Flux limiters percentual difference

Table 5: Median

Flux limiter	σ Mach number	σ Speed	σ Density	σ Pressure
Minmod	2,0211	5,1709	0,2846	13,7347
Yee	1,8021	2,2295	0,3940	7,7921
Superbee	0,1478	0,0799	0,1259	0,4016
Van Leer	0,1025	0,0520	0,0361	0,2683

Table 6: Standard deviation

Flux limiter	σ Mach number	σ Speed	σ Density	σ Pressure
Minmod	2,6396	2,2996	0,4550	0,8862
Yee	1,5442	1,3576	0,2665	0,6889
Superbee	0,2248	0,2003	0,2051	0,2874
Van Leer	0,1395	0,1281	0,1309	0,1593

Analyzing the Tables 5 and 6, it is possible to see that again the Van Leer and superbee limiters provide the best results.

CONCLUSIONS

After analyzing the results for the symmetrical and asymmetrical nozzles, even with the problem boundary conditions not being completely defined, the superbee and Van Leer flux limiters provided consistent results according to the solution obtained by means of isentropic flow equations, while the minmod and yee limiters did not produce satisfactory results, especially for the asymmetrical case. However, it is not possible to say that these limiters that worked well for these cases will have the same performance for all flow conditions and geometries. This article analyzed a very specific case, and the choice of the more suitable flux limiter must be made according to the flow conditions and geometry for every problem.

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