Topological Derivative-Based Hydraulic Fracturing Model in Brittle Material

M. Xavier[1]

N. Van Goethem[2]

A. A. Novotny[1]

marcel@lncc.br

vangoeth@fc.ul.pt

novotny@lncc.br

1-Laboratório Nacional de Computação Científica

Avenida Getúlio Vargas, 333, 25651-075, Petrópolis, Rio de Janeiro, Brasil

2- Universidade de Lisboa, Faculdade de Ciłncias, Departamento de Matemtica, CMAF+CIO

Alameda da Universidade, C6, 1749-016 Lisboa, Portugal

Abstract.

In this work, a novel hydraulic fracturing mechanical model based on the topological derivative concept is proposed. The basic idea consists in adapt the Francfort-Marigo damage model to the context of hydraulic fracturing phenomenon. The Francfort-Marigo damage model describes the behavior of brittle materials under the quasi-static loading assumption, focusing on the evolution of damage regions under an irreversibility constraint. In the context of hydraulic fracture, the loading comes out from a hydrostatic pressure acting within the damaged region used to mimic the hydraulic fracturing process. A shape functional given by the sum of the total potential energy of the system with the Griffith-type dissipated energy term is minimized with respect to a set of ball-shaped pressurized inclusions by using the topological derivative concept. In particular, the topological derivative of the proposed shape functional, with respect to the nucleation of a circular inclusion endowed with non-homogeneous transmission condition on its boundary, is presented. Then, the obtained result is used to devise a simple and efficient topology optimization algorithm specifically designed to simulate the whole nucleation and propagation hydraulic fracturing phenomenon. Finally, a numerical example is presented showing important features associated with hydraulic fracturing process.

Keywords: *Topological Derivative, Hydraulic Fracturing Model, Topology Optimization Algorithm*

1 Introduction

Hydraulic fracturing is a process that involves pumping a mix of water, sand, and chemicals into layers of rock and shale. The purpose is to create from the pre-existing geological faults cracks or fractures into the rock that allow natural gas trapped beneath to escape to the surface. The process starts by perforating a vertical well into the reservoir. As soon as the required depth is reached, a horizontal well is done. Then, the pumping of the mix at an extremely high pressure starts. The pressure inside the geological fault is increased until attains the critical pressure at which occurs the fault activation. In order to start the study of the nucleation and propagation hydraulic fracturing process, we consider an idealized reservoir composed by the pressurization well and the undamaged and damaged regions. It is supposed that the fractured blocks occurs in the cyclic way. Therefore, by adopting convenient boundary conditions, it's enough to consider a single block as reference domain to represent the whole fractured region, see sketch in Figure 1.

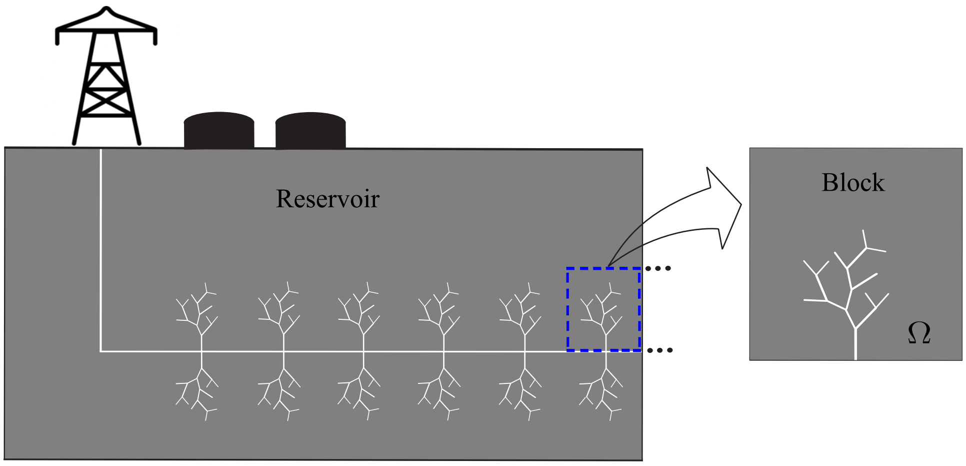


Figure 1: Fractured reservoir.

