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# ANALYTICAL AND NUMERICAL SOLUTIONS OF ONE-DIMENSIONAL NONLINEAR CONSOLIDATION CONSIDERING LOADING RATES AND DOUBLE-LAYERED SOILS

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***Abstract.*** *The one-dimensional consolidation equation, also referred to as Terzaghi's equation, describes the hydraulic behavior of soils after the application of instantaneous loading. However, some shortcomings of the classical Terzaghi consolidation theory were identified and new analytical models were proposed taking into account time-dependent loading, different drainage conditions and multi-layered soils. Recently, more comprehensive models have been developed in order to consider the nonlinear behavior of the soil, becoming more complex the analytical expressions and, consequently, their solution. As an alternative, numerical solutions have also been applied to solve soil consolidation problems. These solutions have become very popular due to their capability to different soil features and to solve problems in two and three-dimensional domains. In this work, we solve the analytical expressions proposed in the literature for one-dimensional nonlinear consolidation and compare these results with those of an in-house simulator for fully coupled hydromechanical problems. The numerical results showed excellent agreement with the closed form solutions and can be used in more rigorous tests in the development of new numerical codes.*

***Keywords:*** *One-dimensional Consolidation, Nonlinear behavior, Analytical solution, Finite Element Method*

# 1 INTRODUCTION

Many of the developments in soil mechanics have been progressively generalized in order to incorporate the effects of new phenomena and new variables to fit real soil behavior. Through the principle of effective stress by Terzaghi, it was recognized that not only the soil itself is problematic; complexity also arises from the interaction between soil and water. When load is applied to an element of saturated soil, excess of pore pressure will be generated in the soil. Along the time, this excess is gradually dissipated due to the drainage of pore water. This time dependent process is known as soil consolidation. The current commonly used consolidation theory was originally introduced by Terzaghi (1943), establishing the birth of modern soil mechanics.

Through the use of several assumptions, Terzaghi introduced an equation that describes one-dimensional consolidation of soils after an instantaneous loading is applied to a draining surface. Since then, several authors have provided analytical solutions of classical Terzaghi's equation for geotechnical applications. However, some shortcomings of Terzaghi's consolidation theory were recognized and some improvements were performed. The hypothesis of constant loading was studied by Schiffman (1958), who obtained analytical expressions considering loadings that increased linearly with time. The hypothesis of constant permeability and compressibility were discussed by Davis and Raymond (1965), who introduced a nonlinear consolidation theory. The hypothesis of homogeneous soil was pointed out by Schiffman and Stein (1970), who presented solutions for consolidation in multi-layered soils. Recently, Xie et al. (2002) developed analytical expressions taking into account the considerations raised by the previous authors.

Another alternative to solve consolidation problems is the use of numerical solutions based on the finite element method (FEM). This approach has become well-known in different engineering applications due to its capabilities to deal with heterogeneous soils and nonlinear behavior, and to solve two and three-dimensional problems. Several computer codes based on FEM have been developed for consolidation problems (Nogueira, 1998; Huang and Griffiths, 2010; NurFirdaus et al., 2012; Quevedo, 2012). Differently from the analytical, the numerical solutions are only approximations that depend on numerical algorithms and strategies to solve a system of equations. In order to validate the accuracy of a numerical approximation, benchmarks based on analytical solutions are necessary. Most numerical solutions for consolidation problems are tested through comparisons with the classical Terzaghi solution. However, it is not possible to test the accuracy and efficiency of the implemented numerical solution regarding the nonlinear behavior of soils, heterogeneity and different boundary conditions.

In this work, the analytical solutions for one-dimensional nonlinear consolidation of double-layered soils under time-dependent loading proposed by Xie et al. (2002) were reviewed and implemented in a computer code. The corresponding results were discussed and compared with those of a Finite Element simulator developed in-house. The excess of pore pressure and degree of consolidation obtained in both solutions showed excellent agreement and can be used in more rigorous tests in the development of new numerical codes.

