PLATE THEORY IN THE RESOLUTION OF THIN-WALLED STRUCTURES

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Abstract. *Plate Structures are present in all spheres of engineering. Concrete dams, car windshields, among other examples that need to be calculated to size the geometry and the optimum materials of your choice. Throughout this article the pertinent formulations and nomenclatures among those studied in the Reference sector were used. The objective of formulating the behavior of a plate based on the finite element theory applied to the Classical Plate Theory was reached for static and transient responses. The deformed structure is contemplated by the Virtual Work Theorem and the Linear Approximation of the displacements in the nodes that define each discretized element. The separation of the terms in the domain and the contour of the plates is done in order to be better studied the phenomena of resistance, stability and vibration of the Plate. Finally this formulation is programmed for better visualization of numerical and graphical results.*

Keywords: *Plates, Finite Element Method, Linear Approximation.*

1. INTRODUCTION

In the world there are works and equipment essential to the human being with components modeled like plates. Plates are structural members with large flat dimensions when compared to their thickness subjected to loading that cause deformation and bending deformations. Because the thickness is much smaller than the other dimensions of the board it is common to size 2D theories instead of 3D in the modeling of this one.

2. CLASSIC PLATE MODEL

2.1 Motion Equations

The Classical Plate Theory (TCP) is, for boards, the extension of Euler-Bernoulli's theory for bars, this because the displacement field is based on the following *Kirchhoff hypotheses* :

1- Lines perpendicular to the middle surface (Transverse Norms, for example) that were straight before the deformation of the plate remain straight after the deformation;

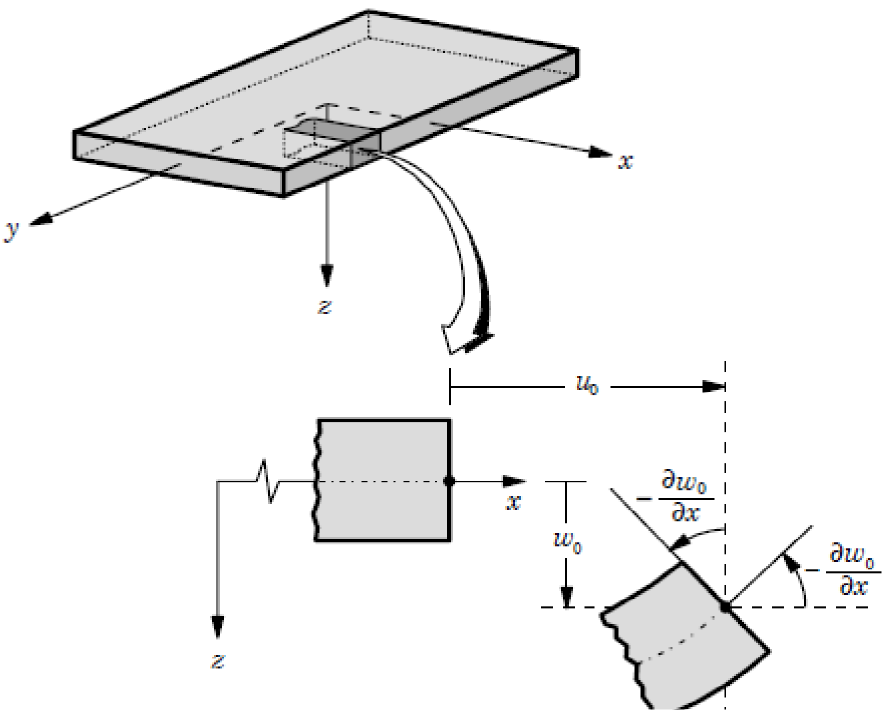

Figure 1 - Deformed Plate Model

2- Transverse Norms do not undergo stretching (they are inextensible);
3- The Transverse Norms rotate so that they remain perpendicular with the average surface after the deformation.

The consequences of hypotheses 1 and 2 are that, since the transverse normals are inextensible, then the normal deformation to the thickness is zero. In this way the displacement w is independent of z and we get in Eq. (1). (Where ε_{zz} is the normal deformation along the z-axis and $\varepsilon_{xz}, \varepsilon_{yz}$ are the transverse deformations along the z-axis)

$$\varepsilon_{zz} = \frac{\partial_w}{\partial_z} = 0 \tag{1}$$

The third hypothesis shows that the shear stresses are zero, with Eq. (2).

$$\varepsilon_{xz} = \frac{\partial_u}{\partial_z} + \frac{\partial_w}{\partial_x} = 0, \qquad \varepsilon_{yz} = \frac{\partial_v}{\partial_z} + \frac{\partial_w}{\partial_y} = 0 \tag{2}$$

The three hypotheses with their consequences show that, since w is independent of z, we arrive at Eq. (3). Where $u_{0(x,y)}$ *e* $v_{0(x,y)}$ represent, respectively, the displacements along the *u* and *v* at the point (x,y,0).

$$u_{(x,y,z)} = -z\frac{\partial_w}{\partial_x} + u_{0(x,y)}, \qquad v_{(x,y,z)} = -z\frac{\partial_w}{\partial_y} + v_{0(x,y),} \qquad w_{(x,y,z)} = w_{0(x,y)} \tag{3}$$

Thus the non-component deformations of the Green Deformation Tensor are: Eq. (4).

$$\begin{aligned}
E_{11} &= \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] \\
E_{22} &= \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] \\
E_{33} &= \frac{\partial w}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] \\
E_{12} &= \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) \\
E_{13} &= \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial z}\right) \\
E_{23} &= \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial w}{\partial y}\right)
\end{aligned} \tag{4}$$

And the gradient of displacements being very small ($|u_{ij}| \ll 1$) the Green Deformation Tensor reduces to the Tensor of infinitesimal deformations:

