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NONLINEAR CONTROL OF A ROTATING SMART BEAM

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Abstract. *In this paper a piezoelectric actuator is used for position and vibration control of a nonlinear rotating system comprising a rigid body hub and a flexible beam-like appendage. The nonlinear control technique named SDRE is applied to the piezoelectric actuator and to the external torque acting in the rotating axis. The nonlinear integro-partial differential governing equations of motion for the system are derived through the Extended Hamilton's Principle. The external torque and the distributed bending moment from the piezoelectric actuator along the beam length is added to the governing equations of motion via work done by external forces. The length of the piezo is a 25% of the beam length. Weak nonlinearities and strong nonlinearities are considered in the mathematical model in order to verify the applicability of the nonlinear control technique investigated here.*

Keywords: *piezoelectric actuators, active vibration control, smart structures, nonlinear control, SDRE control.*

1 INTRODUCTION

Lightweight and flexible structures usually demands low energy consumption and increased performance, and they are easily deployed and assembled. However these benefits, flexible structures are more difficult to control and stabilize because of its susceptibility to vibration. In this work one investigates a flexible beam-like structure connected to a rotating rigid body and subject to mechanical vibration. The main purpose of this research is to investigate the use of piezoelectric material to minimize the beam vibration at the same time as the rigid body is oriented.

Investigations on the use of piezoelectric actuators to control the mechanical vibration of flexible structures are vast (Elmadany et al., 2013).

A cost function can be used in order to determine the optimal placement of the piezoelectric actuator and sensor along the flexible structures (Abreu et al., 2003). According to (Gani et al., 2003), the most effective position to place the piezoelectric element is at the root of the beam.

The piezoelectric ceramic is considered a type of intelligent material due to its properties: direct and inverse piezoelectric effects and the ability to be used as sensor or actuator in active vibration control (Zhang and Wang, 2010).

The use of the piezoelectric elements adhered to the surface of the structure to counteract the effects of mechanical vibration do not add significant weight to the original structure and do not affect its initial flexibility characteristic.

Conventional linear controllers are not sufficiently robust to control nonlinear systems as the one investigated here or system under extreme conditions. State-Dependent Riccati Equation (SDRE) is a nonlinear sub-optimal control technique that synthesizes a nonlinear feedback control law and allows nonlinearities in the system states while additionally offers great design flexibility through state-dependent weighting matrices (Çimen, 2008; Shamma and Cloutier, 2003). The nonlinear system is not linearized and the state-dependent matrices are calculated point to point in the state space (or in each integration time step).

2 GOVERNING EQUATIONS OF MOTION FOR THE ROTATING FLEXIBLE BEAM

The geometric model of the system investigated in this work is presented in Figure 1. This system comprises a rotating flexible beam-like structure connected to a rigid hub.

In this figure, the inertial axis is represented by XY, the moving axis (attached to the slewing axis and moving with it) is represented by xy, the beam deflection (as a space-time variable) is represented by $v(x, t)$ and the slewing angle is represented by $\theta(t)$.

The governing equations of motion are obtained through the energy method named Extended Hamilton's principle (Meirovitch, 1967 ; Fox, 1987). In order to apply this method one needs to know the kinetic and potential (strain) energies stored in the rotating beam-like flexible structure system during its time evolution.

The kinetic energy, T, of the rotating beam system is given by:

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$$T = \frac{1}{2} I_h \dot{\theta}^2 + \frac{1}{2} \int_0^{\ell} \rho A \, [\, (\dot{v} + x\dot{\theta})^2 + (v\dot{\theta})^2 \,] dx \,. \tag{1}$$

In Eq. (1), I_h represents the moment of inertia of the hub about the rotational axis, $\theta(t)$ represents the angular displacement of the system, $v(x, t)$ represents the transversal displacement along the beam, ρ represents the linear mass density of the beam, A represents the cross-section area of the beam and ℓ represents the length of the undeflected beam.

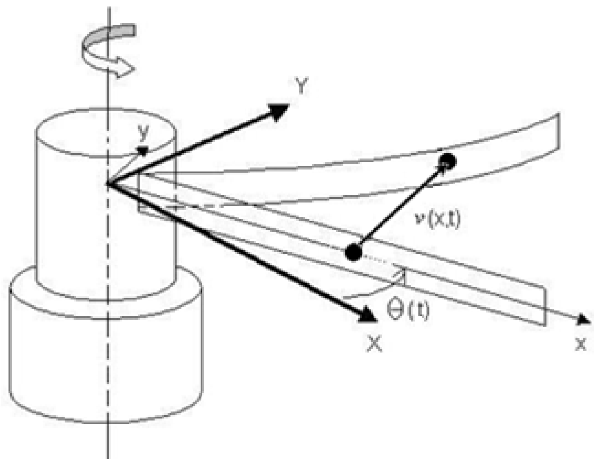

Figure 1. The slewing flexible beam system.

In the mathematical model considered here, linear curvature is assumed in the mathematical modeling of the flexible structure (Fenili, 2000; Popov, 1968; Sah and Mayne, 1990).

The potential energy, V, of the rotating beam system is given by:

$$V = \frac{1}{2} \int_0^{\ell} EI v''^2 dx \,. \tag{2}$$

In Eq. (2), E represents the Young's modulus of the material of the beam, and I represents the moment of inertia of the cross-section area of the beam.

The lagrangian of the system, L, therefore, is given by:

$$L = T - V \,. \tag{3}$$

Substituting Eq. (1) and Eq. (2) in Eq. (3) results:

$$L = \frac{1}{2} I_h \dot{\theta}^2 + \frac{1}{2} \int_0^{\ell} \left\{ \rho A \, [\, (\dot{v} + x\dot{\theta})^2 + (v\dot{\theta})^2 \,] - EI v''^2 \right\} dx \,. \tag{4}$$

Hamilton's principle states that:

$$\delta \int_{t1}^{t2} L \, dt = 0 \,. \tag{5}$$

where $\delta(t_1) = \delta(t_2) == 0$

Substituting the lagrangian given in Eq. (4) in Eq. (5) (and making all the necessary calculations) results the nonlinear integro-partial differential set of equations given by:

$$I_{hub}\ddot{\theta} + \rho A\int_0^L \left(\ddot{\theta}\nu^2 + 2\dot{\theta}\nu\dot{\nu} + \ddot{\theta}x^2 + \ddot{\nu}x \right)dx = T_{rot} \tag{6}$$

$$\ddot{\nu} + \ddot{\theta}x + \frac{EI}{\rho A}\nu^{iv} - \dot{\theta}^2\nu = 0\,. \tag{7}$$

and the boundary condition for the clamped-free as beam given by:

$$\nu(0,t) = 0 \tag{8}$$

$$\nu'(0,t) = 0 \tag{9}$$

$$\nu''(L,t) = 0 \tag{10}$$

$$\nu'''(L,t) = 0 \tag{11}$$

In Eq. (6), the quantity on the right side, T_{rot}, is the external torque applied to the rotating axis. In a real application, it may come from a dc motor, for example.

