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ANALYTICAL STUDY OF THE SHEAR EFFECT ON THE BUCKLING OF COLUMNS ON ELASTIC MEDIUM

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Abstract. *This work studies the shear effect on the buckling of columns embedded in an elastic medium, evidencing the interaction of the column with the foundation. The soil is assumed to be a constant governed by the Winkler model. The effects analyzed in the project are based only on the static mechanical behavior of the column. The study considered the Davisson and Robinson equation, which is based on Euler-Bernoulli's beam theory, where only the effects of axial and bending stiffness are taken into account. It is used for beams with slender profiles. However, on short beams it is expected that the shear effects should be relatively large. In this case, Timoshenko's beam theory is adopted, which considers the addition of the shear effect. Both analyses will develop fourth-order differential equations that govern both theories. For each equation, a numeric method will be implemented and a case study will be analyzed to determine the eigenvalues (critical loads) and buckling eigenvectors (eigenfunctions). At the end of the study, it was perceived that the shear effect on columns in* *elastic foundations exists; however, it is very small and may vary according to the stiffness of the elastic base and length of the column.*

Keywords: *Shear, Buckling, Numeric Method, Euler-Bernoulli, Timoshenko.*

INTRODUCTION

This work aims to develop an analytical study of the shear effect on the buckling of columns in an elastic medium, with the purpose of determining buckling eigenvalues and eigenvectors. Based on beam theories proposed by Euler-Bernoulli and Timoshenko, where mathematically the shear effect is obtained based on the hypothesis of Timoshenko.

In general, beams and columns present similar static mechanical behavior. In both cases, the cross-sectional displacement to the longitudinal axis of the part is governed by an ordinary fourth-order linear differential equation with constant coefficients (ALMEIDA, 2003). For this reason, the application of theories of beams in the study of columns is justified.

According to Veliovich (2015), columns are deep-foundation structural elements that generally transfer loads from buildings to the ground from the lateral friction generated between them and the ground. Bastos (2015) defines beams as being structural elements subject to cross-sectional loads that are generally used to transfer vertical loads.

The main difference between the two structural elements is the presence of an axial compressive load on the columns, which creates a stability problem. Mathematically, there is a problem of eigenvalues in which the eigenvalues are the critical loads of buckling and the associated eigenvectors are the modes of buckling.

Typically, beams and columns are much longer in length than the cross-sectional area, and are considered one-dimensional structures. This justifies the use of the Euler-Bernoulli beam theory, in which shear deformation is not considered and the rotation at any point of the beam is given by the derivative of the cross-sectional displacement depending on the position. (SOARES, 2009).

When the relationship between cross-sectional area and beam length is not so great, the shear deformations become relevant and the use of the Euler-Bernoulli beam theory becomes inadequate. In these cases, Thimoshenko’s beam theory is used, in which the shear deformation at a certain point along the beam is obtained by the difference between the rotation and the derivative of the cross-sectional displacement at that point (TIMOSHENKO, 1922).

Sousa (2005) applies Timoshenko’s beam theory on beams supported by an elastic basis, subjected to dynamic loads, to study the phenomenon of resonance, and the results obtained by Timoshenko’s beam model are more consistent than those obtained with Euller-Bernoulli’s beam model (due to the addition of shear deformations). The elastic base is assumed to be a foundation or some type of cushion, and it is responsible for the change in the main differential equation of deflection.

Timoshenko's beam model is, in certain situations, much closer to reality than the Euler-Bernoulli's theory. This fact can be explained by the contribution of the shear effect, verified in beams subjected to any loads, mainly related to geometric aspects (relation between beam length and section height). Another advantage in relation to Timoshenko’s model regards dynamic analyses, which are essentially used for natural frequency analyses.

As to the application, columns are usually used in a partially set manner. This configuration allows the element to be divided into two lengths, a free length and a foundation length, the latter being the focus of this study.

Thus, this work analyzed the static behavior of columns subjected to an axial compressive load, considering that the columns is completely buried. The soil is considered as a constant physical model presenting a certain elastic rigidity (elastic base theory). Starting from geometric compatibility equations, material constitutive laws and equilibrium relations, the study determined ordinary fourth-order differential equations to govern this behavior based on theories of beams, with the intention to answer the following questions:

Is the shear effect really important in the study of buckling of columns in an elastic medium (foundation)?

What is the influence of the shear effect, the length of the column and the elastic foundation on the critical load value of the column?

Indeed, shear had a very small effect on the buckling of columns in elastic foundations, and this effect varies according to the elastic stiffness of the soil and the length of the columns foundation, and also influences the critical load values.

1 BIBLIOGRAPHICAL REVIEW

Timoshenko (1953) emphasizes that beam theories began to be studied in the XVII century. In the mid-eighteenth century, Bernoulli and especially Euler presented works that can be considered as the threshold of geometric non-linear theory for bars with linear elastic materials. Euler-Bernoulli (1744) and Timoshenko (1921, 1922) developed theories based on the bending of beams.

Popov (1978) points out that the fundamental kinematics hypothesis of the beam deflection problem is the hypothesis that flat sections remain flat during deformation. This hypothesis is considered by the two beam theories used in this study. Hibbeler (2010) and Borges (1996) are good references on the hypotheses by Euler-Bernoulli’s and Timoshenko’s beam theories, respectively. Prakash *et al*. (1989) presented a criterion of classification of beams according to the efforts being employed. In addition, the authors presented several soil resistance parameters and a study of buckling and flexing of partially-set columns.