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 - z\frac{\partial^2 w_0}{\partial x^2} \\
\varepsilon_{xz} &= \frac{1}{2}\left(-\frac{\partial w_0}{\partial x} + \frac{\partial w_0}{\partial x}\right) = 0 \\
\varepsilon_{yy} &= \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2 - z\frac{\partial^2 w_0}{\partial y^2} \\
\varepsilon_{yz} &= \frac{1}{2}\left(-\frac{\partial w_0}{\partial y} + \frac{\partial w_0}{\partial y}\right) = 0 \\
\varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y} - 2z\frac{\partial^2 w_0}{\partial x \partial y}\right) \\
\varepsilon_{zz} &= 0
\end{aligned}
\qquad (5)\ (6)
$$

Since the deformations in Eq. (5) represents the *Deformations of von Karmam* that generate the *Plate Theory of von Karmam.*

Another way of presenting the deformations is presented in Eq. (7) where Eq. (8) are called membrane deformations and Eq. (9) are called flexural deformations.

$$
\begin{aligned}
\varepsilon_{xx} &= \varepsilon_{xx}^0 + z\varepsilon_{xx}^1 \\
\gamma_{xy} &= 2\varepsilon_{xy} = \gamma_{xy}^0 + z\gamma_{xy}^1 \\
\varepsilon_{yy} &= \varepsilon_{yy}^0 + z\varepsilon_{yy}^1
\end{aligned}
\qquad (7)
$$

$$
\begin{aligned}
\varepsilon_{xx}^0 &= \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right) \\
\varepsilon_{yy}^0 &= \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2 \\
\gamma_{xy}^0 &= \frac{\partial w_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}
\end{aligned}
\qquad (8)
$$

$$
\begin{aligned}
\varepsilon_{xx}^1 &= \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 \\
\varepsilon_{yy}^1 &= -z\frac{\partial^2 w_0}{\partial y^2} \\
\gamma_{xy}^1 &= -2z\frac{\partial^2 w_0}{\partial x \partial y}
\end{aligned}
\qquad (9)
$$

The motion equations of the MCP are derived using the principle of virtual displacements. The following considerations are made:

- In the derivations, the thermal effects are considered considering the independence of the properties of the material in relation to the temperature (fact that does not occur in many materials). It is considered that the temperature T is a function known by the position (therefore ∂T). Thus the temperature of the material only enters the constitutive equations;

- The transverse stresses do not enter into the expression of the virtual deformation energy because the transverse displacements of the plate are zero (in this MCP model) and, by kinematic consistency, the virtual transverse displacements must be zero. However, the transverse stresses enter the boundary conditions and equilibrium forces.

The dynamic version of the Principle of Virtual Displacements is given in Eq.(10): Where U is the Deformation Energy, V is the work done by the external forces and K is the Kinetic Energy. With q_b being a force distributed on the underside of the plate (z = -h / 2) and q_t the force distributed at the top (z = h / 2).

$$0 = \int_0^T (\delta U + \delta V - \delta K) dt \tag{10}$$

The virtual deformation energy is given in Eq.(11), but using the forces of the integrated thickness according to Eq.(12) the virtual deformation energy can be rewritten by Eq. (13):

$$\delta U = \int_\Omega \int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{yy}\delta\varepsilon_{yy} + 2\sigma_{xy}\delta\varepsilon_{xy}) dz dx dy = \int_\Omega \left\{ \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\sigma_{xx}(\delta\varepsilon_{xx}^0 + z\delta\varepsilon_{xx}^1) + \sigma_{yy}(\delta\varepsilon_{yy}^0 + z\delta\varepsilon_{yy}^1) + \sigma_{xy}(\delta\gamma_{xy}^0 + z\delta\gamma_{xy}^1)\right] dz \right\} dx dy \tag{11}$$

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} dz \quad , \quad \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} z\, dz \tag{12}$$

$$\delta U = \int_\Omega \left(N_{xx}\delta\varepsilon_{xx}^0 + M_{xx}\delta\varepsilon_{xx}^1 + N_{yy}\delta\varepsilon_{yy}^0 + M_{yy}\delta\varepsilon_{yy}^1 + N_{xy}\delta\gamma_{xy}^0 + M_{xy}\delta\gamma_{xy}^1\right) dx dy \tag{13}$$

Virtual work is expressed in Eq. (14). Where$F_s = -kw_0$; k is the foundation module; $q = q_b + q_t$; u_{0n}; u_{0s}; $w_0$are the displacements along the normal, tangential and transverse directions respectively.

$$\begin{aligned} \delta V = &-\int_\Omega \left[q_t(x,y)\delta w\left(x,y,\frac{h}{2}\right) + q_b(x,y)\delta w\left(x,y,-\frac{h}{2}\right)\right] dx dy \\ &- \int_\Omega F_s(x,y)\delta w\left(x,y,\frac{h}{2}\right) dx dy \\ &- \int_{\Gamma_\sigma} \int_{-\frac{h}{2}}^{\frac{h}{2}} [\hat{\sigma}_{nn}\delta u_n + \hat{\sigma}_{ns}\delta u_s + \hat{\sigma}_{nz}\delta w] dz\, ds \end{aligned} \tag{14}$$

$$= -\int_{\Omega} [q_t(x,y) + q_b(x,y) - kw_0]\delta w_0 dxdy$$

$$-\int_{\Gamma_\sigma} \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\hat{\sigma}_{nn}\left(\delta u_{0n} - z\frac{\partial \delta w_0}{\partial n}\right) + \hat{\sigma}_{ns}\left(\delta u_{0s} - z\frac{\partial \delta w_0}{\partial s}\right) + \hat{\sigma}_{nz}\delta w_0\right] dz\, ds$$

$$=-\int_{\Omega} [q - kw_0]\delta w_0 dxdy - \int_{\Gamma_\sigma}\int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\hat{N}_{nn}\delta u_{0n} - \hat{M}_{nn}\frac{\partial \delta w_0}{\partial n} + \hat{N}_{ns}\delta u_{0s} - \hat{M}_{ns}\frac{\partial \delta w_0}{\partial s} + \hat{Q}_n\delta w_0\right] ds$$