Solving for $\ddot{\nu}$ in Eq. (7) and substituting in Eq. (6) one has an equation with only one second derivative (in this case $\ddot{\theta}$). Solving this new equation for $\ddot{\theta}$ and substituting in Eq. (7) one has an equation with only with the second derivative $\ddot{\nu}$. These resulting governing equations of motion are given by:

$$\ddot{\theta} + 2\dot{\theta}\alpha_1 + \alpha_2\dot{\theta}^2 - \frac{EI}{\rho A}\alpha_3 = \alpha_4 T_{rot} \tag{12}$$

$$\ddot{\nu} + \frac{EI}{\rho A}\left(\nu^{iv} + x\alpha_3 \right) - 2x\dot{\theta}\alpha_1 - \left(\nu + x\alpha_2 \right)\dot{\theta}^2 = -x\alpha_4 T_{rot} \tag{13}$$

where:

$$\alpha_0 = \frac{I_{hub}}{\rho A} + \int_0^L \nu^2 dx \tag{14}$$

$$\alpha_1 = \frac{1}{\alpha_0}\int_0^L \nu\dot{\nu}\,dx \tag{15}$$

$$\alpha_2 = \frac{1}{\alpha_0}\int_0^L \nu x\,dx \tag{16}$$

$$\alpha_3 = \frac{1}{\alpha_0}\int_0^L x\nu^{iv}dx \tag{17}$$

$$\alpha_4 = \frac{1}{\rho A \alpha_0} \tag{18}$$

The time-varying parameters given by Eq. (14) to (18) contain definite integral terms. In order to proceed with the numerical integration of Eq. (12) and (13), these terms are expanded using the method for numerical approximation of definite integrals named Simpson's rule and given by (Abramowitz and Stegun, 1972):

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \ldots + 4f(x_{n-1}) + f(x_n) \right) \tag{19}$$

Consider that the beam is divided in 8 elements (or 9 nodes). The element size is h. Using this approximation rule and this number of elements, Eq. (14) to (17) – since Eq. (18) is not necessary to discuss because it uses Eq. (14) – can be rewriten as:

$$\alpha_0 = \frac{I_{hub}}{\rho A} + \frac{h}{3}\left(\nu_1^2 + 4\nu_2^2 + 2\nu_3^2 + 4\nu_4^2 + 2\nu_5^2 + 4\nu_6^2 + 2\nu_7^2 + 4\nu_8^2 + \nu_9^2 \right) \tag{20}$$

$$\alpha_1 = \frac{1}{\alpha_0}\frac{h}{3}\left(\nu_1\dot{\nu}_1 + 4\nu_2\dot{\nu}_2 + 2\nu_3\dot{\nu}_3 + 4\nu_4\dot{\nu}_4 + 2\nu_5\dot{\nu}_5 + 4\nu_6\dot{\nu}_6 + 2\nu_7\dot{\nu}_7 + 4\nu_8\dot{\nu}_8 + \nu_9\dot{\nu}_9 \right) \tag{21}$$

$$\alpha_2 = \frac{1}{\alpha_0}\frac{h}{3}\left(4h\nu_2 + 4h\nu_3 + 12h\nu_4 + 8h\nu_5 + 20h\nu_6 + 12h\nu_7 + 28h\nu_8 + L\nu_9 \right) \tag{22}$$

$$\begin{aligned}\alpha_3 = \frac{1}{\alpha_0}\frac{h}{3}\Bigg(& 4h\frac{\partial^4\nu_2}{\partial x^4} + 4h\frac{\partial^4\nu_3}{\partial x^4} + 12h\frac{\partial^4\nu_4}{\partial x^4} + 8h\frac{\partial^4\nu_5}{\partial x^4} + 20h\frac{\partial^4\nu_6}{\partial x^4} + 12h\frac{\partial^4\nu_7}{\partial x^4} + \\ & 28h\frac{\partial^4\nu_8}{\partial x^4} + L\frac{\partial^4\nu_9}{\partial x^4} \Bigg)\end{aligned} \tag{23}$$

After the substitution of the numerical approximation presented in Eq. (20) to (23), the resulting nonlinear partial-differential governing equations of motion are transformed into a set of nonlinear ordinary differential equations via a finite difference approach (LeVeque, 2007). The division of the beam in elements is given according to Figure 2.

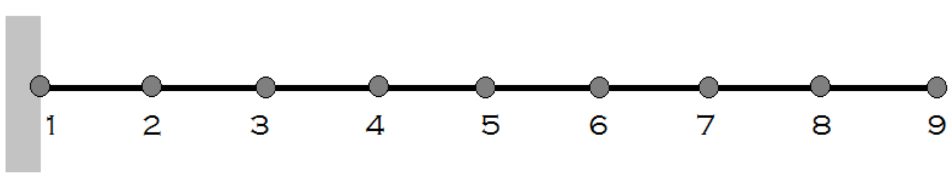

Figure 2. Division of the beam in elements for the application of the finite difference approach.

Considering nodes 2 to 9, the central difference formula using 5 points is considered. This formula for the fourth derivative is given by:

$$\frac{\partial^4 \nu_i(x_i,t)}{\partial x^4} = \frac{\nu(x_i+2h,t) - 4\nu(x_i+h,t) + 6\nu(x_i,t) - 4\nu(x_i-h,t) + \nu(x_i-2h,t)}{h^4} \tag{24}$$

For nodes 2 and 9, the boundary conditions are used in order to eliminate the points located outside the beam. The fourth order partial derivatives in Eq. (23) are also approximated with the formula given in Eq. (24).