# 2 CONSOLIDATION THEORY

Considering the interactions with hydraulic processes, the equation of equilibrium in porous media can be written as

$$\nabla.(\boldsymbol{\sigma}' + \alpha p\boldsymbol{m}) + \boldsymbol{g} = \boldsymbol{0} \tag{1}$$

where $\boldsymbol{\sigma}'$ is the effective stress, $\alpha$ is the Biot coefficient, $p$ is the pore pressure, $\boldsymbol{g}$ is the body force vector and $\boldsymbol{m}$ is a vector with unit values in the normal directions of the stresses and zero in the other components.
The effective stress tensor is related to the deformation ($\boldsymbol{\varepsilon}$) through the generalized Hooke law, given as follows

$$\boldsymbol{\sigma}' = \boldsymbol{D\varepsilon} \tag{2}$$

where $\boldsymbol{D}$ is the constitutive tensor.
Considering the interactions with mechanical processes, a general form of mass balance equation for single-phase fluid flow can be expressed as follows

$$\nabla\boldsymbol{q}_H + \frac{1}{M}\frac{\partial p}{\partial t} + \alpha\frac{\partial}{\partial t}(\boldsymbol{m}^T\boldsymbol{\varepsilon}) = 0 \tag{3}$$

where $\boldsymbol{q}_H$ is the fluid flow and $t$ is the time. The term $1/M$ is known as the Biot modulus.
The fluid flow is related to the pore pressures through Darcy's law as follows

$$\boldsymbol{q}_H = -\boldsymbol{k}\nabla(\frac{p}{\gamma_w} + z) \tag{4}$$

where $\boldsymbol{k}$ is the permeability tensor, which is often called the hydraulic conductivity, $z$ is the elevation head and $\gamma_w$ is the specific weight of the water.

# 3 ANALYTICAL SOLUTION

The analytical solution implemented in this paper corresponds to the consolidation of the double-layered soil presented in Fig. 1. In this figure, $h_i$ is the thickness of the layer $i(i = 1{,}2)$; $H = h_1 + h_2$, the total thickness of the two layers, $q(t)$ is the uniformly distributed load applied on the top surface of the soil, $q_u$ is the ultimate load, and $t_c$ is the construction time.

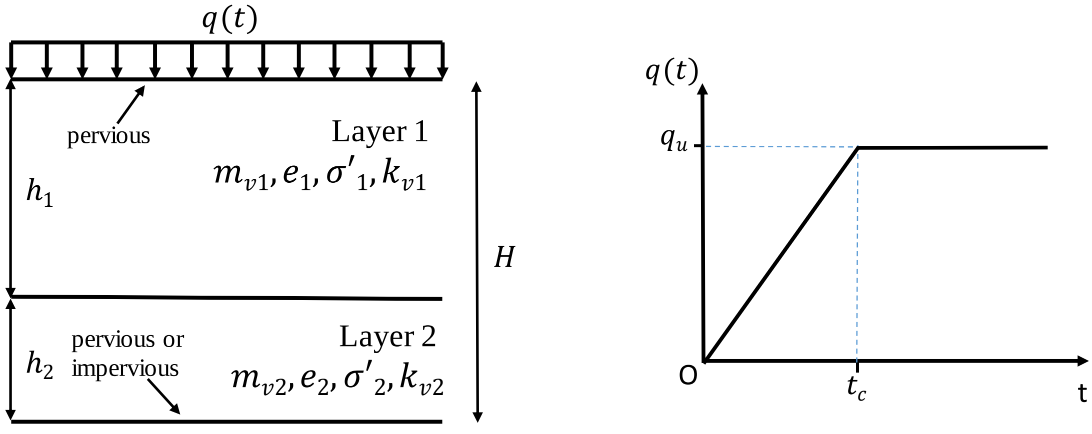



**Figure 1. Double-layered soil considering time-dependent loading (Adapted from Xie et al., 2002)**

Considering $\alpha$ =1 and $1/M$ =0 in a one-dimensional consolidation process, Eq. (3) can be rewritten as

$$\frac{\partial}{\partial z}\left[-\frac{k_v}{\gamma_w}\left(\frac{\partial p}{\partial z}+1\right)\right]=-\frac{\partial \varepsilon}{\partial t} \tag{5}$$

in which $k_v$ is the permeability coefficient and $\varepsilon$ the vertical deformation.

The coefficient of compressibility ($m_v$) of the soil skeleton is given as a function of both the vertical effective stress ($\sigma'$) and the void ratio ($e$) as:

$$m_v=\frac{k}{(1+e)\sigma'} \tag{6}$$

Where $k$ is the swelling index obtained through results of oedometer tests.