The virtual kinetic energy is expressed in Eq. (15). In that the point as an emphasis represents the time derivative ($\dot{u} = \frac{\partial u}{\partial t}$) and ($I_0; I_1; I_2$) represents moments of mass inertia.

$$\begin{aligned}
\delta K &= \int_{\Omega} \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho(\dot{u}\delta\dot{u} + \dot{v}\delta\dot{v} + \dot{w}\delta\dot{w}) dz\, dxdy \\
&= \int_{\Omega} \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho\left[\left(\dot{u}_0 - z\frac{\partial \dot{w}_0}{\partial x}\right)\left(\delta\dot{u}_0 - z\frac{\partial \delta\dot{w}_0}{\partial x}\right) + \left(\dot{v}_0 - z\frac{\partial \dot{w}_0}{\partial y}\right)\left(\delta\dot{v}_0 - z\frac{\partial \delta\dot{w}_0}{\partial y}\right)\right. \\
&\qquad \left. + \dot{w}_0\delta\dot{w}_0\right] dz\, dxdy \\
&= \int_{\Omega} \left[-I_0(\dot{u}_0\delta\dot{u}_0 + \dot{v}_0\delta\dot{v}_0 + \dot{w}_0\delta\dot{w}_0) + I_1\left(\frac{\partial \delta\dot{w}_0}{\partial x}\dot{u}_0 + \frac{\partial \dot{w}_0}{\partial x}\dot{u}_0 + \frac{\partial \delta\dot{w}_0}{\partial y}\dot{v}_0 + \frac{\partial \dot{w}_0}{\partial y}\dot{v}_0\right)\right. \\
&\qquad \left. - I_2\left(\frac{\partial \dot{w}_0}{\partial x}\frac{\partial \delta\dot{w}_0}{\partial x} + \frac{\partial \dot{w}_0}{\partial y}\frac{\partial \delta\dot{w}_0}{\partial y}\right)\right] dxdy
\end{aligned} \tag{15}$$

Replacing Eq. (13), Eq. (14) and Eq. (15) on Eq. (10) and integrating the expression along the thickness of the plaque are obtained Eq. (16), in which the virtual deformations are given in function of the virtual displacements in Eq. (17).

$$0=\int_0^T\left\{\iint_\Omega\left[N_{xx}\delta\varepsilon_{xx}^0+M_{xx}\delta\varepsilon_{xx}^1+N_{yy}\delta\varepsilon_{yy}^0+M_{yy}\delta\varepsilon_{yy}^1+N_{xy}\delta\gamma_{xy}^0+M_{xy}\delta\gamma_{xy}^1\right.\right.$$
$$+[kw_0-q]\delta w_0-I_0(\dot{u}_0\delta\dot{u}_0+\dot{v}_0\delta\dot{v}_0+\dot{w}_0\delta\dot{w}_0)+I_1\left(\frac{\partial\delta\dot{w}_0}{\partial x}\dot{u}_0+\frac{\partial\dot{w}_0}{\partial x}\dot{u}_0+\frac{\partial\delta\dot{w}_0}{\partial y}\dot{v}_0+\frac{\partial\dot{w}_0}{\partial y}\dot{v}_0\right)$$
$$\left.\left.-I_2\left(\frac{\partial\dot{w}_0}{\partial x}\frac{\partial\delta\dot{w}_0}{\partial x}+\frac{\partial\dot{w}_0}{\partial y}\frac{\partial\delta\dot{w}_0}{\partial y}\right)\right]dxdy\right\}dt \tag{16}$$

$$\delta\varepsilon_{xx}^0=\frac{\partial\delta u_0}{\partial x}+\frac{\partial w_0}{\partial x}\frac{\partial\delta w_0}{\partial x},\quad \delta\varepsilon_{xx}^1=-\frac{\partial^2\delta w_0}{\partial x^2}$$
$$\delta\varepsilon_{yy}^0=\frac{\partial\delta v_0}{\partial y}+\frac{\partial w_0}{\partial y}\frac{\partial\delta w_0}{\partial y},\quad \delta\varepsilon_{yy}^1=-\frac{\partial^2\delta w_0}{\partial y^2}$$
$$\delta\gamma_{xy}^0=\frac{\partial\delta u_0}{\partial y}+\frac{\partial\delta v_0}{\partial x}+\frac{\partial\delta w_0}{\partial x}\frac{\partial w_0}{\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial\delta w_0}{\partial y},\quad \delta\gamma_{xy}^1=-2\frac{\partial^2\delta w_0}{\partial y\partial x} \tag{17}$$

Using Eq. (17) on Eq. (16) and integrating in parts, in order to alleviate virtual displacements $(\partial_{u_0},\partial_{v_0},\partial_{w_0})$ on Ω of any differentiation (To use the variable-order lemma), one obtains Eq. (18). Where the comma represents a differentiation with respect to the following parameter: $N_{xx,x}=\frac{\partial N_{xx}}{\partial x}$.