After all these manipulations and substituting the final expressions in Eq. (12) and (13), one obtains a set of nonlinear ordinary differential governing equations of motion. This new set is now up to be put in state space form and numerically integrated using the method named fourth order Runge-Kutta. This new set is finally given by:

$$\ddot{\theta}+\beta_1\dot{\theta}+\beta_2\dot{\theta}^2-\beta_3=\beta_4 T_{rot} \tag{25}$$

$$\ddot{\nu}_2+\beta_0(7\nu_2-4\nu_3+\nu_4)+\beta_3 x_2-\beta_1 x_2\dot{\theta}-(\nu_2+\beta_2 x_2)\dot{\theta}^2=-\beta_4 x_2 T_{rot} \tag{26}$$

$$\ddot{\nu}_3+\beta_0(-4\nu_2+6\nu_3-4\nu_4+\nu_5)+\beta_3 x_3-\beta_1 x_3\dot{\theta}-(\nu_3+\beta_2 x_3)\dot{\theta}^2=-\beta_4 x_3 T_{rot} \tag{27}$$

$$\ddot{\nu}_4+\beta_0(\nu_2-4\nu_3+6\nu_4-4\nu_5+\nu_6)+\beta_3 x_4-\beta_1 x_4\dot{\theta}-(\nu_4+\beta_2 x_4)\dot{\theta}^2=-\beta_4 x_4 T_{rot} \tag{28}$$

$$\ddot{\nu}_5+\beta_0(\nu_3-4\nu_4+6\nu_5-4\nu_6+\nu_7)+\beta_3 x_5-\beta_1 x_5\dot{\theta}-(\nu_5+\beta_2 x_5)\dot{\theta}^2=-\beta_4 x_5 T_{rot} \tag{29}$$

$$\ddot{\nu}_6+\beta_0(\nu_4-4\nu_5+6\nu_6-4\nu_7+\nu_8)+\beta_3 x_6-\beta_1 x_6\dot{\theta}-(\nu_6+\beta_2 x_6)\dot{\theta}^2=-\beta_4 x_6 T_{rot} \tag{30}$$

$$\ddot{\nu}_7+\beta_0(\nu_5-4\nu_6+6\nu_7-4\nu_8+\nu_9)+\beta_3 x_7-\beta_1 x_7\dot{\theta}-(\nu_7+\beta_2 x_7)\dot{\theta}^2=-\beta_4 x_7 T_{rot} \tag{31}$$

$$\ddot{\nu}_8+\beta_0(\nu_6-4\nu_7+5\nu_8-2\nu_9)+\beta_3 x_8-\beta_1 x_8\dot{\theta}-(\nu_8+\beta_2 x_8)\dot{\theta}^2=-\beta_4 x_8 T_{rot} \tag{32}$$

$$\ddot{\nu}_9+\beta_0(2\nu_7-4\nu_8+2\nu_9)+\beta_3 x_9-\beta_1 x_9\dot{\theta}-(\nu_9+\beta_2 x_9)\dot{\theta}^2=-\beta_4 x_9 T_{rot} \tag{33}$$

where:

$$\beta_0=\frac{EI}{\rho A h^4}$$

$$\beta_1=\frac{2h}{3}\left[\frac{4\nu_2\dot{\nu}_2+2\nu_3\dot{\nu}_3+4\nu_4\dot{\nu}_4+2\nu_5\dot{\nu}_5+4\nu_6\dot{\nu}_6+2\nu_7\dot{\nu}_7+4\nu_8\dot{\nu}_8+\nu_9\dot{\nu}_9}{\frac{I_{hub}}{\rho A}+\frac{h}{3}\left(4\nu_2^2+2\nu_3^2+4\nu_4^2+2\nu_5^2+4\nu_6^2+2\nu_7^2+4\nu_8^2+\nu_9^2\right)}\right]$$

$$\beta_2=\frac{4h^2}{3}\left[\frac{\nu_2+\nu_3+3\nu_4+2\nu_5+5\nu_6+3\nu_7+7\nu_8+\frac{L}{4h}\nu_9}{\frac{I_{hub}}{\rho A}+\frac{h}{3}\left(4\nu_2^2+2\nu_3^2+4\nu_4^2+2\nu_5^2+4\nu_6^2+2\nu_7^2+4\nu_8^2+\nu_9^2\right)}\right]$$

$$\beta_3=\frac{EI}{3\rho A h^2}\left[\frac{24\nu_2-32\nu_3+48\nu_4-64\nu_5+80\nu_6+\beta_{31}\nu_7+\beta_{32}\nu_8+\beta_{33}\nu_9}{\frac{I_{hub}}{\rho A}+\frac{h}{3}\left(4\nu_2^2+2\nu_3^2+4\nu_4^2+2\nu_5^2+4\nu_6^2+2\nu_7^2+4\nu_8^2+\nu_9^2\right)}\right]$$

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$$\beta_{31}=\frac{2L}{h}-112$$

$$\beta_{32}=112-\frac{4L}{h}$$

$$\beta_{33}=\frac{2L}{h}-44$$

$$\beta_4=\frac{1}{\rho A\left[\frac{I_{hub}}{\rho A}+\frac{h}{3}\left(4v_2^2+2v_3^2+4v_4^2+2v_5^2+4v_6^2+2v_7^2+4v_8^2+v_9^2\right)\right]}$$

Equation (25) is the governing equation of motion for the angular displacement and velocity. Equations (26) to (33) are the governing equation of motion for nodes 2 to 9.