Following Terzaghi's principle:

$$\partial\sigma'=\partial q-\partial p \tag{7}$$

The vertical deformation can be found according to

$$\partial\varepsilon=m_v(\partial q-\partial p) \tag{8}$$

The coefficient of consolidation ($C_v$) varies much less than the coefficient of compressibility $m_v$ and may be taken as relatively constant. Thus, it is assumed that:

$$\frac{k_v}{\gamma_w}=m_v C_v \tag{9}$$

After replacing Eq. (8) and (9) in Eq. (5), the following equation is obtained:

$$C_v\left(\frac{\partial^2 p}{\partial z^2}+\frac{1}{\sigma'}\left(\frac{\partial p}{\partial z}\right)^2\right)=\left(\frac{\partial p}{\partial t}-\frac{\partial q}{\partial t}\right) \tag{10}$$

by assuming

$$\omega=ln\frac{\sigma'}{\sigma'_0+q} \tag{11}$$

in which $\sigma'_0$ represents the initial value of the vertical effective stress.
Finally, Eq. (10) can be simplified in terms of the function $\omega$:

$$C_v\frac{\partial^2\omega}{\partial z^2}=\frac{\partial\omega}{\partial t}+\frac{1}{\sigma'_0+q}\frac{dq}{dt} \tag{12}$$

Considering two layers, Xie et al. (2002) provided analytical solutions for Eq. (12) taking into account infinite series to represent the function $\omega$ in each layer. In this way, the excess of pore pressure ($u_i$) in layer $i$ can be obtained as follows:

$$u_i=(\sigma'_0+q)\left(1-e^{\omega_i}\right) \tag{13}$$

The corresponding series for special cases considering single and double drained boundaries in combination with constant and linear loadings were introduced by Xie et al.

(2002). Those series were solved in this study by using a computer code written in Maple. The analytical solutions require the following input parameters:

$$c = \frac{h_2}{h_1} \tag{14}$$

$$K = \frac{k_v 2}{k_v 1} \tag{15}$$

$$\mu = \sqrt{\frac{C_{v1}}{C_{v2}}} \tag{16}$$

$$N_\sigma = \frac{\sigma_0' + q_u}{\sigma_0'} \tag{17}$$

$$T_v = \frac{C_{v1} t}{H^2} \tag{18}$$

$$T_{vc} = \frac{C_{v1} t_c}{H^2} \tag{19}$$

Among these parameters, $N_\sigma$ defines the level of non-linearity. When this parameter equals 1, linear behavior of the soil is considered; otherwise, nonlinear behavior is taken into account by the solution. The parameter $T_{vc}$ defines the kind of loading. When this parameter equals 0, constant loading is applied; otherwise, linear loading is considered. Finally, observe that with the combination of parameters $\mu = 1$, $K = 1$, $T_{vc} = 0$ and $N_\sigma = 1$ the classical Terzaghi consolidation solution is recovered.

## 4 NUMERICAL SOLUTION

Based on the finite element method, the matrix form of the equilibrium equation after discretization can be expressed as follows

$$\boldsymbol{K_M \dot{U}} + \boldsymbol{L_{MH} \dot{P}} = \boldsymbol{\dot{W}_{Mext}} \tag{20}$$

in which

$$\boldsymbol{K_M} = \int_{\boldsymbol{V}} \boldsymbol{B_M^T D B_M dV} \tag{21}$$

$$\boldsymbol{L_{MH}} = \int_{\boldsymbol{V}} \boldsymbol{B_M^T m N dV} \tag{22}$$

In those equations, $\boldsymbol{\dot{U}}$, $\dot{\mathbf{P}}$ and $\boldsymbol{\dot{W}_{Mext}}$ represent the nodal rates of displacements, pore pressures and external forces, respectively. $\boldsymbol{K_M}$ is known as the stiffness matrix and $\boldsymbol{L_{MH}}$ as the coupling matrix; $\boldsymbol{B_M}$ is the matrix that relates deformations and displacements and $\boldsymbol{N}$ represent the shape functions of the finite element.

In a similar way, the matrix form of the mass balance equation after discretization can be expressed as follows:

$$\boldsymbol{S_H}\dot{\boldsymbol{P}} + \dot{\boldsymbol{K}}_{\boldsymbol{H}}\boldsymbol{P} + \boldsymbol{L_{HM}}\dot{\boldsymbol{U}} = \dot{\boldsymbol{W}}_{\boldsymbol{Hext}} \tag{23}$$

in which

$$\boldsymbol{S_H} = \int_V \boldsymbol{N_H^T S_H N dV} \tag{24}$$

$$\dot{\boldsymbol{K}}_{\boldsymbol{H}} = \int_V \boldsymbol{B_H^T} \frac{\boldsymbol{k_H}}{\boldsymbol{\gamma_f}} \boldsymbol{B_H dV} \tag{25}$$

$$\boldsymbol{L_{HM}} = \boldsymbol{L_{MH}^T} \tag{26}$$

In those equations, $\boldsymbol{P}$ and $\dot{\boldsymbol{W}}_{Hext}$ represent the nodal pore pressures and external flows, respectively. $\boldsymbol{S}_H$ is known as the mass matrix and $\dot{\boldsymbol{K}}_H$ as the flow matrix; $\boldsymbol{B}_H$ is the matrix that relates hydraulic gradients and pore pressures.