$$\begin{aligned}0=\int_0^T\Bigg\{\iint_\Omega\Bigg[&-\left(N_{xx,x}+N_{xy,x}\right)\delta u_0\\
&-\Bigg[M_{xx,xx}+M_{yy,yy}+2M_{xy,xy}+\frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w_0}{\partial x}\right)+\frac{\partial}{\partial x}\left(N_{xy}\frac{\partial w_0}{\partial y}\right)\\
&+\frac{\partial}{\partial y}\left(N_{yy}\frac{\partial w_0}{\partial y}\right)+\frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w_0}{\partial y}\right)\Bigg]\delta w_0-\left(N_{yy,y}+N_{xy,x}\right)\delta v_0\\
&+[kw_0-q]\delta w_0+I_0(\ddot{u}_0\delta u_0+\ddot{v}_0\delta v_0+\ddot{w}_0\delta w_0)\\
&-I_2\left(\frac{\partial^2\ddot{w}_0}{\partial x^2}+\frac{\partial^2\ddot{w}_0}{\partial y^2}\right)\delta w_0\Bigg]dxdy\\
&+\oint_\Gamma\Bigg[N_{xx}n_x\delta u_0+\left(N_{xx}\frac{\partial w_0}{\partial x}\right)n_x\delta w_0-M_{xx}n_x\frac{\partial\delta w_0}{\partial x}+M_{xx,x}n_x\delta w_0\\
&+N_{yy}n_y\delta v_0+\left(N_{yy}\frac{\partial w_0}{\partial y}\right)n_y\delta w_0-M_{yy}n_y\frac{\partial\delta w_0}{\partial y}+M_{yy,y}n_y\delta w_0\\
&-M_{xy}n_x\frac{\partial\delta w_0}{\partial y}+M_{xy,x}n_y\delta w_0-M_{xy}n_y\frac{\partial\delta w_0}{\partial x}+M_{xy,y}n_x\delta w_0\\
&+N_{xy}n_y\delta u_0+N_{xy}n_x\delta v_0+N_{xy}\frac{\partial w_0}{\partial y}n_x\delta w_0+N_{xy}\frac{\partial w_0}{\partial x}n_y\delta w_0\Bigg]ds\\
&-\int_{\Gamma_\sigma}\left(\widehat{N}_{nn}\delta u_{0n}-\widehat{M}_{nn}\frac{\partial\delta w_0}{\partial n}+\widehat{N}_{ns}\delta u_{0s}-\widehat{M}_{ns}\frac{\partial\delta w_0}{\partial s}+\widehat{Q}_n\delta w_0\right)ds\\
&+\oint_\Gamma I_2\left(\frac{\partial\ddot{w}_0}{\partial x}n_x+\frac{\partial\ddot{w}_0}{\partial y}n_y\right)\delta w_0 ds\Bigg\}dt\end{aligned} \tag{18}$$

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Collecting the coefficients of virtual displacements ($\partial_{u_0}, \partial_{v_0}, \partial_{w_0}$) and noting that these are zero as t = 0, T = 0 and in the contour Γ_u we arrive in Eq. (19), where Eq. (20).

$$
\begin{aligned}
0 = \int_0^T \Bigg\{ \iint_\Omega \Bigg[& -\left(N_{xx,x} + N_{xy,y} - I_0 \ddot{u}_0\right)\partial u_0 - \left(N_{xy,x} + N_{yy,y} - I_0 \ddot{v}_0\right)\partial v_0 \\
& - \Bigg(M_{xx,xx} + 2M_{xy,xy} + M_{yy,yy} + \mathcal{N}(u_0, v_0, w_0) - k w_0 + q - I_0 \ddot{w}_0 \\
& + I_2 \frac{\partial^2 \ddot{w}_0}{\partial x^2} + I_2 \frac{\partial^2 \ddot{w}_0}{\partial y^2} \Bigg) \partial w_0 \Bigg] dxdy \\
& + \int_{\Gamma_\sigma} \Bigg[\left(N_{xx} n_x + N_{xy} n_y\right)\delta u_0 + \left(N_{xy} n_x + N_{yy} n_y\right)\delta v_0 \\
& + \Bigg(M_{xx,x} n_x + M_{xy,y} n_x + M_{yy,y} n_y + M_{xy,x} n_y + \mathcal{P}(u_0, v_0, w_0) \\
& + I_2 \frac{\partial^2 \ddot{w}_0}{\partial x^2} n_x + I_2 \frac{\partial^2 \ddot{w}_0}{\partial y^2} n_y \Bigg) \delta w_0 - \left(M_{xx} n_x + M_{xy} n_y\right) \frac{\partial \delta w_0}{\partial x} - (M_{xy} n_x \\
& + M_{yy} n_y) \frac{\partial \delta w_0}{\partial y} \Bigg] ds \\
& - \int_{\Gamma_a} \left[\widehat{N}_{nn} \delta u_{0n} + \widehat{N}_{ns} \delta u_{0s} - \widehat{M}_{nn} \frac{\partial \delta w_0}{\partial n} - \widehat{M}_{ns} \frac{\partial \delta w_0}{\partial s} + -\hat{Q}_n \delta w_0 \right] ds \Bigg\} dt
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
\mathcal{N}(u_0, v_0, w_0) &= \frac{\partial}{\partial x}\left(N_{xx} \frac{\partial w_0}{\partial x} + N_{xy} \frac{\partial w_0}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_{xy} \frac{\partial w_0}{\partial x} + N_{yy} \frac{\partial w_0}{\partial y}\right) \\
\mathcal{P}(u_0, v_0, w_0) &= \left(N_{xx} \frac{\partial w_0}{\partial x} + N_{xy} \frac{\partial w_0}{\partial y}\right) n_x + \left(N_{xy} \frac{\partial w_0}{\partial x} + N_{yy} \frac{\partial w_0}{\partial y}\right) n_y
\end{aligned}
\tag{20}
$$

Thus, we arrive at the Euler-Lagrange equations by zeroing the coefficients $\partial_{u_0}, \partial_{v_0}$ e ∂_{w_0} over the domain Ω, separately. Arriving in Eq. (21), Eq. (22), e Eq. (23). On Eq. (23) there is the particularity of the term I_2 that is called *Rotary Inertia (or Rotary)* which contributes to high frequencies of vibration.

$$
\delta u_0 : \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0 \frac{\partial^2 u_0}{\partial t^2} \tag{21}
$$

$$
\delta v_0 : \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = I_0 \frac{\partial^2 v_0}{\partial t^2} \tag{22}
$$

$$
\begin{aligned}
\frac{\partial^2 M_{xx}}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} + \mathcal{N}(u_0, v_0, w_0) - k w_0 + q \\
= I_0 \frac{\partial^2 w_0}{\partial t^2} - I_2 \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2}\right)
\end{aligned}
\tag{23}
$$

2.2 Contour Conditions

In the Energy Principles are the Natural Contour Conditions and identify the variables involved in the specification of the Essential Contour Conditions. Quantities with variation in Contour Integrals are the *Primary Variables* and its specifications constitute the Geometric Contour Conditions. The expressions that are coefficients of the variable quantities are the Secondary Variables, and their specifications constitute the Natural Contour Conditions.