3 GOVERNING EQUATIONS OF MOTION FOR THE ROTATING FLEXIBLE BEAM + THE PIEZOELECTRIC ACTUATOR

The work done by the piezoelectric forces on the beam is given by:

$$W=\int_0^L f(x,t)\,v(x,t)\,dx \tag{34}$$

Equation (5) can now be expanded in order to include the work of external forces as given in Eq. (34). The Hamilton's principle now states that:

$$\delta\int_{t1}^{t2}(T-V+W)\,dt=0 \tag{35}$$

Substituting Eq. (1), (2) and (34) in Eq. (35) there is no change Eq. (6) but the governing equation (7) is now rewritten as:

$$\ddot{v}+\ddot{\theta}x+\frac{EI}{\rho A}v^{iv}-\dot{\theta}^2v=\frac{1}{\rho A}f(x,t) \tag{36}$$

The term f(x,t) represents the distributed bending moment acting along the beam due to the piezoelectric actuator and is given by:

$$f(x,t)=\frac{\partial^2 M_{piezo}}{\partial x^2} \tag{37}$$

where (Abreu et al., 2003; Zhang and Wang, 2010):

$$M_{piezo}=C_aV_a(x,t) \tag{38}$$

In Eq. (38), $V_a(x, t)$ is the control voltage applied to the actuator and Ca is a constant dependent on the piezo-actuator and beam geometries and is given by:

$$C_a = \frac{1}{2} E_c d_{31} b (t_b + t_c) \tag{39}$$

where E_c is the elastic modulus of the piezoceramic, d_{31} is a piezoelectric constant, b is the piezoceramic length, t_b is the beam thickness and t_c is the piezoceramic thickness.

Repeating the same steps as before, Eq. (12) and (13) are now substituted by the new governing equations of the smart structure (structure + piezoelectric actuator) given by:

$$\ddot{\theta} + 2\dot{\theta}\alpha_1 + \alpha_2\dot{\theta}^2 - \frac{EI}{\rho A}\alpha_3 = \alpha_4 T_{rot} - \frac{C_a}{\rho A \alpha_0}\int_0^L x V_a'' dx \tag{40}$$

$$\ddot{v} + \frac{EI}{\rho A}\left(v^{iv} + x\alpha_3 \right) - 2x\dot{\theta}\alpha_1 - \left(v + x\alpha_2 \right)\dot{\theta}^2 = -x\alpha_4 T_{rot} + \frac{C_a}{\rho A}\left(V_a'' + \frac{x}{\alpha_0}\int_0^L x V_a'' dx \right) \tag{41}$$

Using again the Simpson´s rule, the integral in the right sides of Eq. (40) and (41) can be rewriten as:

$$\begin{aligned}\int_0^L x V_a'' dx = \frac{h}{3}\Bigg(& x_1 \frac{\partial^2 V_{a1}}{\partial x^2} + 4x_2 \frac{\partial^2 V_{a2}}{\partial x^2} + 2x_3 \frac{\partial^2 V_{a3}}{\partial x^2} + 4x_4 \frac{\partial^2 V_{a4}}{\partial x^2} + 2x_5 \frac{\partial^2 V_{a5}}{\partial x^2} + 4x_6 \frac{\partial^2 V_{a6}}{\partial x^2} + \\ & 2\, x_7 \frac{\partial^2 V_{a7}}{\partial x^2} + 4x_8 \frac{\partial^2 V_{a8}}{\partial x^2} + x_9 \frac{\partial^2 V_{a9}}{\partial x^2} \Bigg)\end{aligned} \tag{42}$$

The finite difference approach considering the central difference formula using 3 points is considered in order to rewrite the second derivatives in Eq. (42). This formula is given by:

$$\frac{\partial^2 V_{an}(x,t)}{\partial x^2} = \frac{V_{an-1} - 2V_{an} + V_{an+1}}{h^2} \tag{43}$$

The final expression, then, for Eq. (42) is:

$$\int_0^L x V_a'' dx = \frac{4}{3}V_{a1} - \frac{4}{3}V_{a2} + \frac{8}{3}V_{a3} - 4V_{a4} + \frac{16}{3}V_{a5} - \frac{20}{3}V_{a6} + 8V_{a7} - 12V_{a8} + 4V_{a9} \tag{44}$$

In this work only two piezoelements (regarding three nodes) are considered, according to Fig. 3.

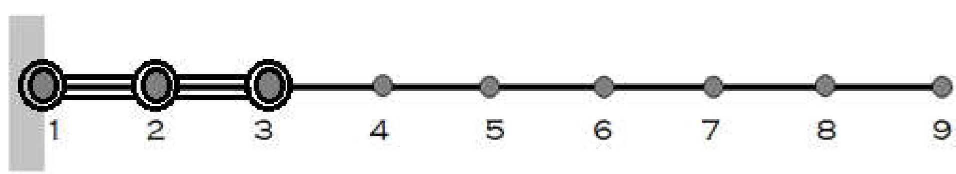

Figure 3. Division of the beam in elements with the location of the piezoelectric actuator (nodes 1 to 3).

Considering only two elements near the beam root, Eq. (44) van be reduced to:

$$\int_0^L x V_a'' dx = \frac{4}{3} V_{a1} - \frac{4}{3} V_{a2} + \frac{8}{3} V_{a3} \tag{45}$$

Finally, the complete set of nonlinear ordinary differential governing equations of motion for the flexible structure under rotation with piezoelectric elements attached (and considering Fig. 1 to 3) is given by:

$$\ddot{\theta} + \beta_1 \dot{\theta} + \beta_2 \dot{\theta}^2 - \beta_{3a} \nu_2 - \beta_{3b} \nu_3 - \beta_{3c} \nu_4 - \beta_{3d} \nu_5 - \beta_{3e} \nu_6 - \beta_{3f} \nu_7 - \beta_{3g} \nu_8 - \beta_{3h} \nu_9 = \beta_4 T_{rot} - \beta_5 (V_{a1} - V_{a2} + 2V_{a3}) \tag{46}$$

$$\ddot{\nu}_2 + (7\beta_0 + \beta_{3a} x_2) \nu_2 - (4\beta_0 - \beta_{3b} x_2) \nu_3 + (\beta_0 + \beta_{3c} x_2) \nu_4 + \beta_{3d} x_2 \nu_5 + \beta_{3e} x_2 \nu_6 + \beta_{3f} x_2 \nu_7 + \beta_{3g} x_2 \nu_8 + \beta_{3h} x_2 \nu_9 - \beta_1 x_2 \dot{\theta} - (\nu_2 + \beta_2 x_2) \dot{\theta}^2 = - \beta_4 x_2 T_{rot} + \left(\frac{\beta_6}{h^2} + \beta_5 x_2 \right) V_{a1} - \left(\frac{2\beta_6}{h^2} + \beta_5 x_2 \right) V_{a2} + \left(\frac{\beta_6}{h^2} + 2\beta_5 x_2 \right) V_{a3} \tag{47}$$