In this simplified context, a new hydraulic fracturing mechanical model based on the topological derivative concept is proposed. The idea consists in adapt the damage model proposed by Francfort & Marigo (1993), to the context of hydraulic fracturing phenomenon. This damage model describes the behavior of brittle materials taking into account the quasi-static evolution of pre-existing damaged regions. The main idea of this model consists in minimize a shape functional, which is given by the sum of the total potential energy of the system with the so called Griffith-type dissipated energy term, with respect to the distribution of the healthy and damaged phases, under an irreversibility constraint. In the present case, the loading comes out from a hydrostatic pressure acting within the damaged region used to mimic the hydraulic fracturing process.

A natural approach to deal with such a problem consists in use the topological derivative concept. Topological derivative is a scalar field that represents a first order correction of the topological asymptotic expansion of a given shape functional with respect to the introduction of infinitesimal perturbations such as holes, inclusions, source terms or cracks. This concept was introduced by Sokołowski & Żochowski (1999), and since then, has been successfully used to solve many physical and engineering problems, in particular, in damage and fracture mechanics. For more details concerning theory and applications of topological derivative see for instance the book by Novotny & Sokołowski (2013) and, in particular, for the applications in damage and fracture mechanics, see Xavier *et al.* (2017) and Van Goethem & Novotny (2010).

The methodology is outlined as follows. First of all, the topological derivative of the shape functional proposed in the context of hydraulic fracturing, with respect to the nucleation of a circular inclusion endowed with non-homogeneous transmission condition on its boundary, is presented. Then, the presented result is used to propose a topology optimization algorithm in order to simulate the hydraulic fracturing process. In this context, the topological derivative is used as descent direction to minimize the proposed shape functional indicating, in each iteration, the regions that have to be damaged. Finally, a numerical example is presented showing important features associated with hydraulic fracturing process.

The work is organized as follows. The Francfort-Marigo and the hydraulic fracturing mechanical models are presented in Section 2. The topological derivative of the proposed shape functional is presented in Section 3. The proposed algorithm is shown in details in Section 4. The numerical result is driven in Section 5, where the mentioned aspects related with hydraulic fracturing process are presented. Finally, some concluding remarks are presented in Section 6.

2 Mechanical Model

In this section we present the main aspects concerning Francfort-Marigo damage model together with its adapted version to the context of hydraulic fracture phenomenon.

2.1 Francfort-Marigo damage model

The Francfort-Marigo damage model initially proposes that a damaged elastic body, represented by a domain Ω, is composed by two distinct materials, the first one, denoted by ω, representing the damaged region, and the other one, denoted by $\Omega \setminus \overline{\omega}$, representing the undamaged part. See Figure 2.

From this configuration, the damage evolution occurs only if the specific deformation energy, ϵ, overcomes a certain material-dependent threshold. In other words, the damage evolution occurs if

$$\epsilon > \kappa, \tag{1}$$

where κ is a material property that represents the damage toughness.

Finally, the model proposes a functional, $\mathcal{F}_\omega(u_i)$, to be minimized at each time step t_i, defined as

$$\mathcal{F}_\omega(u_i) = \mathcal{J}(u_i) + \kappa|\omega|, \tag{2}$$

where u_i is the displacement field and $|\omega|$ is the Lebesgue measure of ω. The first term on the right side of (2) represents the total potential energy of the system whereas the second term is the so called Griffith-type dissipated energy term.

In addition, two conditions are assumed in this model. The first one consider that the healthy material should be more stiffer than the damaged material to characterize the stiffness loss associated to the crack growth. The other one assumes that the crack is irreversible, meaning that healing is precluded. For a complete description of the model see Francfort & Marigo (1993) and for its application in brittle materials modeling, see Allaire *et al.* (2011) and Xavier *et al.* (2017), for instance.