Analyzing the equations Eq. (19) and Eq. (21) to Eq. (23) it is noted that the integral over Ω is zero, thus leaving the integral Eq. (24) along the contour, also null. Furthermore, in Eq. (21) to Eq. (23) the equations are expressed in terms of displacements $(u_0, v_0\ e\ w_0)$, which contains second order spatial distributions along x and y and fourth order along z. In this way the MCP is a *Eighth order theory*, therefore it must present 4 natural and 4 essential contour conditions.

$$\begin{aligned}
0 = \int_0^T \Bigg\{ \int_{\Gamma^e} \Bigg[& \left(N_{xx}n_x + N_{xy}n_y\right)\delta u_0 + \left(N_{xy}n_x + N_{yy}n_y\right)\delta v_0 \\
& + \Bigg(M_{xx,x}n_x + M_{xy,y}n_x + M_{yy,y}n_y + M_{xy,x}n_y + \mathrm{P}(u_0, v_0, w_0) + I_2\frac{\partial \ddot{w}_0}{\partial x}n_x \\
& + I_2\frac{\partial \ddot{w}_0}{\partial y}n_y\Bigg)\delta w_0 - \left(M_{xx}n_x + M_{xy}n_y\right)\frac{\partial \delta w_0}{\partial x} \\
& - \left(M_{xy}n_x + M_{yy}n_y\right)\frac{\partial \delta w_0}{\partial y}\Bigg] ds \\
& - \int_{\Gamma^e} \left(\widehat{N}_{nn}\delta w_{0n} + \widehat{N}_{ns}\delta w_{0s} - \widehat{M}_{nn}\frac{\partial \delta w_0}{\partial n} - \widehat{M}_{ns}\frac{\partial \delta w_0}{\partial s} + \hat{Q}_n \delta w_0\right) ds \Bigg\} dt
\end{aligned} \tag{24}$$

Thus, Eq. (24) shows that quantities with variation are$u_0, v_0, \frac{\partial w_0}{\partial x}\ e\ \frac{\partial w_0}{\partial y}$, therefore, these are the primary variables for a plate with edges parallel to the x and y axes. However this formulation can be evolved with the following considerations:

2.2.1 – Forces and Moments along the Edge:

Having the unitary normal Eq. (25), the displacements are given by Eq. (26), the normal and tangential derivatives given by Eq. (27), the tensions Eq. (28), the Normal ones by Eq. (29), the moments for Eq. (30) and equation Eq. (24) can be rewritten in Eq. (33) due to the changes explained in Eq. (31) and Eq. (32).

$$\hat{n} = n_x\hat{e}_x + n_y\hat{e}_y = \cos\theta\ \hat{e}_x + \sin\theta\ \hat{e}_y \tag{25}$$

$$\begin{Bmatrix} u_0 \\ v_0 \\ w_0 \end{Bmatrix} = \begin{bmatrix} n_x & -n_y & 0 \\ n_y & n_x & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_{0n} \\ u_{0s} \\ w_0 \end{Bmatrix} \tag{26}$$

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$$\begin{Bmatrix} \dfrac{\partial w_0}{\partial x} \\ \dfrac{\partial w_0}{\partial y} \end{Bmatrix} = \begin{bmatrix} n_x & -n_y \\ n_y & n_x \end{bmatrix} \begin{Bmatrix} \dfrac{\partial w_0}{\partial n} \\ \dfrac{\partial w_0}{\partial s} \end{Bmatrix} \tag{27}$$

$$\begin{Bmatrix} \sigma_{nn} \\ \sigma_{ns} \end{Bmatrix} = \begin{bmatrix} n_x^2 & n_y^2 & 2n_x n_y \\ -n_x n_y & n_x n_y & n_x^2 - n_y^2 \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} \tag{28}$$

$$\begin{Bmatrix} N_{nn} \\ N_{ns} \end{Bmatrix} = \begin{bmatrix} n_x^2 & n_y^2 & 2n_x n_y \\ -n_x n_y & n_x n_y & n_x^2 - n_y^2 \end{bmatrix} \begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} \tag{29}$$

$$\begin{Bmatrix} M_{nn} \\ M_{ns} \end{Bmatrix} = \begin{bmatrix} n_x^2 & n_y^2 & 2n_x n_y \\ -n_x n_y & n_x n_y & n_x^2 - n_y^2 \end{bmatrix} \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} \tag{30}$$

$$\begin{aligned} &\left(N_{xx}n_x + N_{xy}n_y\right)\delta u_0 + \left(N_{xy}n_x + N_{yy}n_y\right)\delta v_0 \\ &\quad = \left(N_{xx}n_x + N_{xy}n_y\right)\left(n_x \delta u_{0n} - n_y \delta u_{0s}\right) + \left(N_{xy}n_x + N_{yy}n_y\right)\left(n_y \delta u_{0n} - n_x \delta u_{0s}\right) \\ &\quad = \left(N_{xx}n_x^2 + 2N_{xy}n_x n_y + N_{yy}n_y^2\right)\delta u_{0n} \\ &\quad + \left[-N_{xx}n_x n_y + N_{yy}n_x n_y + N_{xy}\left(n_x^2 - n_y^2\right)\right]\delta u_{0s} = N_{nn}\delta u_{0n} + N_{ns}\delta u_{0s} \end{aligned} \tag{31}$$