Davisson and Robinson (1963) developed a differential equation that describes the behavior of foundation columns, considering only the effect of deflection based on Euler-Bernoulli’s beam theory. In this study, the authors considered partially-set columns, with a free length and another in the foundation. They modeled the soil behavior according to Winkler's elastic base theory.

Davisson *et al*. (1985) developed a procedure to determine the actual bending length of partially crimped columns. The work by Sousa (2005), Serebrenick (2004) and Pereira (2002) presented a differential equation of dynamic equilibrium of cables and beams, considering the effects of geometric friction, shear, buckling and inertia. In these works, the authors were interested in studying these effects in the natural frequencies of beams and cables in different support configurations.

Euler-Bernoulli’s theory determines the behavior of bent beams under a given static or dynamic load. According to Carrera *et al*. (2011), this theory is based on the physical hypotheses that planes perpendicular to a neutral line of the beam remain flat and perpendicular after deformation. The angle of rotation of the cross-section, for the Euler-

Bernoulli theory, is the same as the angle of rotation of the referential axis. This rotation at any point of the beam is obtained from the derivative of the cross-sectional displacement, as shown in the following equation: (Sousa *et al.*, 2011).

$$\theta_B = v'(x) \tag{1}$$

where θ_B is the cross-section rotation of the beam for Euler-Bernoulli's theory, and $v(x)$ is the cross-sectional displacement of the beam.

The Euler-Bernoulli beam hypothesis, also known as classical beam theory, is the most commonly used theory for studying the behavior of bent beams. However, Euler-Bernoulli's theory disregards some effects of the behavior of beams which, depending on the situation, may be important in the behavior of beams. Timoshenko (1922) proposed a beam theory based on Euler-Bernoulli's beam theory, but adding the shear effect.

Timoshenko's beam theory is a complement to the classical Euler-Bernoulli beam theory, that is, it takes into account the effects of the bending moment and lateral displacement on the beams. However, Timoshenko attributes the shear distortions effects (Timoshenko, 1922). This beam theory is based on the fact that the sections perpendicular to the neutral line remain flat, but not necessarily perpendicular, after deformation, since the rotation angle of the cross section is no longer the same as the referential axis. This angle also presents a small twist, caused by the shear distortions, as shown by the equation below: (Sousa *et al.*, 2011).

$$\theta_T = v'(x) - \gamma_{xy} \tag{2}$$

where θ_T is the cross-section rotation of the beam for Timoshenko's theory, and $v(x)$ is the cross-sectional displacement of the beam, and γ_{xy} is the shear deformation.

The study of beams on elastic bases seeks to evaluate the effects of this elastic base on the static and/or dynamic behavior of beams. In order to do this, it uses physical models of soil behavior, among which Winkler's elastic base theory can be highlighted. The fundamental hypothesis of this theory is that the reactive forces of the base at a point of the beam are proportional to the deflection at that point (Silva, 1998).

According to Silva (1998), Winkler's elastic base model is equivalent to a layer of springs that are closely spaced and independent of each other. These springs will be treated as independent elastic bases. Equation (3) represents the hypothesis adopted in the Winkler model, where the force distributed per unit of length (*rb*) is obtained from the product of the elastic stiffness constant of the soil (*k*), which can vary with depth (x), with the cross-sectional displacement of the referential axis of the beam for a given coordinate along its length. That is, the normal reaction at a given point in the structure is directly proportional to the deflection exerted at that point.

$$rb = kv(x) \tag{3}$$

The work by Davisson and Robinson (1963) is taken as starting point. This work uses Euler-Bernoulli's classical beam theory, in which only the effects of axial loads and bending are considered, neglecting the effects of mass and moment of inertia of rotation, and considering Winkler's elastic base model. The critical buckling load in this case is obtained from the solution of the eigenvalues problem of a fourth-order homogeneous linear differential equation with constant coefficients.

Timoshenko's model is more indicated for problems with short beams, because in this model the shear effect is taken into consideration. Since columns are beams under axial compressive loads, the shear effect is expected to be important. Analytically, shear stiffness arises from Timoshenko's beam hypotheses. In this way, a theoretical study of the shear effect on the value of the critical buckling load of columns on an elastic base will be performed.

Silva (2013) analyzed the slope stability assessing the effect that the shear strength of the pile has on its safety factors. Costa (2006) used Timoshenko's model on elastic beams with the objective of determining the Newtonian formulation of the Timoshenko model, through fundamental response and modal analysis. Mendonça (2008) performed a dynamic analysis of a Timoshenko beam on an elastic basis using integral equations. The work of Prakash (1990) and Lundgren (1979) presented the study by Davisson and Robinson (1963) to determine the foundation length of partially set columns.

2 BEAM THEORIES AND THEIR DIFFERENTIAL EQUATIONS

The following hypotheses were considered to determine the equilibrium equations:

- The column is considered to be inextensible;

- A reasoning that considers small displacements is applied;

- Because of the hypothesis above, there is a linear compatibility relation between the components of the deformation stresses and the horizontal (u) and vertical (v) displacement fields;

- The material considered here is subject to linear elastic constitutive relations;

- The soil is assumed to be an elastic medium based on Winkler's hypothesis;

- Cross-sections to the length of the beam remain flat after deflection.