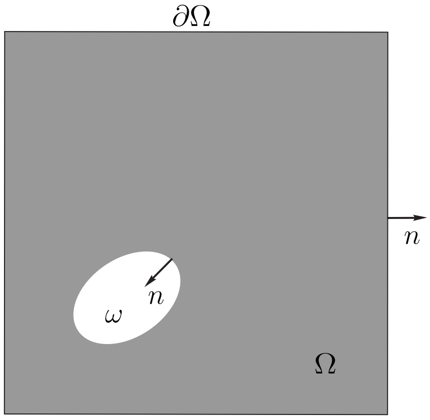


Figure 2: Unperturbed problem.

2.2 Hydraulic fracturing mechanical model

Now, in order to introduce the hydraulic fracturing mechanical model we consider that the damaged region ω is submitted to hydrostatic pressure. In addition, the pressure inside the damaged region ω, depending on the time instant t_i, is given by

$$p_i = p_{i-1} + \Delta p_i \tag{3}$$

where Δp_i is the increment. Therefore, the total applied pressure p is computed as the sum

$$p = p_0 + \sum_{i=1}^{N} \Delta p_i \ , \tag{4}$$

where p_0 is the initial pressure and N the total number of increments. Note that, for each increment of pressure p_i a new displacement field u_i is induced. Then, in order to simplify future notations we will omit the subscript i in the displacement field u_i.

Therefore, the hydraulic fracturing model has the same structure of Francfort-Marigo model and, in particular, proposes the following shape functional:

$$\mathcal{F}_\omega(u) = \mathcal{J}(u) + \kappa|\omega|, \tag{5}$$

where the total potential energy $\mathcal{J}(u)$ is given by

$$\mathcal{J}(u) = \frac{1}{2}\int_\Omega \sigma(u) \cdot \nabla u^s - \int_\omega p_i \operatorname{div}(u). \tag{6}$$

where p_i is the current hydrostatic pressure assumed to be constant in $\omega \subset \Omega$. The vector function u is the displacement field induced by the pressure p_i and solution to the following variational problem: Find $u \in \mathcal{U}$, such that

$$\int_\Omega \sigma(u) \cdot \nabla \eta^s = \int_\omega p_i \operatorname{div}(\eta), \ \forall \eta \in \mathcal{V}. \tag{7}$$

Some terms in the above variational equation require explanation. The stress tensor $\sigma(\varphi)$ is given by

$$\sigma(\varphi) = \rho \mathbb{C} \nabla \varphi^s \ , \tag{8}$$

where the parameter ρ, introduced to characterize the damage distribution, is defined as

$$\rho = \rho(x) := \begin{cases} 1, & \text{if } x \in \Omega \setminus \overline{\omega}, \\ \rho_0, & \text{if } x \in \omega, \end{cases} \tag{9}$$

with $0 < \rho_0 \ll 1$. We consider isotropic material, so that the elasticity tensor $\mathbb{C}$ can be represented by the Lamé's coefficients μ and λ in the following form

$$\mathbb{C} = 2\mu\mathbb{I} + \lambda(\mathrm{I} \otimes \mathrm{I}) \;, \tag{10}$$

where I and $\mathbb{I}$ are the second and fourth identity tensors, respectively. The strain tensor $\nabla\varphi^s$ is given by the symmetric part of the gradient of φ, namely

$$\nabla\varphi^s = \frac{1}{2}(\nabla\varphi + (\nabla\varphi)^\top) \;. \tag{11}$$

The set $\mathcal{U}$ and the space $\mathcal{V}$ are defined as:

$$\mathcal{V} := \mathcal{U} := \left\{\varphi \in H^1(\Omega) : (\varphi \cdot n)_{|\partial\Omega} = 0\right\} . \tag{12}$$