The matrix form of the full system considering Eq. (21) and (24) is

$$\begin{bmatrix} \mathbf{K_M} & \mathbf{L_{MH}} \\ \mathbf{L_{HM}} & \mathbf{S_H} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{U}} \\ \dot{\mathbf{P}} \end{Bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \dot{\mathbf{H}}_{\mathbf{M}} \end{bmatrix} \begin{Bmatrix} \mathbf{U} \\ \mathbf{P} \end{Bmatrix} = \begin{Bmatrix} \dot{\mathbf{W}}_{\mathbf{Mext}} \\ \dot{\mathbf{W}}_{\mathbf{Hext}} \end{Bmatrix} \tag{27}$$

Observe that $\boldsymbol{K}_M$ and $\dot{\boldsymbol{H}}_M$ are considered dependent on the effective stress state. This feature gives the nonlinear behavior of the soil during the consolidation process.

Eq. (28) can be represented as follows:

$$\boldsymbol{C}_{(\boldsymbol{X})}\dot{\boldsymbol{X}} + \dot{\boldsymbol{K}}_{(\boldsymbol{X})}\boldsymbol{X} = \dot{\boldsymbol{W}}_{\boldsymbol{ext}} \tag{28}$$

where $\boldsymbol{X}$ is $\{\boldsymbol{U} \quad \boldsymbol{P}\}^T$ and $\dot{\boldsymbol{X}}$ is $\{\dot{\boldsymbol{U}} \quad \dot{\boldsymbol{P}}\}^T$.

The numerical solution of this fully coupled system has been implemented in an in-house simulator (Quevedo, 2012). This computer code is based on schemes of automatic time stepping and Newton-Raphson iterations. Together, those schemes attempt to choose time step size that constrains the temporal integration error in the primary variables to a prescribed tolerance. Unlike other techniques, these algorithms compute not only the primary variables, but also their derivatives with respect to time. The knowledge of these rates in previous steps allows the prediction of the next time step size, improving both the efficiency and the accuracy of the solution.

# 5 PRACTICAL APPLICATIONS

In order to check the implemented solutions, this study assess two sets of consolidation problems. For each group, diagrams of computed results have been prepared taking into consideration the temporal evolution of the degree of consolidation and profiles of the excess of pore water pressure along the soil column.

In the first group, we considered single drainage condition and the initial parameters given in table 1. Observe that with those values, the swelling index in layer 1 and 2 are defined as 0.00154 and 0.00307, respectively. The thickness values of the layers were adopted as 0.1m and 0.9m. A loading equal to 200kPa was applied on the top of layer 1 which allows drainage of pore water pressure. Observe that with that loading value $N_\sigma = 3.5$. For the numerical solution, it was considered a mesh with bi-linear quadrilateral elements. In total 20 elements equally spaced in vertical direction were used.

**Table 1. Initial parameters adopted in the first group of consolidation problems**

| Parameter | Layer 1 | Layer 2 | Units |
|---|---|---|---|
| $m_{v0}$ | $1.2\text{x}10^{-5}$ | $2.4\text{x}10^{-5}$ | $\text{kPa}^{-1}$ |
| $e_0$ | 0.6 | 0.6 | - |
| $\sigma'_0$ | 80 | 80 | kPa |
| $k_{v0}$ | $10^{-8}$ | $2\text{x}10^{-8}$ | m/s |

In total, three models were considered in this first group by varying $t_c$ and, consequently, $T_{vc}$. Figure 2 shows the evolution of the degree of consolidation considering for $T_{vc}$ the values 0, 1 and 10. The numerical and analytical result show excellent agreement.