The main difference between the beam hypotheses used in this study is the addition of the shear effect:

- Euler-Bernoulli's beam theory: The shear deformation is not considered;

- Timoshenko's beam theory: The shear deformation is considered.

The difference between both beam theories is a small change in the kinematics hypothesis, resulting in the consideration, or not, of the shear effect. Figure 1 presents a cross-section *AB* throughout the length of an intact beam which, after being bent, goes to position *A'B'* due to a rotation of $\theta(x)$ (Popov, 1978).

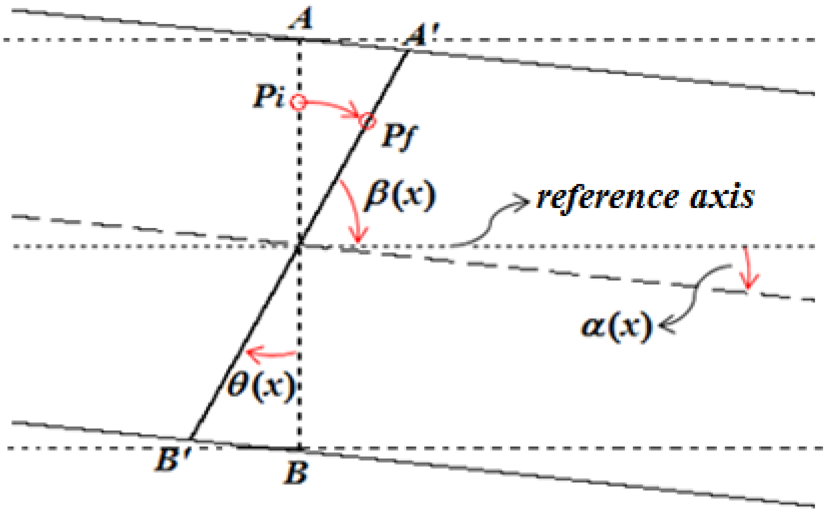

Figure 1. Detail of a deformed section of the column

2.1 Compatibility equations

Verifying Figure 1, it can be seen that the final position (*Pf*) if obtained from the initial position (*Pi*) of any point throughout the length of a certain cross-sectional section. This comes from considering that the points on the referential axis do not change their y coordinate. From the coordinates (x,y) of the point *Pi* and (x_f,y_f) the coordinates of point *Pf*, it is possible to describe the fields of horizontal $u(x,y)$ and vertical $v(x,y)$ displacement at any point of the column, expressed by:

$$u(x,y)=u_0(x)+y\,\mathrm{sen}(-\theta(x))\therefore u(x,y)=u_0(x)-y\,\mathrm{sen}(\theta(x)) \tag{4}$$

$$v(x,y)=v_0(x)-y[1-\cos(-\theta(x))]\therefore v(x,y)=v_0(x)-y[1-\cos(\theta(x))] \tag{5}$$

Applying the hypothesis of small displacements, and considering that the column is not extendable, Equations (4) and (5) can be rewritten, resulting in:

$$u(x,y)=-y\theta(x) \tag{6}$$

$$v(x,y)=v_0(x) \tag{7}$$

The following notation will be used: $v(x)\Leftrightarrow v_0(x)$.

As a consequence of this hypothesis, Equations (8), (9) and (10) are obtained, and represent the linear compatibility relationships between the displacement fields and the components of the deformation tensioner.

$$\varepsilon_{xx}(x,y)=\frac{\partial u(x,y)}{\partial x} \tag{8}$$

$$\varepsilon_{yy}(x,y)=\frac{\partial v(x,y)}{\partial y} \tag{9}$$

$$\gamma_{xy}(x,y)=2\varepsilon_{xy}(x,y)=\frac{\partial u(x,y)}{\partial y}+\frac{\partial v(x,y)}{\partial x} \tag{10}$$

where $\varepsilon_{xx}(x,y)$ and $\varepsilon_{yy}(x,y)$ are the longitudinal and cross-sectional unit deformations, respectively, $\gamma_{xy}(x,y)$ is the shear deformation.

2.2 Constitutive equations

From the linear elastic constitutive hypotheses of the materials result Equations (11) and (12):

$$\sigma_{xx}(x,y) = E\varepsilon_{xx}(x,y) \tag{11}$$

$$\sigma_{xy}(x,y) = G\gamma_{xy}(x,y) \tag{12}$$

2.3 Equilibrium equations between stresses and tensor components of the stresses

From the equilibrium equations, results:

$$Q(x) = \int_A [\sigma_{xy}(x,y)]dA \tag{13}$$

$$M(x) = -\int_A dFy \therefore dF = \sigma_{xx}(x,y)dA \therefore M(x) = -\int_A [y\sigma_{xx}(x,y)]dA \tag{14}$$

Developing the derivatives of Equations (8) and (10) from the considerations used for Equations (4) and (5), the results on Equations (11) and (12) are replaced and, finally, the stress expressions obtained with Equations (13) and (14) are used. Consequently, the study found that the equations of shear stress and bending moment depend on the characteristic of the material, expressed by Equations (15) and (16), respectively:

$$Q(x) = GA\gamma_{xy}(x) \tag{15}$$

$$M(x) = EI\frac{d}{dx}\theta(x) \tag{16}$$

2.4 Considerations applied to the beam theories

After defining the equilibrium conditions, the beam hypotheses by Euler-Bernoulli and Timoshenko were applied, where the fundamental difference between them is the rotation angle (θ).