Finally, the strong formulation associated with the variational problem (7) is given by: Find u, such that:

$$\left\{ \begin{array}{rcll} \operatorname{div}\sigma(u) & = & 0 & \text{in} \quad \Omega, \\ \sigma(u) & = & \rho\mathbb{C}\nabla u^s, & \\ u \cdot n & = & 0 & \text{on} \quad \partial\Omega, \\ [\![u]\!] & = & 0 & \\ [\![\sigma(u)]\!]\, n & = & -p_i n & \end{array} \right. \left.\begin{array}{l} \\ \\ \\ \\ \\ \end{array}\right\} \text{on} \quad \partial\omega \tag{13}$$

where the operator $[\![\varphi]\!]$ is used to denote the jump of the function φ on the interface $\partial\omega$, namely $[\![\varphi]\!] = \varphi_{|\Omega} - \varphi_{|\omega}$ on $\partial\omega$. The transmission condition on the interface $\partial\omega$ comes out from the variational formulation (7).

Critical Pressure

In hydraulic fracturing process, the critical pressure is the specific value of the pressure from which occurs the activation of the pre-existing geological fault. The difficulty to modelling this kind of problems by using the Francfort-Marigo model, is that, due the stress singularities, any value of the pressure rises the strain energy density locally to unbounded values and consequently above any finite value. In this sense, it's necessary to verify, trough some numerical strategy, if the proposed hydraulic fracturing model allows the characterization such a critical pressure. There exists some remedies available in the literature to contour this problem, see for instance Allaire *et al.* (2011). Here, we adopt the same strategy proposed by Xavier *et al.* (2017). The idea consists in introduce a new material property κ_s used together with a scaling factor associated with the width δ of the initial damage. In particular, we introduce a modified energy release parameter κ_δ defined by

$$\kappa_\delta = \frac{\kappa_s}{\delta} \;. \tag{14}$$

Therefore, when δ decreases, the parameter κ_δ increases in a similar way as the energy density and the critical pressure converges to a finite nonzero value. Will be shown in Section 5 that this strategy was enough to ensure the characterization of the critical pressure.

2.3 Optimization Problem Statement

The minimization problem can be defined in the following way: for each time increment t_i,

$$\underset{\omega\subset\Omega}{\text{Minimize}}\ \mathcal{F}_\omega(u),\ \text{subject to } (13)\ , \tag{15}$$

where $\mathcal{F}_\omega(u)$ is given by (5).

A natural approach to deal with such a minimization problem consists in use the topological derivative concept. The basic idea consists in evaluate the topological derivative of the shape functional (5) with respect to the nucleation of a small circular pressurized inclusion. In this sense, the topological derivative can be used as a descent direction to solve the minimization problem (15) indicating, in each iteration, the regions that have to be damaged.

3 The Topological Derivative Formula

The topological derivative of the shape functional (5) with respect to the nucleation of a circular inclusion endowed with non-homogeneous transmission condition on its boundary is given by the sum

$$D_T\mathcal{F}_\omega(x) = D_T\mathcal{J}(x) + D_T\kappa|\omega| \quad \forall x\in\Omega\ . \tag{16}$$

The topological derivative of the Griffith-type dissipated energy term, $\kappa D_T|\omega|$, is given by

$$\kappa D_T|\omega|(x) = \begin{cases} +\kappa, & \text{if } x\in\Omega\setminus\overline{\omega}, \\ -\kappa, & \text{if } x\in\omega. \end{cases} \tag{17}$$

The topological derivative of the total potential energy of the system, $D_T\mathcal{J}(x)$, is given by

$$D_T\mathcal{J}(x) = -\mathbb{P}_\gamma\sigma(u)(x)\cdot\nabla u^s(x) - \frac{1+\alpha}{1+\alpha\gamma}p_i\,\mathrm{div}(u)(x) - \frac{1}{2\rho\mu}\frac{p_i^2}{(1+\alpha\gamma)}, \tag{18}$$

where γ is the contrast in the material property and $\mathbb{P}_\gamma$ is a fourth order isotropic tensor given by

$$\mathbb{P}_\gamma = \frac{1}{2}\frac{1-\gamma}{1+\beta\gamma}\left((1+\beta)\mathbb{I} + \frac{1}{2}(\alpha-\beta)\frac{1-\gamma}{1+\alpha\gamma}\mathrm{I}\otimes\mathrm{I}\right), \tag{19}$$

with the coefficients α and β defined as

$$\alpha = \frac{\lambda+\mu}{\mu} \quad \text{and} \quad \beta = \frac{\lambda+3\mu}{\lambda+\mu}. \tag{20}$$