$$\ddot{\nu}_3 - (4\beta_0 - \beta_{3a} x_3) \nu_2 + (6\beta_0 + \beta_{3b} x_3) \nu_3 - (4\beta_0 - \beta_{3c} x_3) \nu_4 + (\beta_0 + \beta_{3d} x_3) \nu_5 + \beta_{3e} x_3 \nu_6 + \beta_{3f} x_3 \nu_7 + \beta_{3g} x_3 \nu_8 + \beta_{3h} x_3 \nu_9 - \beta_1 x_3 \dot{\theta} - (\nu_3 + \beta_2 x_3) \dot{\theta}^2 = - \beta_4 x_3 T_{rot} + (\beta_5 x_3) V_{a1} + \left(\frac{\beta_6}{h^2} - \beta_5 x_3 \right) V_{a2} - \left(\frac{2\beta_6}{h^2} - 2\beta_5 x_3 \right) V_{a3} \tag{48}$$

$$\ddot{\nu}_4 + (\beta_0 + \beta_{3a} x_4) \nu_2 - (4\beta_0 - \beta_{3b} x_4) \nu_3 + (6\beta_0 + \beta_{3c} x_4) \nu_4 - (4\beta_0 - \beta_{3d} x_4) \nu_5 + (\beta_0 + \beta_{3e} x_4) \nu_6 + \beta_{3f} x_4 \nu_7 + \beta_{3g} x_4 \nu_8 + \beta_{3h} x_4 \nu_9 - \beta_1 x_4 \dot{\theta} - (\nu_4 + \beta_2 x_4) \dot{\theta}^2 = - \beta_4 x_4 T_{rot} \tag{49}$$

$$\ddot{\nu}_5 + \beta_{3a} x_5 \nu_2 + (\beta_0 + \beta_{3b} x_5) \nu_3 - (4\beta_0 - \beta_{3c} x_5) \nu_4 + (6\beta_0 + \beta_{3d} x_5) \nu_5 - (4\beta_0 - \beta_{3e} x_5) \nu_6 + (\beta_0 + \beta_{3f} x_5) \nu_7 + \beta_{3g} x_5 \nu_8 + \beta_{3h} x_5 \nu_9 - \beta_1 x_5 \dot{\theta} - (\nu_5 + \beta_2 x_5) \dot{\theta}^2 = - \beta_4 x_5 T_{rot} \tag{50}$$

$$\ddot{\nu}_6 + \beta_{3a} x_6 \nu_2 + \beta_{3b} x_6 \nu_3 + (\beta_0 + \beta_{3c} x_6) \nu_4 - (4\beta_0 - \beta_{3d} x_6) \nu_5 + (6\beta_0 + \beta_{3e} x_6) \nu_6 - (4\beta_0 - \beta_{3f} x_6) \nu_7 + (\beta_0 + \beta_{3g} x_6) \nu_8 + \beta_{3h} x_6 \nu_9 - \beta_1 x_6 \dot{\theta} - (\nu_6 + \beta_2 x_6) \dot{\theta}^2 = - \beta_4 x_6 T_{rot} \tag{51}$$

$$\ddot{\nu}_7 + \beta_{3a} x_7 \nu_2 + \beta_{3b} x_7 \nu_3 + \beta_{3c} x_7 \nu_4 + (\beta_0 + \beta_{3d} x_7) \nu_5 - (4\beta_0 - \beta_{3e} x_7) \nu_6 + (6\beta_0 + \beta_{3f} x_7) \nu_7 - (4\beta_0 - \beta_{3g} x_7) \nu_8 + (\beta_0 + \beta_{3h} x_7) \nu_9 - \beta_1 x_7 \dot{\theta} - (\nu_7 + \beta_2 x_7) \dot{\theta}^2 = - \beta_4 x_7 T_{rot} \tag{52}$$

$$\ddot{\nu}_8 + \beta_{3a} x_8 \nu_2 + \beta_{3b} x_8 \nu_3 + \beta_{3c} x_8 \nu_4 + \beta_{3d} x_8 \nu_5 + (\beta_0 + \beta_{3e} x_8)\nu_6 - (4\beta_0 - \beta_{3f} x_8)\nu_7 + (5\beta_0 + \beta_{3g} x_8)\nu_8 - (2\beta_0 - \beta_{3h} x_8)\nu_9 - \beta_1 x_8 \dot{\theta} - (\nu_8 + \beta_2 x_8)\dot{\theta}^2 = -\beta_4 x_8 T_{rot} \quad (53)$$

$$\ddot{\nu}_9 + \beta_{3a} x_9 \nu_2 + \beta_{3b} x_9 \nu_3 + \beta_{3c} x_9 \nu_4 + \beta_{3d} x_9 \nu_5 + \beta_{3e} x_9 \nu_6 + (2\beta_0 + \beta_{3f} x_9)\nu_7 - (4\beta_0 - \beta_{3g} x_9)\nu_8 + (2\beta_0 + \beta_{3h} x_9)\nu_9 - \beta_1 x_9 \dot{\theta} - (\nu_9 + \beta_2 x_9)\dot{\theta}^2 = -\beta_4 x_9 T_{rot} \quad (54)$$

where:

$$\beta_{3a} = \frac{24EI}{3\rho A h^2}\left[\frac{1}{\frac{I_{hub}}{\rho A} + \frac{h}{3}(4\nu_2^2 + 2\nu_3^2 + 4\nu_4^2 + 2\nu_5^2 + 4\nu_6^2 + 2\nu_7^2 + 4\nu_8^2 + \nu_9^2)}\right]$$

$$\beta_{3b} = \frac{-32EI}{3\rho A h^2}\left[\frac{1}{\frac{I_{hub}}{\rho A} + \frac{h}{3}(4\nu_2^2 + 2\nu_3^2 + 4\nu_4^2 + 2\nu_5^2 + 4\nu_6^2 + 2\nu_7^2 + 4\nu_8^2 + \nu_9^2)}\right]$$