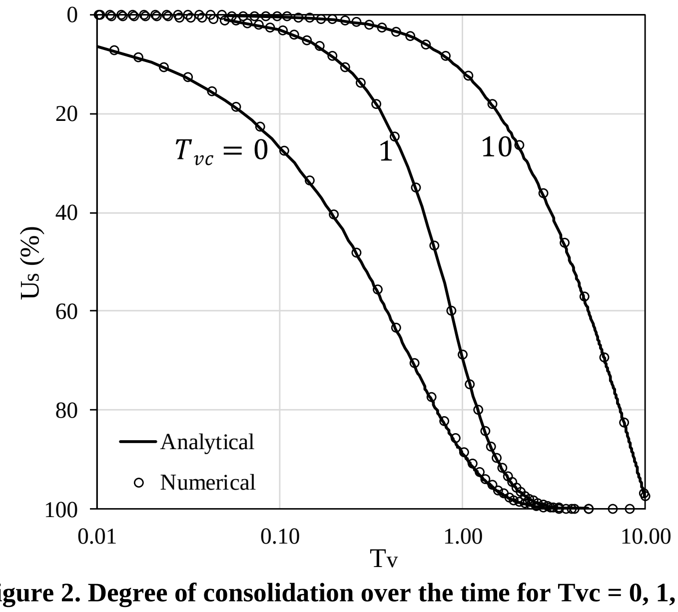



**Figure 2. Degree of consolidation over the time for Tvc = 0, 1, 10.**

Figure 3 shows the dissipation of the excess of pore pressure along the depth for the case of constant loading ($T_{vc} = 0$). Those excess of pore pressures are plotted for four instants: $T_v$ =0.01, 0.1, 0.5 and 1.0. Once again, excellent agreement between the analytical and the numerical solutions is present.

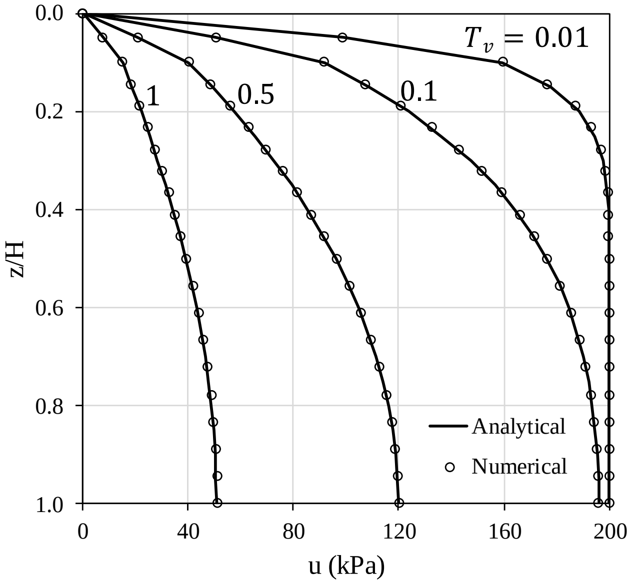
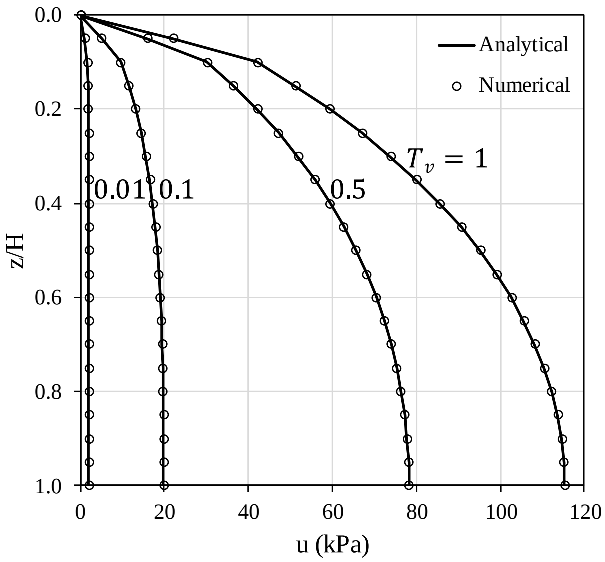
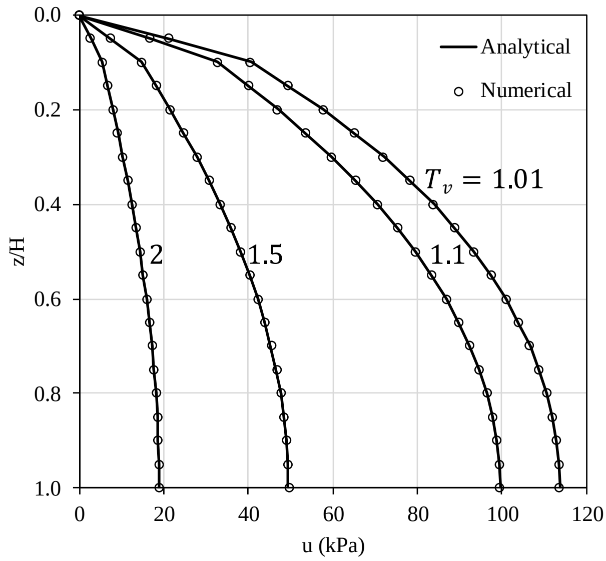