$$\begin{aligned} &\left(M_{xx}n_x + M_{xy}n_y\right)\delta u_0 + \left(M_{xy}n_x + M_{yy}n_y\right)\delta v_0 \\ &\quad = \left(M_{xx}n_x + M_{xy}n_y\right)\left(n_x \delta u_{0n} - n_y \delta u_{0s}\right) + \left(M_{xy}n_x + M_{yy}n_y\right)\left(n_y \delta u_{0n} - n_x \delta u_{0s}\right) \\ &\quad = \left(M_{xx}n_x^2 + 2M_{xy}n_x n_y + M_{yy}n_y^2\right)\delta u_{0n} \\ &\quad + \left[-M_{xx}n_x n_y + M_{yy}n_x n_y + M_{xy}\left(n_x^2 - n_y^2\right)\right]\delta u_{0s} \\ &\quad = M_{nn}\delta u_{0n} + M_{ns}\delta u_{0s} \end{aligned} \tag{32}$$

$$\begin{aligned} 0 = \int_0^T \int_{\Gamma^e} \Bigg[&\left(N_{nn} - \widehat{N}_{nn}\right)\delta u_{0n} + \left(N_{ns} - \widehat{N}_{ns}\right)\delta u_{0s} \\ &+ \left(M_{xx,x}n_x + M_{xy,y}n_x + M_{yy,y}n_y + M_{xy,x}n_y + \mathrm{P}(u_0, v_0, w_0) + I_2 \frac{\partial \ddot{w}_0}{\partial x} n_x \right. \\ &\left. + I_2 \frac{\partial \ddot{w}_0}{\partial y} n_y - \widehat{Q}_n \right) \delta w_0 - \left(M_{nn} - \widehat{M}_{nn}\right) \frac{\partial \delta w_0}{\partial n} \\ &- \left(M_{ns} - \widehat{M}_{ns}\right) \frac{\partial \delta w_0}{\partial s} \Bigg] ds\, dt \end{aligned} \tag{33}$$

Integrating the tangential derivative by parts comes in Eq. (34). When the end points of the contour curve coincide, or when $M_{ns} = 0$, the term $[M_{ns}\partial_{w0}]_\Gamma$ is null. When$M_{ns} \neq 0$ the force of magnitude Eq. (35) acts at the contour cornersΓ of the polygonal plate. Factor 2 is due to M_{ns} the two sides of the corner of the plate. Here it assumes $M_{ns} = 0$.

Joining then Eq. (34) with the coefficient of ∂_{w0} into Eq. (33) we arrive in Eq. (36) that is called *Kirchhoff free edge condition.* So Eq. (33) can be rewritten by Eq. (37), where you have Eq. (38) and the primary and secondary MCP variables are given in Eq. (39).

$$\oint_{\Gamma} M_{ns}\frac{\partial \delta w_0}{\partial s} ds = \oint_{\Gamma} \frac{\partial M_{ns}}{\partial s} w_0 \, ds - [M_{ns}\delta w_0]_{\Gamma} \tag{34}$$

$$F_c = -2M_{ns} \tag{35}$$

$$V_n \equiv M_{xx,x}n_x + M_{xy,y}n_x + M_{yy,y}n_y + M_{xy,x}n_y + \frac{\partial M_{ns}}{\partial s} + \mathrm{P}(u_0, v_0, w_0) + I_2\frac{\partial \ddot{w}_0}{\partial x}n_x + I_2\frac{\partial \ddot{w}_0}{\partial y}n_y \tag{36}$$

$$0 = \int_0^T \oint_{\Gamma^e} \Big[\big(N_{nn} - \widehat{N}_{nn}\big)\delta u_{0n} + \big(N_{ns} - \widehat{N}_{ns}\big)\delta u_{0s} + \big(V_n - \widehat{V}_n\big)\delta w_0 - \big(M_{nn} - \widehat{M}_{nn}\big)\frac{\partial \delta w_0}{\partial n}\Big] ds dt \tag{37}$$

$$\widehat{V}_n \equiv \widehat{Q}_n + \frac{\partial \widehat{M}_{ns}}{\partial s} \tag{38}$$

$$Primary\ Variables:\ u_{0n}, \quad u_{0s}, \quad w_0, \quad \frac{\partial w_0}{\partial n}$$

$$Secondary\ Variables: N_{nn}, \quad N_{ns}, \quad V_n, \quad M_{nn} \tag{39}$$

2.3 Finite elements in the MCP

One of the complexities of the finite element models of plates is found in the approximations that this model makes when representing the plate. Among these are:

- Two-dimensional problems with geometrically complex regions are described by partial differential equations. The outline Γ of the domain Ω in two dimensions is usually a curve, which needs finite elements of simple geometry to approximate this domain. Therefore, the method presents domain approximation errors.

- The finite element method presents errors of approximation of the equations that govern the whole studied body and represents it as a group of small elements that work together with interpolations of the field of displacements inside the element.

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Thus, for a finite elemento Ω^e, using the MCP, the Virtual Work Theorem is represented in (40). Where $(n_x; n_y)$ represent the directional cosines of the unitary normal of the element Γ^e, $(M_{xx}; M_{xy}; M_{yy})$ represent the bending moments and $(\widehat{N}_{xx}; \widehat{N}_{xy}; \widehat{N}_{yy})$ represent the compression normals and the cuttings along the plane of the plate for buckling studies of the plate.