4 Resulting Algorithm

The minimization problem (15) is solved by using the algorithm originally proposed by Xavier *et al.* (2017). For the sake of completeness, the adapted version of the algorithm to the context of hydraulic fracturing is presented below and for more details see Xavier *et al.* (2017).

4.1 Algorithm description

The present algorithm is based on the introduction of an inclusion of finite size at the region where the topological derivative is negative. If the size of the inclusion is small enough, to corroborate with theoretical point of view, but at the same time large enough to be treated numerically, it is expected that the shape functional (5) decreases. The size of inclusion is associated with the region ω^* where the topological derivative field is negative, i.e.,

$$\omega^* := \{x \in \Omega : D_T\mathcal{F}_\omega(x) < 0\} \ . \tag{21}$$

The algorithm can be design either by nucleating only at those points where the topological derivative achieves its minimum, or at all points were it is negative. In other hand, an intermediate choice would be to calibrate the size of the inclusion to be nucleated according to the characteristic size of the previously damaged region. This choice will be provided by the model parameter $\beta \in (0, 1)$, with the extreme choices given by $\beta = 0$ (minimum points only), and $\beta = 1$ (the whole negative region), respectively. To this aim, let us introduce the quantity

$$D_T\mathcal{F}^*_\omega := \min_{x\in\omega^*} D_T\mathcal{F}_\omega(x) \ , \tag{22}$$

which allows us to define the inclusion to be nucleated $\omega^\beta \subset \omega^*$ as follows

$$\omega^\beta := \{x \in \omega^* : D_T\mathcal{F}_\omega(x) \leq (1-\beta)D_T\mathcal{F}^*_\omega\} \ , \tag{23}$$

where $\beta \in (0, 1)$ is chosen such that $|\omega^\beta| \approx (\pi l^2)/4$ (with $l \leq \delta$), so that the size of the inclusion to be nucleated is here related to the width δ of the initial damage. Therefore, if the initial damage is crack-like (δ small), β will be taken as small as to satisfy $|\omega^\beta| \leq (\pi l^2)/4$. By this choice, a damage will evolve like a crack. As a matter of fact, the parameter β induces a threshold for the topological derivative $D_T\mathcal{F}_\omega(x)$ and the volume of the inclusion will only depend on l, while its shape will depend on the contour lines (level-sets) of $D_T\mathcal{F}_\omega$. We will show through a numerical experiment that this strategy ensures a decreasing of the proposed functional in each iteration, provided that the size of the inclusion to be nucleated ω^β is small enough.

The algorithm can be outlined as follows. Given the solution of the linear elasticity system (13), the associated topological derivative field (16) is evaluated. If the field is positive everywhere or $|\omega^*| < (\pi l^2)/4$, a perturbation of size $(\pi l^2)/4$ at any point of the domain is likely to increase the value of the functional. In this case, the algorithm will not propagate the damage, and it is possible to increase the pressure p_i further and run a new analysis. On the contrary, if the topological derivative field is negative in some undamaged region and the condition $|\omega^*| \geq (\pi l^2)/4$ is fulfilled, a damage ω^β will be nucleated inside ω^*, with $\beta : |\omega^\beta| \approx (\pi l^2)/4$. Since the nucleation of a new damage ω^β modifies the problem, the solution to the elasticity system associated with the new topology have to be computed again. Finally, the new topological derivative field is evaluated and the process is repeated until the condition $|\omega^*| \geq (\pi l^2)/4$ is not fulfilled anymore for any load increment.