**Figure 3. Excess pore pressure for constant loading (Tvc = 0)**

In a similar way, Figures 4 and 5 show the excess of pore pressures considering linear loadings by adopting $T_{vc} = 1$ and $T_{vc} = 10$, respectively. In those Figures, the results were separated in two plots in order to reflect better the results after reaching the maximum loading.

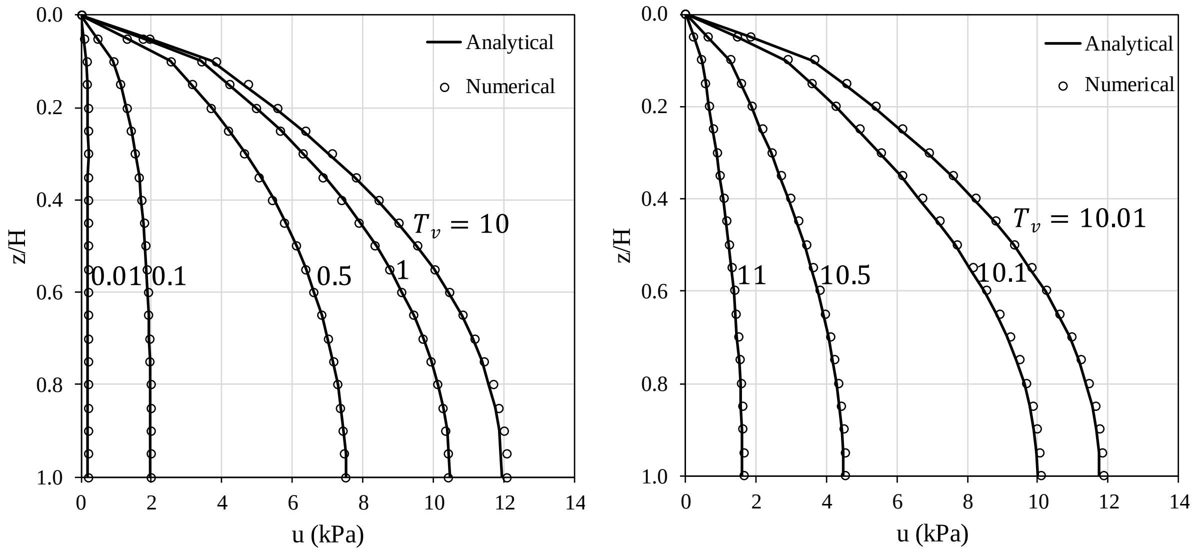



**a)**




**b)**

**Figure 4. Excess pore pressure for linear loading (Tvc = 1) when Tv ≤ Tvc (a) and Tv > Tvc (b).**




**a)** **b)**

**Figure 5. Excess pore pressure for linear loading (Tvc = 10) when Tv ≤ Tvc (a) and Tv > Tvc (b).**

In the second group, we considered double drainage condition and the initial parameters given in table 2. Observe that with those values, the swelling index in layer 1 and 2 are defined as 0.00256 and 0.00384, respectively. The thickness of the layers was adopted equal to 0.5m. A loading equal to 200kPa was applied on the top of layer 1. Observe that with that loading value it is obtained $N_\sigma = 2.5$. The drainage condition was applied on the top of layer 1 and on the bottom of the layer 2. The numerical solution was obtained with the same mesh used in the first group.

**Table 2. Initial parameters adopted in the second group of consolidation problems**

| Parameter | Layer 1 | Layer 2 | Units |
|---|---|---|---|
| $m_{v0}$ | $1.2\text{x}10^{-5}$ | $1.8\text{x}10^{-5}$ | $kPa^{-1}$ |
| $e_0$ | 0.6 | 0.6 | - |
| $\sigma'_0$ | 133.33 | 133.33 | kPa |
| $k_{v0}$ | $10^{-8}$ | $4\text{x}10^{-8}$ | m/s |

Three models were considered by varying $t_c$ and consequently $T_{vc}$. Figure 6 shows the evolution of the degree of consolidation considering $T_{vc}$ =0, 0.5 and 1. It can be observed excellent agreement between the analytical and numerical results.