Euler-Bernoulli's theory assumes that the rotation angle of the referential axis is the same in the cross-sectional section, which is expressed by Equation (1). Applying it to Equation (6), the result is:

$$Q(x) = GA\gamma_{xy}(x) \quad \text{com} \quad \gamma_{xy}(x) = 0 \tag{17}$$

$$M(x) = EI\frac{d}{dx}v'(x) \Leftrightarrow M(x) = EIv''(x) \tag{18}$$

In Timoshenko's beam theory, the rotation angle of the referential axis is not the same in the cross-sectional axis. Due to this, the displacement fields are not altered, in the same manner as Equations (15) and (16), which relates efforts with deformation and characteristic of the material.

2.5 Differential equations – Equilibrium

The differential equations of equilibrium are obtained by the direct method. In this case, equilibrium is imposed through the universal static equations. Figure 2 presents an infinitesimal element of the column of length d*x*, where it is under the action of axial and cross-sectional forces and bending moments and elastic foundation.

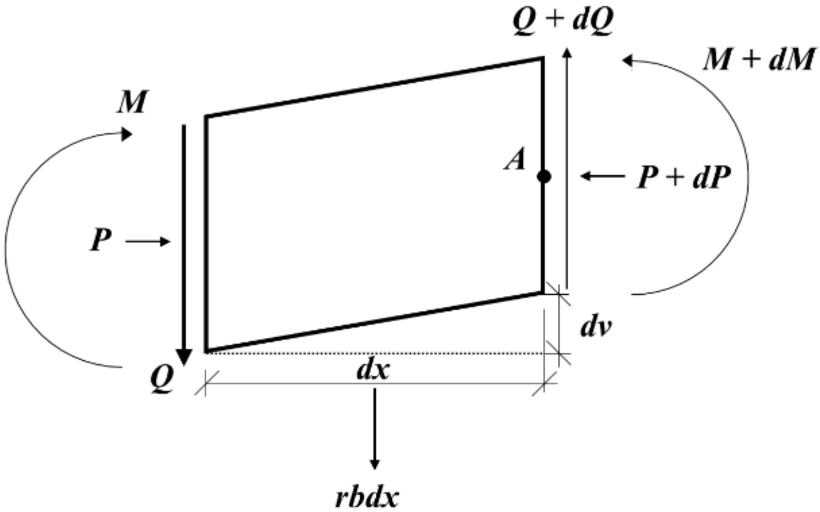

Figure 2. Equilibrium of the infinitesimal element considering the bending stiffness and elastic base

Where *P* is the compression force, *M* is the bending moment, *Q* is the shear effort, *rb* is the reaction of the elastic base, d*x* is the length of the infinitesimal element and d*v* is the increase in vertical displacement.

From the sum of the vertical forces and moment around point A which are equal to zero, considering Winkler's elastic base model (Equation 3). Equation (18) is used to obtain the differential equation of equilibrium considering Euler-Bernoulli's beam theory.

$$v^{iv}(x)+\frac{P}{EI}v''(x)+\frac{k}{EI}v(x)=0 \tag{19}$$

In the same way, the differential equation of equilibrium is obtained considering Timoshenko's beam theory, where the shear effect is considered. For that, It is necessary to replace Equations (15) and (16) in the sum of forces and moments, obtaining:

$$v^{iv}(x)+\left(\frac{P}{EI}-\frac{k}{GA}\right)v''(x)+\frac{k}{EI}v(x)=0 \tag{20}$$

When analyzing this expression above, it can be perceived that it is a generalization of Euler-Bernoulli's differential equation of static equilibrium. For this, the shear stiffness must tend to infinity, and Equation (20) will fall on Equation (19).

3 GENERIC ANALYTICAL SOLUTION OF THE DIFFERENTIAL EQUATION OF THE PROBLEM

As seen in the previous chapter, regardless of the adopted beam theory, the differential equations of static equilibrium have the same general form. Equation (50) presents the general aspect of the equilibrium equations used for this work. The difference between the two cases is in the coefficient of the second-order derivative (B_2).

$$v^{iv}(x)+0v'''(x)+B_2v''(x)+0v'(x)+B_0v(x)=0 \tag{21}$$

where B_2 and B_0 are the constant coefficients of the displacement derivatives. If the concept of differencial operator $(D_x \Leftrightarrow \mathrm{d}/\mathrm{d}x)$ is used, the equation above will be as follows:

$$\left[D_x^{\ 4}+0D_x^{\ 3}+B_2D_x^{\ 2}+0D_x+B_0\right]v(x)=0 \tag{22}$$

Equation (22) is a fourth-degree polynomial. Avoiding the trivial solution $v(x)=0$, to find the solution for the Equation above it is necessary to determine the roots (r_1, r_2, r_3, r_4) of the fourth-degree polynomial. Expressing the polynomial as dependent on its roots, we obtain a system of equations, whose solution is:

$$r_1=+ia,\ \ r_2=-ia,\ \ r_3=+b,\ \ r_4=-b \tag{23}$$

In which $i=\sqrt{-1}$, *a* and *b* are expressed by:

$$a=\sqrt{\frac{B_2+\sqrt{B_2^{\ 2}-4B_0}}{2}} \tag{24}$$

$$b=\sqrt{\frac{-B_2+\sqrt{B_2^{\ 2}-4B_0}}{2}} \tag{25}$$

Obtaining the polynomial roots (r_1, r_2, r_3, r_4) results in the differential equation decreased from the boundary conditions, which is as follows:

$$v(x)=C_1\cos(ax)+C_2\,\mathrm{sen}(ax)+C_3\cosh(bx)+C_4\,\mathrm{senh}(bx) \tag{26}$$