5 Numerical Experiment

The reference domain Ω represents one block of the reservoir, i.e., a limited region which contains a initial damage representing a pre-existing geological fault, see Figure 3. The topology is identified by the elastic material distribution and the compliant material is used to represent the geological fault. We assume that the structure is under plane strain assumption and the total intensity of the pressure p was divided into $N = 100$ uniform increments. The mechanical problem is discretized into linear triangular finite elements only and a concentrated refinement is always done around the crack tip before the damage evolution.

5.1 Example

In this example, the hold-all domain Ω is given by a square with dimension $(5 \times 5)m^2$ as shown in Figure 3. On the top and bottom sides we consider a confinement and symmetry conditions, respectively, while on the left and right sides we consider periodicity condition. A pre-existing geological fault, represented by an initial damage of length h and width δ, is located at the center of the botton side immediately above the pressurization well. The material properties modulus of elasticity E, Poisson ratio ν and the correspondent energy release rate κ_s correspond to the same used by Pereira *et al.* (2014). In addition, the inclusion is made of a material with an elasticity modulus $\rho_0 E$ and its diameter is specified by the parameter l. All these data are summarized in the Table bellow.

Table 1: Example 1. Parameters.

Parameter	Value	Parameter	Value
h	1,0m	E	30GPa
δ	0,05m	ρ_0	10^{-6}
l	$1/5\delta$	ν	0,3
p	4MPa	κ_s	975,0 J/m

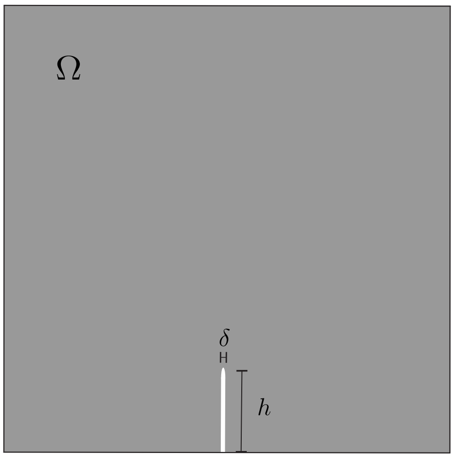


Figure 3: Geometry and boundary conditions.

Critical pressure

As mentioned in Section 2, it is necessary to verify the characterization of the critical pressure. To this aim, five tests were made with different values for the initial width of the damage, namely $\delta \in \{\frac{1}{20}, \frac{1}{40}, \frac{1}{80}, \frac{1}{160}, \frac{1}{320}\}$[m]. The critical pressure p_c was selected as the value of the current pressure which allows the nucleation of the first inclusion, that is, when the condition $|\omega^*| \geq \pi\delta^2/4$ holds for the first time. Figure 4 illustrates the critical pressure obtained for the different experiments, which are normalized according to the first estimate found for the critical pressure p_c^0. As expected, with the decrease of the width δ, the energy density at the

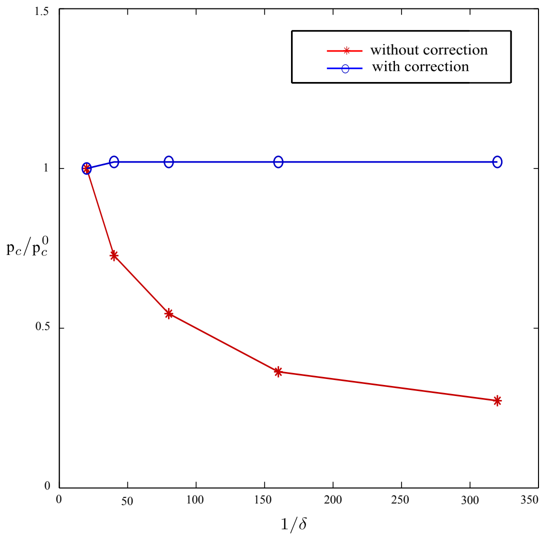
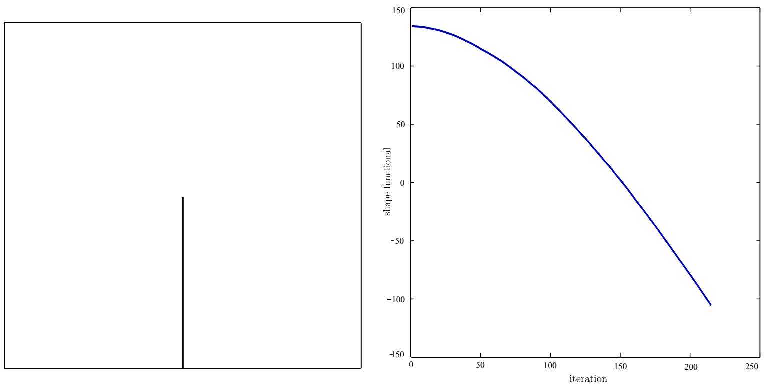