$$\begin{aligned}
0 = \int_0^T \int_{\Omega^e} \Bigg[& -\frac{\partial^2 \delta w_0}{\partial x^2} M_{xx} - 2\frac{\partial^2 \delta w_0}{\partial x \partial y} M_{xy} - \frac{\partial^2 \delta w_0}{\partial y^2} M_{yy} + k\delta w_0 w_0 \\
& + \frac{\partial \delta w_0}{\partial x}\left(\widehat{N}_{xx}\frac{\partial w_0}{\partial x} + \widehat{N}_{xy}\frac{\partial w_0}{\partial y}\right) + \frac{\partial \delta w_0}{\partial y}\left(\widehat{N}_{xy}\frac{\partial w_0}{\partial x} + \widehat{N}_{yy}\frac{\partial w_0}{\partial y}\right) + I_0 \delta w_0 \ddot{w}_0 \\
& + I_2\left(\frac{\partial \delta w_0}{\partial x}\frac{\partial \ddot{w}_0}{\partial x} + \frac{\partial \delta w_0}{\partial y}\frac{\partial \ddot{w}_0}{\partial y}\right) - \delta w_0 q \Bigg] dxdydt \\
& + \int_0^T \oint_{\Gamma^e} \Bigg[-\delta w_0 \left(\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} + \widehat{N}_{xx}\frac{\partial w_0}{\partial x} + \widehat{N}_{xy}\frac{\partial w_0}{\partial y}\right) n_x \\
& - \delta w_0 \left(\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} + \widehat{N}_{xy}\frac{\partial w_0}{\partial x} + \widehat{N}_{yy}\frac{\partial w_0}{\partial y}\right) n_y \\
& + I_2 \delta w_0 \left(\frac{\partial \ddot{w}_0}{\partial x} n_x + \frac{\partial \ddot{w}_0}{\partial y} n_y\right) + \frac{\partial w_0}{\partial x}\left(M_{xx} n_x + M_{xy} n_y\right) \\
& + \frac{\partial w_0}{\partial y}\left(M_{xy} n_x + M_y n_y\right) \Bigg] dsdt
\end{aligned} \tag{40}$$

It is noted in Eq. (40) with $w_0, \frac{\partial_{w0}}{\partial_x}$ and $\frac{\partial_{w0}}{\partial_y}$ are the primary variables (generalized displacements) and Eq. (40), Eq. (40) the secondary variables (generalized forces). Therefore, finite elements based on the MCP require continuity of the transverse deflections and their normal derivatives along the contour of the elements. In addition, to satisfy the requirements of constant displacements (rigid body mode) and constant deformations, the polynomial expansion of w_0 must be cubic full.

$$T_x \equiv M_{xx} n_x + M_{xy} n_y, \quad T_y \equiv M_{xy} n_x + M_{yy} n_y \tag{41}$$

$$\begin{aligned}
Q_n \equiv & \left(\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} + \widehat{N}_{xx}\frac{\partial w_0}{\partial x} + \widehat{N}_{xy}\frac{\partial w_0}{\partial y} - I_2 \frac{\partial^3 w_0}{\partial x \partial t^2}\right) n_x \\
& + \left(\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} + \widehat{N}_{xy}\frac{\partial w_0}{\partial x} + \widehat{N}_{yy}\frac{\partial w_0}{\partial y} - I_2 \frac{\partial^3 w_0}{\partial y \partial t^2}\right) n_y
\end{aligned} \tag{42}$$

Thus, assuming the approximation of finite elements in the form of Eq. (40), where Δ_j^e are the values of w_0 and their derivatives at nodes (x, y), φ_j^e, and are interpolation functions, their form depends on the element geometry and interpolated degrees of freedom.

Substituting Eq. (40) into Eq. (40) yields Eq. (40) where $i, j = 1,2, \dots, n$ and the coefficients of the Stiffness, Stability, Mass, and Force matrices are given in Eq. (40) where Eq. (40). Thus Eq. (40), which is the finite element displacement model, based on equations of motion expressed in terms of displacements, and generalized displacements are the primary nodal degrees of freedom.

$$w_0(x,y,t) = \sum_{j=1}^{n} \Delta_j^e(t)\varphi_j^e(x,y) \tag{43}$$

$$\sum_{j=1}^{n}\left[(K^e + G^e)\Delta^e + M^e\ddot{\Delta}^e\right] = F^e + F^{eT} \tag{44}$$

$$K_{ij}^e = \int_{\Omega^e}\left[D_{11}T_{ij}^{xxxx} + D_{12}\left(T_{ij}^{xxyy} + T_{ij}^{yyxx}\right) + 4D_{66}T_{ij}^{xyxy} + D_{22}T_{ij}^{yyyy} + kS_{ij}^{00}\right]dxdy$$

$$T_{ij}^{\xi\eta\zeta\mu} = \frac{\partial^2\varphi_i^e}{\partial\xi\partial\eta}\frac{\partial^2\varphi_j^e}{\partial\zeta\partial\mu}dxdy, \qquad S_{ij}^{\xi\eta} = \frac{\partial\varphi_i^e}{\partial\xi}\frac{\partial\varphi_j^e}{\partial\eta}dxdy \tag{46}$$

$$G_{ij}^e = \int_{\Omega^e}\left[\hat{N}_{xx}S_{ij}^{xx} + \hat{N}_{xy}(S_{ij}^{xy} + S_{ij}^{yx}) + \hat{N}_{yy}S_{ij}^{yy}\right]dxdy$$

$$M_{ij}^e = \int_{\Omega^e}\left[I_0S_{ij}^{00} + I_2(S_{ij}^{xx} + S_{ij}^{yy}\right]dxdy \tag{45}$$

$$F_i^e = \int_{\Omega^e} q\varphi_i^e dxdy + \oint_{\Gamma^e}\left(Q_n\varphi_i^e - T_x\frac{\partial\varphi_i^e}{\partial x} - T_y\frac{\partial\varphi_i^e}{\partial y}\right)ds$$

$$F_i^{eT} = \int_{\Omega^e}\left(\frac{\partial^2\varphi_i^e}{\partial x^2}M_{xy}^T + 2\frac{\partial^2\varphi_i^e}{\partial xy}M_{xy}^T + \frac{\partial^2\varphi_i^e}{\partial y^2}M_{yy}^T\right)dxdy$$

$$([K^e] + [G^e])\{\Delta^e\} + [M^e]\{\ddot{\Delta}^e\} = \{F^e\} + \{F^{eT}\} \tag{47}$$