After determining the vertical displacement solution, the rotation can be expressed depending on the considered beam theory. If the Euller-Bernoulli's beam theory is being considered, rotation (θ_B) will derive from the displacement, whose result is obtained replacing Equation (26) for Equation (1), obtaining Equation (27). In the same manner, considering the equation for the rotation in Timoshenko's beam theory (θ_T), present in Equation (2), results in Equation (28).

$$\theta_B=-aC_1\,\mathrm{sen}(ax)+aC_2\cos(ax)+bC_3\,\mathrm{senh}(bx)+bC_4\cosh(bx) \tag{27}$$

$$\theta_T(x)=v'(x)-\frac{k}{GA}\int v(x)dx \tag{28}$$

To determine the eigenfunctions of the behavior of the Euler-Bernoulli's and Timoshenko's functions, it is necessary to determine the boundary conditions of the problem. Based on the beam-displacement equation represented by Equation (26), the boundary conditions are used for the two-part column case presented below:

$$\begin{aligned} & v(0) = 0 \\ & M(0) = \theta'(0) = 0 \\ & v(L) = 0 \\ & M(L) = \theta'(L) = 0 \end{aligned} \tag{29}$$

Equations (30) and (31) present Euler-Bernoulli's and Timoshenko's eigenfunctions, respectively:

$$\text{sen}(a_B L) = 0 \tag{30}$$

$$\text{sen}(a_T L) = 0 \tag{31}$$

where L is the length of the column, a_B and a_T are the roots of the Euler- Bernoulli's and Timoshenko's polynomials, respectively.

Equation (32) is the characteristic function or autofunction of the problem that only considered the bending stiffness (Euller-Bernoulli's beam theory) and the respective boundary conditions.

$$v(x) = C_2 \left[\text{sen}(a_B x) - \text{senh}(b_B x) \frac{\text{sen}(a_B L)}{\text{senh}(b_B L)} \right] \tag{32}$$

Equation (33) is the characteristic function or autofunction of the problem that only considered the bending stiffness plus the shear effect (Timoshenko's beam theory) and the respective boundary conditions.

$$v(x) = C_2 \left[\text{sen}(a_T x) - \text{senh}(b_T x) \frac{\text{sen}(a_T L)}{\text{senh}(b_T L)} \right] \tag{33}$$

4 CASE STUDY

After obtaining the eigenfunctions for the case in which the bending stiffness and bending stiffness with shear effect were considered, the study sought to analyze the behavior of the eigenfunctions and their eigenvectors for a practical case.

To perform the following analyses, it is necessary to determine the properties of the elastic base (stiffness of the soil - k); of the material (longitudinal elasticity - E and cross-sectional elasticity - G); and the geometric properties (moment of inertia - I, length of the column - L and area of the cross-sectional section - A).

According to Gerdau (2015), metal columns have outstanding applications as foundation elements. Thus, this case study used a metal column with a type-I cross-sectional section, classified by its manufacturer as W200 x 86 of A-36 steel, with a cross-sectional section (A)

of 0.01109 m^2 and moment of inertia (I) of $3.139x10^{-5}$ m^4. The longitudinal elasticity (E) and cross-sectional elasticity (G) of the steel are, respectively, $2x10^{11}$ N/m^2 and $77x10^9$ N/m^2.

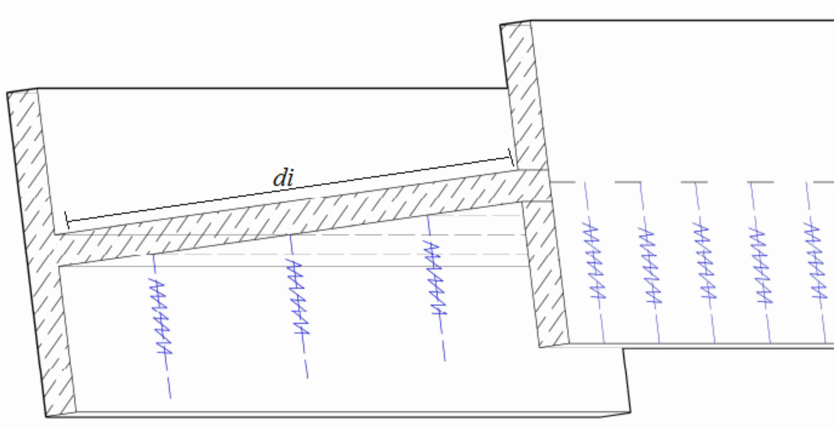

Figure 3: Configuration of the beam used in elastic medium with the length of the core (*di*).