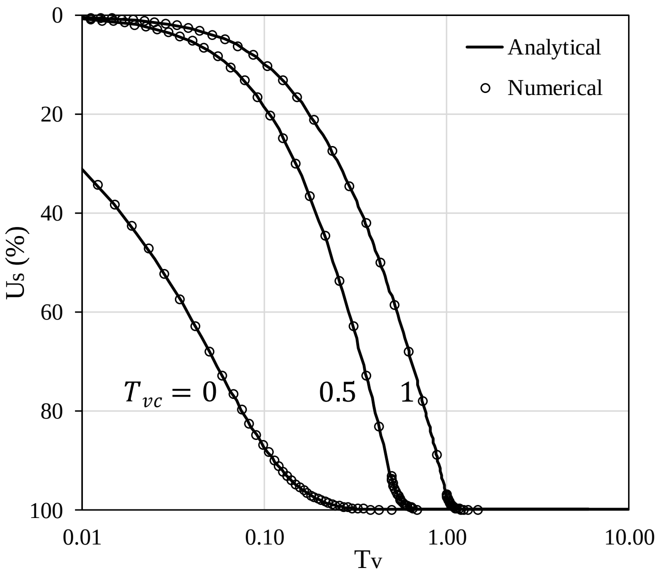



**Figure 6. Degree of consolidation over the time for Tvc = 0, 0.5, 1.**

Figure 7 shows the dissipation of the excess of pore pressure along the depth for the case of constant loading ($T_{vc} = 0$). Those excess of pore pressures are plotted for four instants $T_v$ =0.01, 0.05, 0.1 and 0.15. Once again, it can be observed the excellent agreement between the analytical and the numerical solutions.

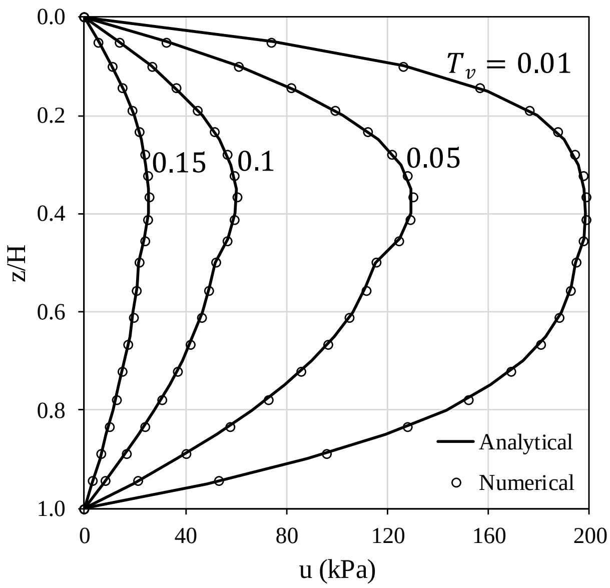
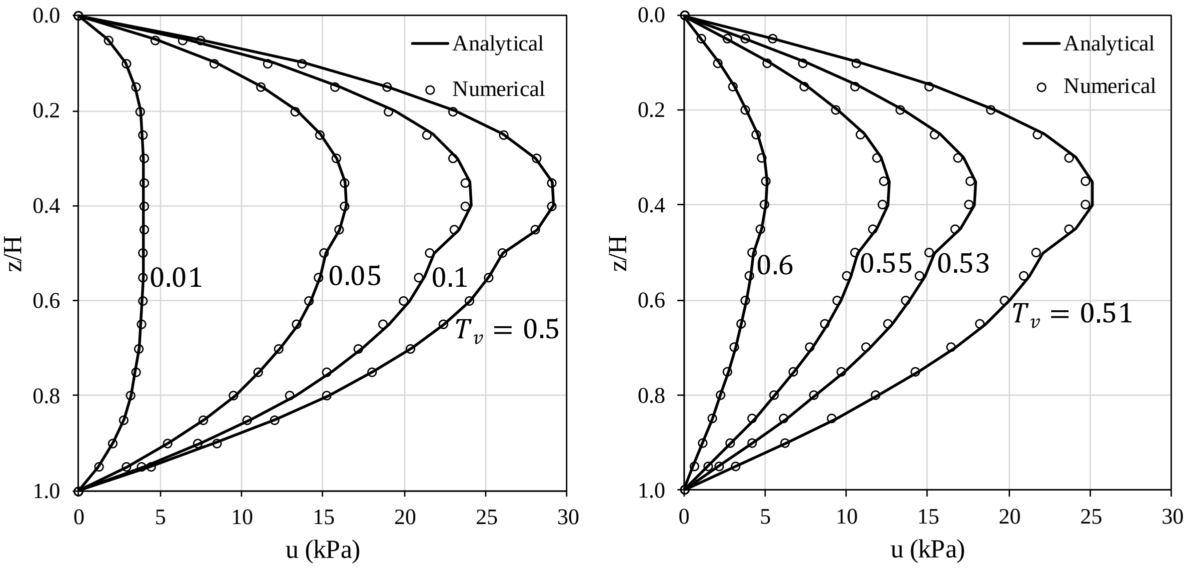