$$\beta_{3c} = \frac{48EI}{3\rho A h^2}\left[\frac{1}{\frac{I_{hub}}{\rho A} + \frac{h}{3}(4\nu_2^2 + 2\nu_3^2 + 4\nu_4^2 + 2\nu_5^2 + 4\nu_6^2 + 2\nu_7^2 + 4\nu_8^2 + \nu_9^2)}\right]$$

$$\beta_{3d} = \frac{-64EI}{3\rho A h^2}\left[\frac{1}{\frac{I_{hub}}{\rho A} + \frac{h}{3}(4\nu_2^2 + 2\nu_3^2 + 4\nu_4^2 + 2\nu_5^2 + 4\nu_6^2 + 2\nu_7^2 + 4\nu_8^2 + \nu_9^2)}\right]$$

$$\beta_{3e} = \frac{80EI}{3\rho A h^2}\left[\frac{1}{\frac{I_{hub}}{\rho A} + \frac{h}{3}(4\nu_2^2 + 2\nu_3^2 + 4\nu_4^2 + 2\nu_5^2 + 4\nu_6^2 + 2\nu_7^2 + 4\nu_8^2 + \nu_9^2)}\right]$$

$$\beta_{3f} = \frac{EI}{3\rho A h^2}\left(\frac{2L}{h} - 112\right)\left[\frac{1}{\frac{I_{hub}}{\rho A} + \frac{h}{3}(4\nu_2^2 + 2\nu_3^2 + 4\nu_4^2 + 2\nu_5^2 + 4\nu_6^2 + 2\nu_7^2 + 4\nu_8^2 + \nu_9^2)}\right]$$

$$\beta_{3g} = \frac{EI}{3\rho A h^2}\left(112 - \frac{4L}{h}\right)\left[\frac{1}{\frac{I_{hub}}{\rho A} + \frac{h}{3}(4\nu_2^2 + 2\nu_3^2 + 4\nu_4^2 + 2\nu_5^2 + 4\nu_6^2 + 2\nu_7^2 + 4\nu_8^2 + \nu_9^2)}\right]$$

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$$\beta_{3h} = \frac{EI}{3\rho A h^2}\left(\frac{2L}{h} - 44\right)\left[\frac{1}{\frac{I_{hub}}{\rho A} + \frac{h}{3}\left(4\nu_2^2 + 2\nu_3^2 + 4\nu_4^2 + 2\nu_5^2 + 4\nu_6^2 + 2\nu_7^2 + 4\nu_8^2 + \nu_9^2\right)}\right]$$

$$\beta_5 = \frac{4C_a}{3\rho A \alpha_0}$$

$$\beta_6 = \frac{C_a}{\rho A}$$

4 THE NONLINEAR CONTROL SDRE (STATE DEPENDENT RICATTI EQUATION)

The SDRE (State Dependent Ricatti Equation) approach to nonlinear system control relies on representing a nonlinear system's dynamics with state-dependent matrices to be inserted into state-dependent Riccati equations in order to generate a feedback law (Çimen, 2008; Shamma and Cloutier, 2003).

The main idea of this method is to represent the nonlinear system:

$$\dot{x} = f(x) + B(x)u \tag{55}$$

in the form:

$$\dot{x} = A(x)\,x + B(x)u \tag{56}$$

The feedback law is then given by (Çimen, 2008; Shamma and Cloutier, 2003):

$$u = -R^{-1}(x)\,B^T(x)\,P(x)\,x \tag{57}$$

where P(x) is obtained from the SDRE:

$$P(x)A(x) + A^T(x)P(x) + Q(x) - P(x)B(x)R^{-1}(x)B^T(x)P(x) = 0 \tag{58}$$

In Eq. (58), Q(x) and R(x) are design parameters that satisfy the positive definiteness conditions Q(x) > 0 and R(x) > 0.

In this work, the control feedback law is applied to the torque T_{rot} and the voltages V_{a1}, V_{a2}, and V_{a3}.

5 NUMERICAL SIMULATIONS

The parameters used in the numerical simulations are presented in Table 1.

The piezoactuator width is the same as the beam width. The time step in the Runge-Kutta numerical integration of the governing equations is 0.001s.

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Table 1. Parameters used in the numerical simulations.

Constitutive relation	Nomenclature	Value
Linear mass density (beam)	ρ	2700 kg/m^3
Elastic modulus (beam)	E	0.700*10^{11} Pa
Beam length	L	1.500 m
Beam thickness	b	1.000*10^{-3} m
Beam width	w	2.000*10^{-2} m
Elastic modulus (piezoactuator)	E_c	0.714*10^{11} Pa
Piezoactuator strain constant	d_{31}	200*10^{-12} m/V
Piezoactuator thickness	t_c	0.305*10^{-3} m
Piezoactuator length	b	0.1875 m

The weighting matrices used in the control law are:

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.01 & 0 & 0 \\ 0 & 0 & 0.01 & 0 \\ 0 & 0 & 0 & 0.01 \end{bmatrix}$$

$$Q = \begin{bmatrix} 70 & 0 & 0 & 0 & \dots & 0 \\ 0 & 70 & 0 & 0 & \dots & 0 \\ 0 & 0 & 140 & 0 & \dots & 0 \\ 0 & 0 & 0 & 140 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 140 \end{bmatrix}_{18x18}$$

Two different initial conditions are considered for the angular displacement: 1^o and 120^o.

The objective of the nonlinear control is to bring the angular displacement back to 0^o and at the same time to eliminate the vibration on the flexible structure. These two cases are chosen in order that the nonlinear terms in the governing equations of motion may assume different orders of magnitude (from a weakly nonlinear system - small angles, velocities, deflections,… - to a strongly nonlinear system - large angles, velocities, deflections,…).

The results for the first case are depicted in Fig. 4 to 9 and the results for the second case are depicted in Fig. 10 to 15.

The linear curvature assumption is made for the mathematical modeling of the beam-like structure. For this assumption to be valid, the simulations developed here considered that the maximum amplitude of deflection does not surpass 15% of the beam length in the worst case. The larger amplitudes occur at node 9. Only the results for nodes 2 (near the clamped end of the beam) and 9 (in the free end of the beam) are presented here.