The analyses vary between elastic bases of medium clay soils with 2.0 kgf/cm^3 ($1.691x10^7$ N/m^3) and hard clay with 10.0 kgf/cm^3 ($9.807x10^7$ N/m^3), according to Table 1. However, these values are multiplied by the length of the core (di) of 0.157 m, obtaining k = 2.65 x 10^6 N/m^2 for medium-clay soils and k = 1.54 x 10^7 N/m^2 for hard-clay soils. Three column lengths were used: 10 m, 25 m and 50 m.

Table 1. Stiffness values for clayey soils in kgf/cm^3(k)

	SOFT CLAY	MEDIUM CLAY	STIFF CLAY	HARD CLAY
DRY OR SATURATED	0 to 1.5	2	5	10

4.1 Eigenvalues

This section will determine the eigenvalue (critical load) for both types of elastic bases (medium clay and hard clay) used previously for length columns (L) of 10, 25 and 50 m. Thus, six cases were defined: Case 1 – Medium clay with L = 10 m, Case 2 – Medium clay with L = 25 m, Case 3 – Medium clay with L = 50 m, Case 4 – Hard clay with L = 10 m, Case 5 – Hard clay with L = 25 m and Case 6 – Hard clay with L = 50 m. The following subsections will determine the relative error of Timoshenko's autofunction in relation to the Euler-Bernoulli's autofunction from Equation (34).

$$ER = \left(\frac{|P_B - P_T|}{P_B} \right) \times 100 \tag{34}$$

4.1.1 Medium Clay (k = 2.65 x 10^6 N/m^2)

Tables 2, 3 and 4 present the values of the six first eigenvalues of the eigenfunctions that correspond to the theories by Euler-Bernoulli and Timoshenko for Cases 1, 2 and 3.

Table 2. Six first eigenvalues for medium clay soils with a foundation length of 10 m (Case 1)

EIGENVALUES	EULER-BERNOULLI P_B (N)	TIMOSHENKO P_T (N)	RELATIVE ERROR (%)
1	8.749 x 10^6	8.769 x 10^6	0.228
2	1.193 x 10^7	1.195 x 10^7	0.167
3	1.709 x 10^7	1.711 x 10^7	0.117
4	2.381 x 10^7	2.383 x 10^7	0.083
5	3.194 x 10^7	3.196 x 10^7	0.062
6	4.142 x 10^7	4.144 x 10^7	0.048

Table 3. Six first eigenvalues for medium clay soils with a foundation length of 25 m (Case 2)

EIGENVALUES	EULER-BERNOULLI P_B (N)	TIMOSHENKO P_T (N)	RELATIVE ERROR (%)
1	8.448 x 10^6	8.467 x 10^6	0.224
2	9.189 x 10^6	9.199 x 10^6	0.108
3	1.038 x 10^7	1.040 x 10^7	0.096
4	1.194 x 10^7	1.195 x 10^7	0.083
5	1.381 x 10^7	1.382 x 10^7	0.072
6	1.594 x 10^7	1.595 x 10^7	0.062

Table 4. Six first eigenvalues for medium clay soils with a foundation length of 50 m (Case 3)

EIGENVALUES	EULER-BERNOULLI P_B (N)	TIMOSHENKO P_T (N)	RELATIVE ERROR (%)
1	8.448 x 10^6	8.467 x 10^6	0.224
2	8.760 x 10^6	8.769 x 10^6	0.102
3	9.189 x 10^6	9.199 x 10^6	0.108
4	9.748 x 10^6	9.749 x 10^6	0.083
5	1.038 x 10^7	1.040 x 10^7	0.096
6	1.119 x 10^7	1.119 x 10^7	0.017