Figure 4: Convergence analysis for the critical pressure.

crack tip increases. Note that without a scale factor correction, the critical pressure decreases towards zero. On the other hand, the use of the factor δ leads to an asymptotic behavior for the critical pressure.

Damage evolution

The final result is obtained after 380 iterations, then, the program is interrupted. Note that the crack trajectory occurs in a straight line, see Figure 5(a). The pressure has been incremented 80 times from which the propagation criteria was satisfied. Therefore, the observed critical pressure is $p_c = 3.2$MPa. Note that the model dissipates energy in all iterations, see Figure 5(b).



(a) final result (b) shape functional

Figure 5: Example 1. Results.

6 Concluding Remarks

This paper has demonstrated that the linear and variational fracture model introduced by Francfort and Marigo may be applied to hydraulic fracturing process, by means of an original use of topological derivative concept. In particular, a novel hydraulic fracture model obtained by adapting the one introduced by Francfort and Marigo to the context of hydraulic fracturing process has been proposed. Then, the topological derivative associated with the proposed shape functional was presented. Finally, a numerical example was presented showing important features associated with hydraulic fracturing process.

ACKNOWLEDGEMENTS

This research was partly supported by CNPq (Brazilian Research Council), CAPES (Brazilian Higher Education Staff Training Agency), ANP (National Agency of Petroleum, Natural Gas and Biofuels) and FAPERJ (Research Foundation of the State of Rio de Janeiro). These supports are gratefully acknowledged. Second author was supported by the FCT Starting Grant " Mathematical theory of dislocations: geometry, analysis, and modelling" (IF/00734/2013).

REFERENCES

Allaire, G., Jouve, F., & Van Goethem, N. 2011. Damage and fracture evolution in brittle materials by shape optimization methods. *Journal of Computational Physics*, **230**(12), 5010–5044.

Francfort, G. A., & Marigo, J. J. 1993. Stable damage evolution in a brittle continuous medium. *European Journal of Mechanics, A/Solids*, **12**(2), 149–189.

Novotny, A. A., & Sokołowski, J. 2013. *Topological derivatives in shape optimization*. Interaction of Mechanics and Mathematics. Springer-Verlag, Berlin, Heidelberg.

Pereira, L. C., Guimarães, L. J. N., Horowitz, B., & Sánchez, M. 2014. Coupled hydro-mechanical fault reactivation analysis incorporating evidence theory for uncertainty quantification. *Computers and Geotechnics*, **56**, 202–215.

Sokołowski, J., & Żochowski, A. 1999. On the topological derivative in shape optimization. *SIAM Journal on Control and Optimization*, **37**(4), 1251–1272.

Van Goethem, N., & Novotny, A. A. 2010. Crack Nucleation Sensitivity Analysis. *Mathematical Methods in the Applied Sciences*, **33**(16), 1978–1994.

Xavier, M., Fancello, E. A., Farias, J. M. C., Goethem, N. Van, & Novotny, A. A. 2017. Topological Derivative-Based Fracture Modelling in Brittle Materials: A Phenomenological Approach. *Engineering Fracture Mechanics*, **179**, 13–27.