Considering the numerical simulation results as presented in Fig. 4 to 15, for both the weakly nonlinear system and the strongly nonlinear system the desired final states were reached.

It is very important to compare the orders of magnitude of the amplitudes of the beam vibration, angular displacement and velocity and control torque and piezoelectric voltages, as presented in the following figures and considering the weakly and strongly nonlinear cases.

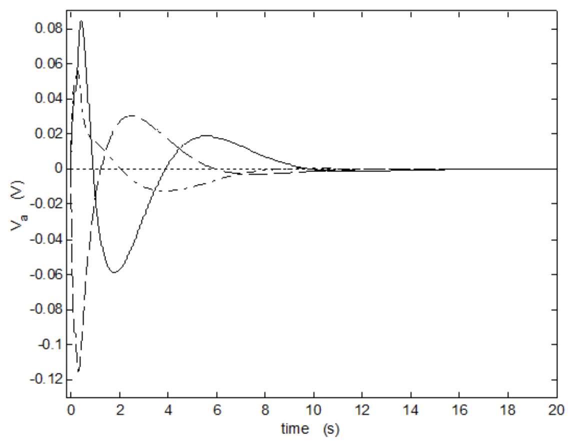

Figure 4. Weakly nonlinear system: control voltages applied to the piezoelectric element V_{a1} (- . - .), Va_2 (- - -) and V_{a3} (——).

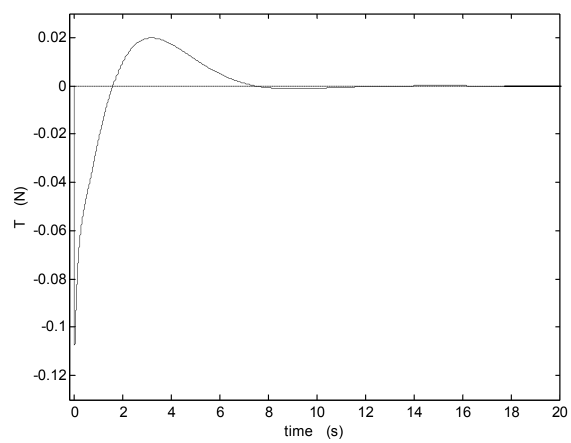

Figure 5. Weakly nonlinear system: control torque applied to the rotating axis.

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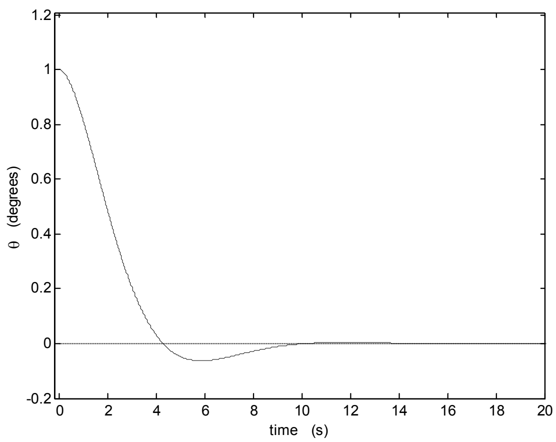

Figure 6. Weakly nonlinear system: angular displacement.

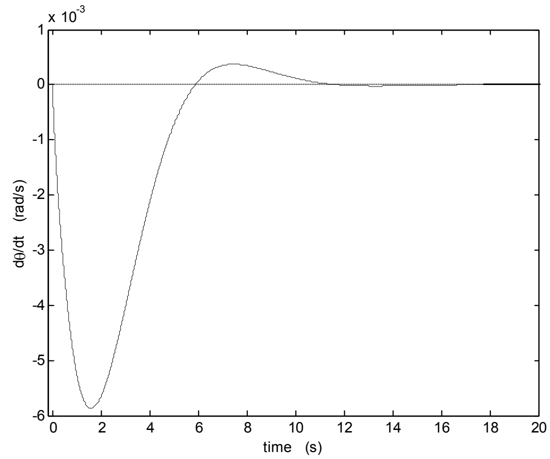

Figure 7. Weakly nonlinear system: angular velocity.

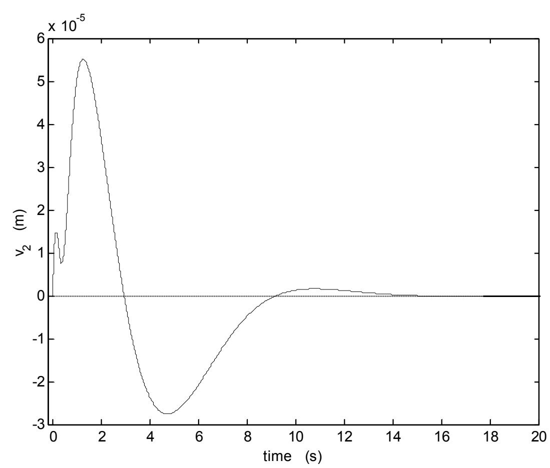

Figure 8. Weakly nonlinear system: deflection near the clamped end of the beam (node 2).

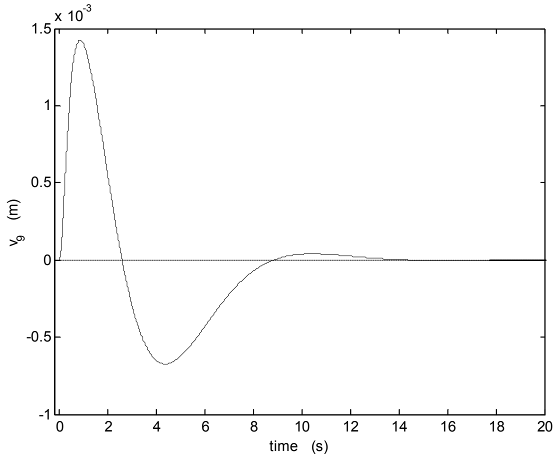

Figure 9. Weakly nonlinear system: deflection at the free end of the beam (node 9).