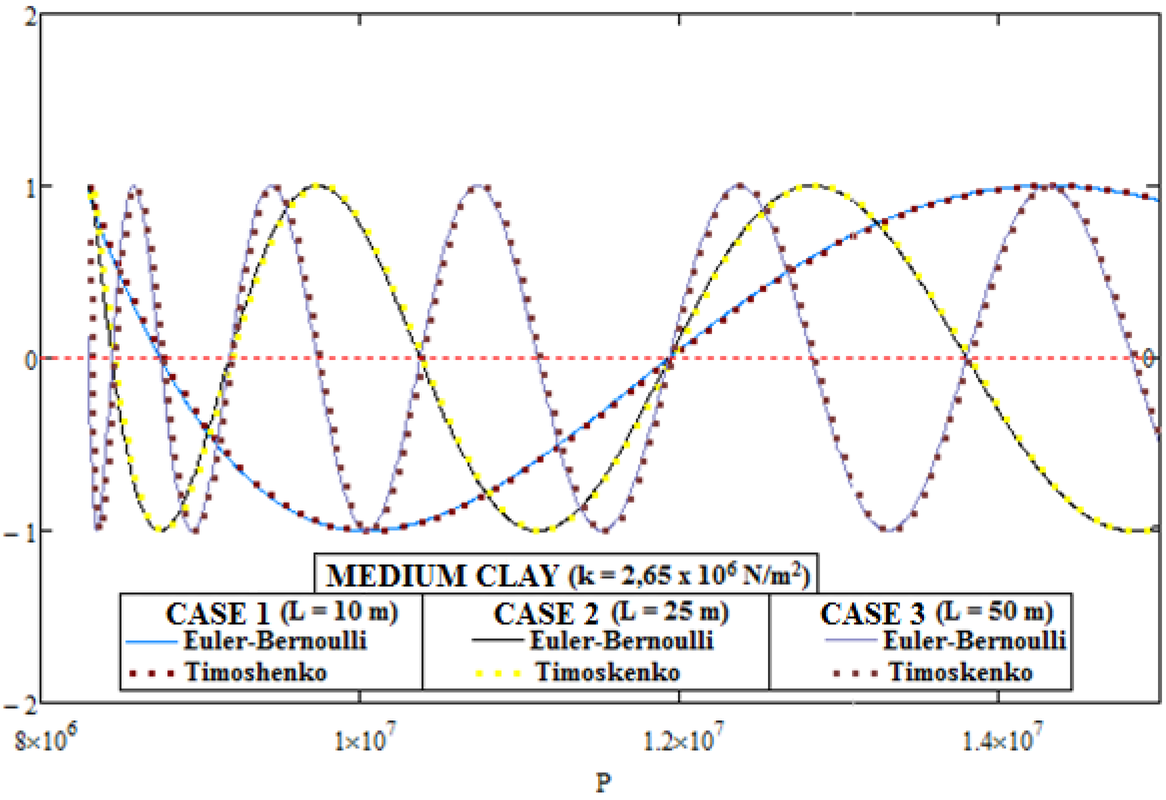

Figure 4: Behavior of the Euler-Bernoulli and Timoshenko eigenfunctions for the medium clay soils in Cases 1, 2 and 3.

4.1.2 Hard Clay (k = 1.54 x 10^7 N/m^2)

Tables 5, 6 and 7 present the values of the six first eigenvalues of the eigenfunctions that correspond to the theories by Euler-Bernoulli and Timoshenko for Cases 4, 5 and 6.

Table 5. Six first eigenvalues for hard clay soils with a foundation length of 10 m (Case 4)

EIGENVALUES	EULER-BERNOULLI P_B (N)	TIMOSHENKO P_T (N)	RELATIVE ERROR (%)
1	2.000 x 10^7	2.012 x 10^7	0.600
2	2.226 x 10^7	2.236 x 10^7	0.449
3	2.739 x 10^7	2.750 x 10^7	0.401
4	3.458 x 10^7	3.469 x 10^7	0.318
5	4.340 x 10^7	4.351 x 10^7	0.253
6	5.385 x 10^7	5.393 x 10^7	0.148

Table 6. Six first eigenvalues for hard clay soils with a foundation length of 25 m (Case 5)

EIGENVALUES	EULER-BERNOULLI	TIMOSHENKO	RELATIVE

	P_B (N)	P_T (N)	ERROR (%)
1	2.000×10^7	2.012×10^7	0.600
2	2.046×10^7	2.057×10^7	0.537
3	2.153×10^7	2.164×10^7	0.510
4	2.309×10^7	2.321×10^7	0.519
5	2.506×10^7	2.517×10^7	0.438
6	2.740×10^7	2.751×10^7	0.401

Table 7. Six first eigenvalues for hard clay soils with a foundation length of 50 m (Case 6)

EIGENVALUES	EULER-BERNOULLI P_B (N)	TIMOSHENKO P_T (N)	RELATIVE ERROR (%)
1	2.000×10^7	2.012×10^7	0.600
2	2.014×10^7	2.026×10^7	0.595
3	2.046×10^7	2.057×10^7	0.537
4	2.093×10^7	2.104×10^7	0.525
5	2.153×10^7	2.164×10^7	0.510
6	2.226×10^7	2.237×10^7	0.494