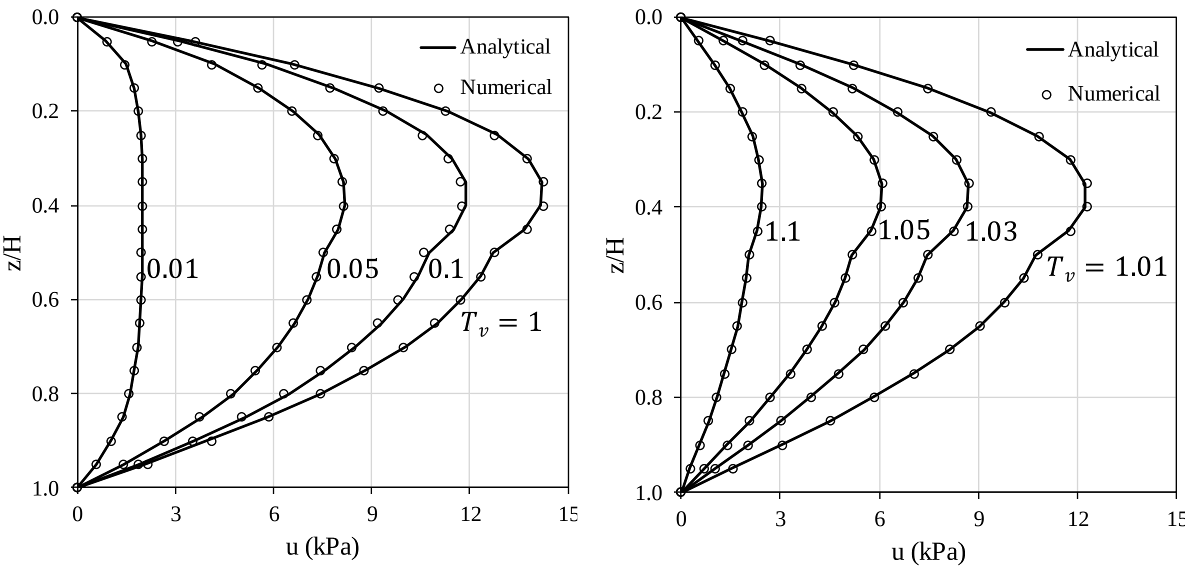

**Figure 7. Excess pore pressure for constant loading (Tvc = 0)**

In a similar way, Figures 8 and 9 show the excess of pore pressure considering linear loadings by adopting $T_{vc} = 0.5$ and $T_{vc} = 1$, respectively. In those Figures, the results were separated in two plots.




a) b)

**Figure 8. Excess pore pressure for linear loading (Tvc = 0.5) when Tv ≤ Tvc (a) and Tv > Tvc (b).**




a) b)

**Figure 9. Excess pore pressure for linear loading (Tvc = 1) when Tv ≤ Tvc (a) and Tv > Tvc (b).**

# 6 CONCLUSION

The paper has examined the theory of consolidation in order to describe the governing equations based on equilibrium of forces and balance of mass. Taken into account the assumptions proposed by Xie et al. (2002), explicit analytical expressions for special cases of one-dimensional nonlinear consolidation were discussed and implemented in a computer code using Maple. In addition, a numerical formulation for geotechnical applications based on the finite element method was introduced. In order to check the analytical and numerical results, several consolidation problems in double layered soils were presented considering different loading rates and different levels of non-linearity. Results corresponding to the degree of consolidation and the excess of pore pressure along the time showed good accuracy of the numerical solution, proving the efficiency of a simulator when analyzing nonlinear consolidation problems. These results can be used as benchmarks for numerical simulators, providing more rigorous tests for checking accuracy of solution algorithms. The use of numerical simulators enable the possibility to model problems in two and three dimensions including multi-layered formations and different kind of loadings.

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