2.4Discretized Finite Element Models

2.4.1 Static curvature

In case of bending the plate with static loading the equation Eq. (47) is reduced to the equation Eq. (48) where terms with derivatives over time are null. The methodology in solving these problems is to find the displacements generalized by the equations that assemble the problem. The deformations and the tensions are found differentiating the displacements in each element.

$$([K^e] + [G^e])\{\Delta^e\} = \{F^e\} + \{F^{eT}\} \tag{48}$$

2.4.2 Buckling

In the case of buckling of the plates due to the compression along the plane of this and cutting the edges the equation Eq. (47) is reduced to Eq. (49) Where Eq. (50).

$$\left([K^e] + \lambda\left[\hat{G}^e\right]\right)\{\Delta^e\} = \left\{\hat{F}\right\} \tag{49}$$

$$\begin{aligned} &\lambda = -\frac{\widehat{N}_{xx}}{\widehat{N}_{xx}^0} = -\frac{\widehat{N}_{yy}}{\widehat{N}_{yy}^0} = -\frac{\widehat{N}_{xy}}{\widehat{N}_{xy}^0} \\ &\hat{G}_{ij}^e = \int_{\Omega^e} \left[\widehat{N}_{xx}^0 S_{ij}^{xx} + \widehat{N}_{xy}^0\left(S_{ij}^{xy} + S_{ij}^{yx}\right) + \widehat{N}_{yy}^0 S_{ij}^{yy}\right] dxdy \\ &\bar{F}_k = \int_{\Gamma^e} \left(Q_n \varphi_k^e - T_x \frac{\partial \varphi_k^e}{\partial x} - T_y \frac{\partial \varphi_k^e}{\partial y}\right) ds \end{aligned} \tag{50}$$

2.4.3 Natural Vibration

For Natural Vibration, it is assumed that the response of the plate is periodic and the Eq. (47) is reduced to Eq.(51). Where ω represents the natural vibration frequency.

$$([K^e] - \omega^2[M^e])\{\Delta^e\} = \{\bar{F}^e\} \tag{51}$$

2.5 - Rotation and Power

The rotation used is based on the Rodrigues formulas where the rotation tensor, the symmetrical tensor of the inclinations, tensor Γ and the gradient of the deformations are given respectively by: Eq. (52), Eq. (53), Eq. 54) and Eq. (56).

Where the envelope "R" refers to the plate in the reference state and without the superscript in the post-deformation state. The abbreviations "α" refer to the median plane of the plate. The subscripts "i" refer to the orthogonal directing vectors that govern the system.

$$\begin{aligned} Q &= I + h_1(\theta)\Theta + h_2(\theta)\Theta^2 \\ h_1(\theta) &= \frac{\sin\theta}{\theta} \\ h_2(\theta) &= \frac{1}{2}\left(\frac{\sin\theta/2}{\theta/2}\right)^2 \end{aligned} \tag{52}$$

$$\Theta = skew(\theta) \tag{53}$$

$$\begin{aligned} \Gamma &= I + h_2(\theta)\Theta + h_3(\theta)\Theta^2 \\ h_3(\theta) &= \frac{1 - h_1(\theta)}{\theta^2} \\ h_2(\theta) &= \frac{1}{2}\left(\frac{\sin\theta/2}{\theta/2}\right)^2 \\ \Gamma &= Q^T\Gamma Q = Q\Gamma Q^T \end{aligned} \tag{54}$$

After differentiation of the final position after the deformation Eq. (55), where "a" is the directional vector of the mean surface after deformation:

$$\mathrm{x} = z + a \tag{55}$$

$$\mathrm{F} = QF^R = Q[I + \gamma_\alpha^r \otimes e_\alpha^r] = Q[I + (\eta_\alpha^r + \kappa_\alpha^r \times a^r) \otimes e_\alpha^r]$$
$$\eta_\alpha^r = Q^T z_{,\alpha} - e_\alpha^r \tag{56}$$
$$\kappa_\alpha^r = \Gamma^{\mathrm{T}} \theta_{,\alpha}$$

Differentiating Eq. (56), one arrives at Eq. (57): (Where the tensor $\Omega = \dot{Q}Q^T$represents the directorial spin)

$$\dot{\mathrm{F}} = \Omega\mathrm{F} + \mathrm{Q}(\dot{\gamma}_\alpha^r \otimes e_\alpha^r) \tag{57}$$

Where the deformation vector Eq. (58) and its derivative Eq. (59):

$$\gamma_\alpha^{\mathrm{r}} = \eta_\alpha^r + \kappa_\alpha^r \times a^r \tag{58}$$

$$\dot{\gamma}_\alpha^{\mathrm{r}} = \dot{\eta}_\alpha^r + \dot{\kappa}_\alpha^r \times a^r \tag{59}$$

Thus, by introducing the first Piola-Kirchhoff tensor and the tensions per unit length along the cross section of the plate in Eq. (60), we arrive at Eq. (61) because the tensors are energetically conjugated. Thus, by integrating the product of the tensile and strain tensors along the thickness of the plate and then along the domain $\Omega \subset \mathbb{R}^2$, the internal power of the plate is reached in Eq. (62).

$$P = QP^r = Q\tau_i^r \otimes e_i^r$$
$$n_\alpha^r = \int_H \tau_\alpha^r dz \qquad \text{and} \qquad m_\alpha^r = \int_H a^r \times \tau_\alpha^r dz \tag{60}$$

$$P \colon \dot{F} = \tau_\alpha^r \dot{\gamma}_\alpha^r \tag{61}$$

$$P_{int} = \int_\Omega \left[\int_H \tau_\alpha^r \dot{\gamma}_\alpha^r dz \right] d\Omega = \int_\Omega n_\alpha^r . \dot{\eta}_\alpha^r + m_\alpha^r . \dot{\kappa}_\alpha^r d\Omega = \int_\Omega \sigma_\alpha . \dot{\varepsilon}_\alpha \, d\Omega \tag{62}$$

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