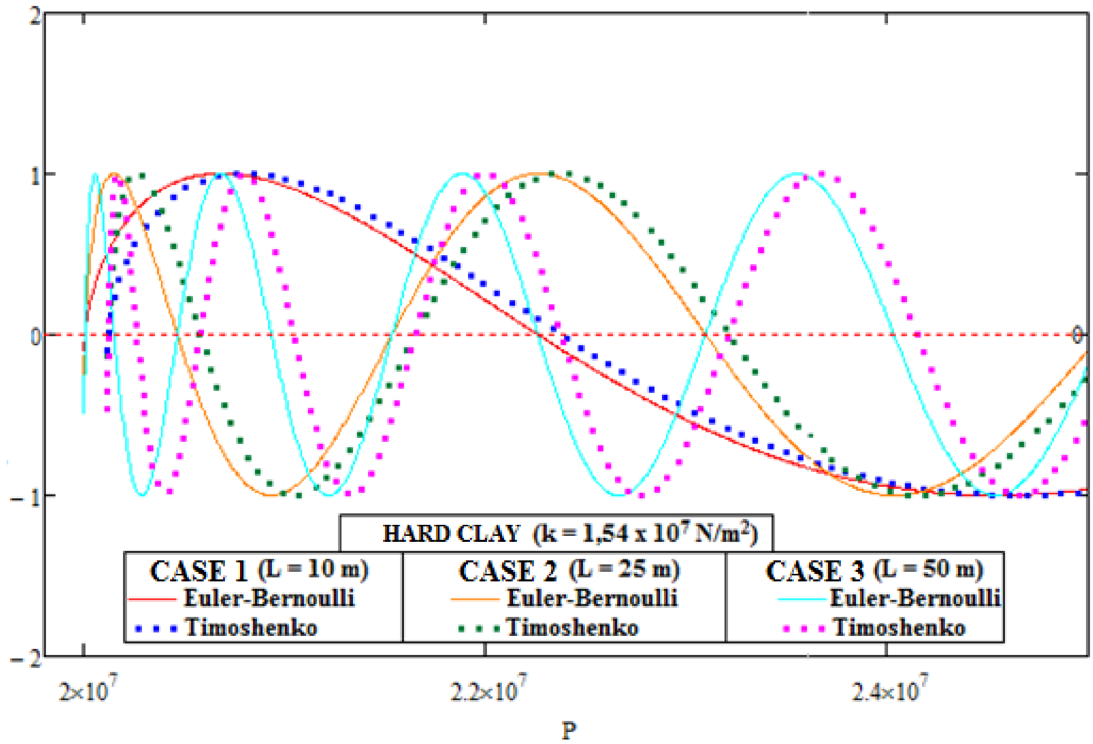

Figura 5: Behavior of the Euler-Bernoulli and Timoshenko eigenfunctions for the hard clay soils in Cases 4, 5 and 6.

4.2 Result Analyses

From Tables 2, 3, 4, 5, 6 and 7, and Figures 4 and 5, it is possible to perceive a small shear effect on the eigenvalues for the six cases, which was expected, since Equation (19), determined from the Euler-Bernoulli's beam theory, was obtained from Equation (20) (Timoshenko's theory) when it did not consider the deformation caused by the shear effect, which corresponds to considering infinite stiffness ($GA \rightarrow \infty$).

On cases 1, 2 and 3 (medium clay), the study found that, when not considering the shear effect, an error of close to 0.22% on the critical value of the column occurs. Since the analyzed cases correspond to a medium soil with respective lengths equal to 10, 25 and 50 m, the study concluded that due to the presence of an elastic base the length had a small, almost insignificant influence on the first eigenvalue.

As to cases 4, 5 and 6 (hard), the study found that the first eigenvalue obtained with the Euler-Bernoulli's theory and Timoshenko's theory were the same for the three cases, that is, $P_B = 2.000 \times 10^7$ N and $P_T = 2.012 \times 10^7$ N. When the shear effect was not considered for the hard soil, an error of 0.60% was verified. Since cases 4, 5 and 6 correspond to a hard soil with respective lengths equal to 10, 25 and 50 m, the study concluded that due to the presence of an elastic base the length did not influence the first eigenvalue.

Other important observations are:

- The stiffness effect of the elastic base is more important than the length effect. This conclusion can be drawn from the fact that the error, when not considering the shear effect for the medium clay soil, was 0.22%, and for the hard clay soil it was 0.60%, regardless of the length of the stake;

- The shear effect decreases as the order of eigenvalues increases. This tendency was verified for the six cases. However, the literature explains that the first buckling mode (n = 1) is the most important for buckling cases and, generally, in a practical case, the structures do not present a second buckling mode without them presenting large deflections, which causes them to collapse. (Hibbeler, 2010; Popov, 1978).

5 CONCLUSION

This work studied the static behavior of columns/beams subjected to axial compressive loads, characterizing the problem of the buckling of columns under the effect of an elastic base, in which only bending stiffness should be considered according the beam theory by Euler-Bernoulli. This work uses the article by Davisson and Robinson (1963) as a starting point and adds the shear effect from Timoshenko's beam theory. Therefore, two beam theories are used: Euler-Bernoulli's theory (classical beam theory), and Timoshenko's, in which the shear effect is considered.

Considering the two beam theories together with the geometric compatibility equations, elastic linear constitutive law and equilibrium equations, the study found, in both cases, a fourth order linear differential equation with constant coefficients, where the only coefficients different from zero were those of the fourth derivative ($B_4 = 1$) and second order (B_2) and the independent term (B_0). Mathematically, the difference between the two theories appears on coefficient B_2 when considering the effect of the elastic base. This work also presented the analytical solution of the differential equations of the problem of buckling according to the two beam theories and independently of the boundary conditions.

Only one boundary condition was analyzed in this work, a two-part column, obtaining two eigenfunctions corresponding to the beam theories by Euler-Bernoulli and Timoshenko, which caused a problem of eigenvalue, from which the study sought to obtain a nontrivial solution of the roots of the characteristic equation of the problem, which are the critical buckling loads. The study identified that the eigenvalues of the eigenfunctions referring to the two theories are not periodic, presenting nonlinear terms as exponential and trigonometric functions. Therefore, these eigenvalues were determined from an algorithm based on the bisection method.

Four cases were analyzed in this study. Cases 1, 2 and 3 considered a medium clay soil with lengths equal to 10, 25 and 50 m, respectively. However, cases 4, 5 and 6 considered a hard clay soil (stiffness greater than that of the medium clay) with the same lengths adopted for the previous cases. In general, in all cases the shear effect was very small, which was expected, since the columns from constructions with a long time of use would have presented some type of structural problem.

Case 6, with a more rigid soil and length of 50 m was the one that presented the greatest shear effect for all calculated eigenvalues (six). The largest error occurred for the critical load (first eigenvalue), with a value of 0.60%. In all cases, a tendency was observed for the shear effect to decrease as eigenvalues increase. Another point observed was that the length had no significant influence on the critical load for the studied soil types. In relation to the effect of soil rigidity, it was found that when soil stiffness increases, eigenvalues increase, which is physically expected, since the higher the soil stiffness, the greater the resistance to lateral displacement of the column.

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