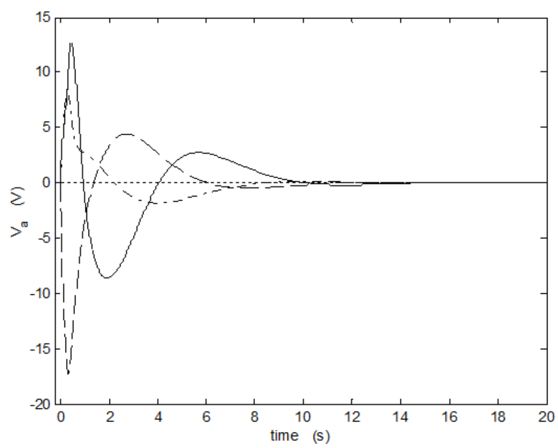

Figure 10. Strongly nonlinear system: control voltages applied to the piezoelectric element V_{a1} (- . - .), Va_2 (- - -) and V_{a3} (——).

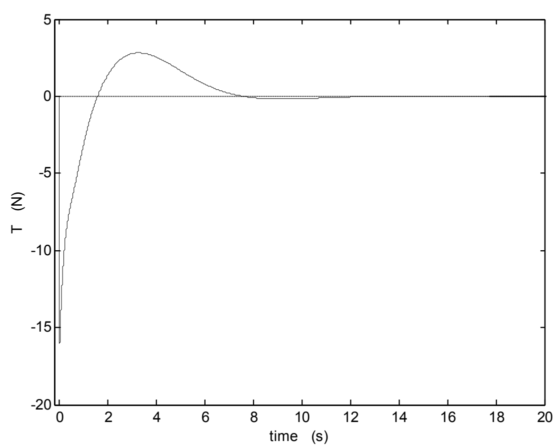

Figure 11. Strongly nonlinear system: control torque applied to the rotating axis.

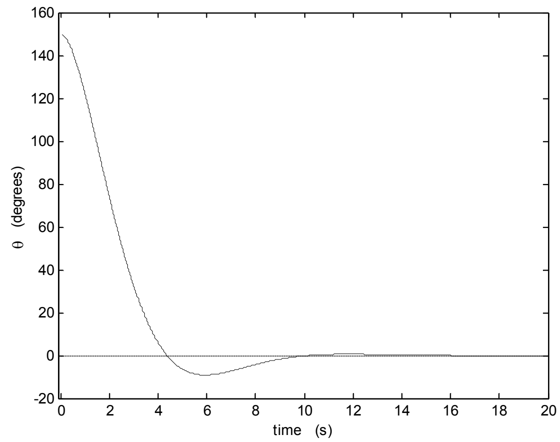

Figure 12. Strongly nonlinear system: angular displacement.

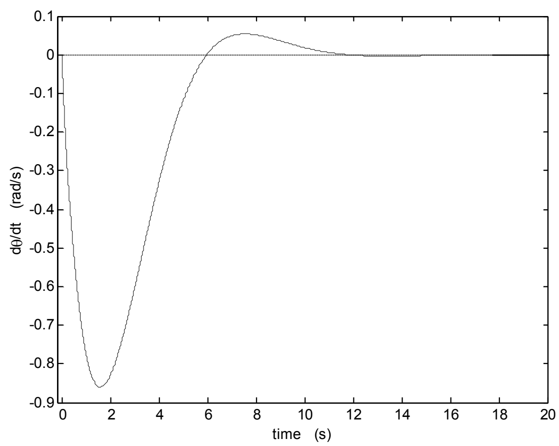

Figure 13. Strongly nonlinear system: angular velocity.

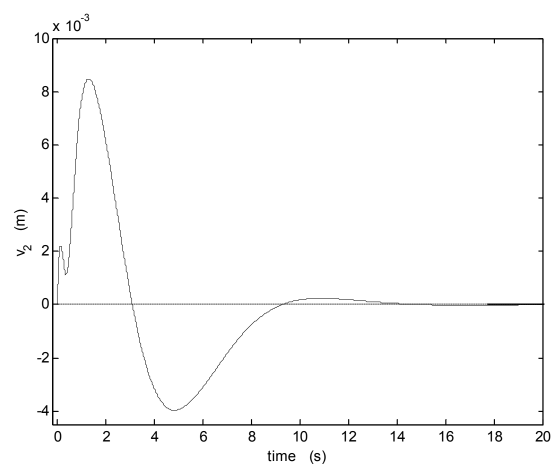

Figure 14. Strongly nonlinear system: deflection near the clamped end of the beam (node 2).

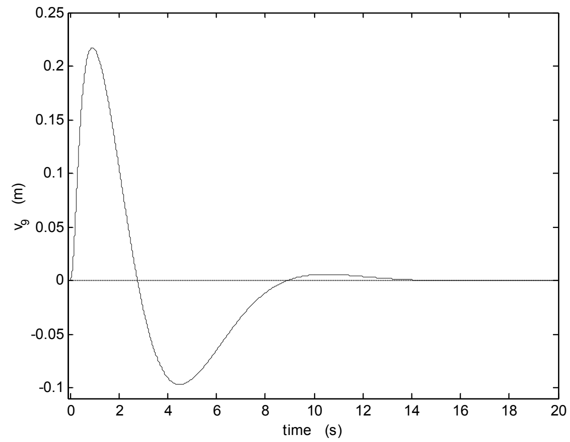

Figure 15. Strongly nonlinear system: deflection at the free end of the beam (node 9).

The results presented in these figures can be improved, for example, by adding more piezoelectric elements along the beam. In this case, one has concentrated bending moments acting as external control moments in more nodes along the beam.

6 CONCLUSIONS

According to the numerical solutions presented in this work, the nonlinear control law named SDRE is sufficiently effective and robust when used for position control and for the attenuation of vibration in a rotating flexible beam actuated by piezoelectric elements. Two cases are studied here: the weakly nonlinear case (small angles, velocities, deflections,…) and the strongly nonlinear case (large angles, velocities, deflections,…). For the last case, the main objective of this investigation, the results are as good and as stable as for the former case (easier to be controlled by considering simpler linear control laws). In order to make the stronger the effect of the nonlinear terms in the governing equations of motion one need necessarily to assume a different mathematical model – nonlinear curvature – for the flexible beam-like structure. One way to improve the results presented here is by adding more piezoelectric elements along the beam.

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