# QUANTIFICATION OF UNCERTAINTIES IN DYNAMIC STRUCTURAL SYSTEMS USING THE INTERVAL ANALYSIS AND THE FINITE ELEMENT METHOD

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***Abstract.*** *Due to the great importance of quantifying the uncertainties present in the mechanical systems, in this work, we will present the interval eigenvalue problem, with parametric uncertainties in the global masses and stiffness matrices of the system. Numerical examples were presented, solved computationally through a proposed method, using Interval Analysis and Finite Element Method (FEM). In order to support the solution of the problems, the basic fundamentals of the eigenvalue problem and the interval analysis were presented. After solving the proposed problems, the eigenvalues and natural frequencies of the system were presented, as well as frequency response graphs. To compare results, numerical simulations were performed using the Monte Carlo method, with the presentation of results by means of table and graphs.*

***Keywords:*** *Interval Analysis, Vibrating Systems, Eigenvalue Problem, Finite Elements.*

# 1 INTRODUCTION

Errors are inherent in all known processes. They can be caused by measurement errors, approximations, rounding, among others. Thus, the purpose of an interval analysis is to provide an interval with lower and upper limits where the exact value of the variable under analysis is contained, that is, interval in which the errors present in the system are computed. In Moore et al. (2009) can be seen several examples of the use of interval mathematics in the treatment of uncertainties.

The approach is important both in the design and in the rehabilitation of truss structures, as they are widely used in stadium coverings, transmission towers and general application sheds. The uncertainties must be quantified to ensure a longer life of the structure, avoiding sudden failures, and thus ensuring the safety of facilities and especially of people.

To ensure robustness and agility in structural analysis, computational techniques are increasingly employed. A widely used technique is the Finite Element Method (FEM), which allows good implementation and malleability in solving often complex problems. In conjunction with the interval analysis, FEM can be applied to quantify uncertainties in structural systems. A fairly complete approach to FEM can be seen in Hutton (2004).

Parameters of great importance to be determined in a dynamic analysis of structures are the natural frequencies, mainly in the design phase, whose objective is to move away the natural frequencies from those of excitation, thus avoiding the unwanted phenomenon of resonance. Theoretical studies, showing the invariance of the eigenvector signals, were performed by Deif (1991) in order to determine the eigenvalues for the standard interval eigenvalue problem for symmetric and asymmetric dynamic matrices. Later, Qiu et al. (1995) presented the generalized interval eigenvalue problem, using the Deif (1991) method and iterative process with Rayleigh quotient. Another method of solution called the inclusion principle using FEM was presented by Qiu et al. (2005) with the solution of several numerical examples. Modal analysis using the Deif (1991) method was used in Sim et al. (2007) to solve the interval eigenvalue problem, being performed simulations using the Monte Carlo method and presenting graphical results using the Frequency Response Function (FRF). For the solution of the standard problem, in Hladík et al. (2011) several algorithms were presented for the case of symmetric matrices. Using the matrix perturbation theory and FEM, Albuquerque (2015) studied the interval eigenvalue problem. Further, Li et al. (2017) used the Taylor series of the second order to compute the eigenvalues of space truss and plate, with uncertain parameters.

# 2 REVIEW OF THE EIGENVALUE PROBLEM AND FRF

It was spoken by Chapra and Canale (2010) that, eigenvalue problems are a special class of boundary-value problems that are common in engineering problem contexts involving vibrations, elasticity, and other oscillating systems. They are also used in a wide variety of engineering problems, in addition to the boundary-value problems.

The linear differential equation of motion with constant coefficients for non-damped mechanical systems with *n* degrees of freedom is presented by Eq. (1).

$$[M]\{\ddot{u}(t)\} + [K]\{u(t)\} = \{f(t)\} \tag{1}$$

where $[M]$ and $[K]$ are the symmetric matrices $n \times n$ of global mass and stiffness of the system, respectively, $u$ is the column vector of dimension $n$ with the generalized coordinates, and $f$ is the external excitation.

For systems that are not subject to external excitation, the equation of motion is given by Eq. (2).

$$[M]\{\ddot{u}(t)\} + [K]\{u(t)\} = \{0\} \tag{2}$$

As the system is conservative, the solutions of Eq. (2) are expected to be periodic, of the type of Eq. (3),

$$\{u(t)\} = \{x\}e^{j\omega_n t} \tag{3}$$

where $\{x\}$ is a column vector of constants and $\omega_n$ is the natural frequency of the system.

Substituting Eq. (3) into Eq. (2), leads to

$$\left(-\omega_n^2[M]\{x\} + [K]\{x\}\right) e^{j\omega_n t} = \{0\} \tag{4}$$

As $e^{j\omega_n t} \neq 0$ for every instant of time, arrives to

$$\left([K] - \omega_n^2[M]\right)\{x\} = \{0\} \tag{5}$$

For the non-trivial solution, it is required that

$$\det\left([K] - \lambda[M]\right) = \{0\} \tag{6}$$

with $\lambda = \omega_n^2$, where $\lambda$ are the eigenvalues of the system.

For the determination of the eigenvectors or vibration modes $\{x\}$, associated with eigenvalues, one must use Eq. (7), which is known as the Generalized Eigenvalue Problem.

$$[K]\{x\} = \lambda[M]\{x\} \tag{7}$$

Applying the Fourier Transform in Eq. (1), is found

$$\{u(t)\} \xrightarrow{\mathcal{F}} \{U(j\omega)\} \tag{8}$$

$$\{\dot{u}(t)\} \xrightarrow{\mathcal{F}} j\omega\{U(j\omega)\} \tag{9}$$

$$\{\ddot{u}(t)\} \xrightarrow{\mathcal{F}} (j\omega)^2\{U(j\omega)\} \tag{10}$$

$$\{f(t)\} \xrightarrow{\mathcal{F}} \{F(j\omega)\} \tag{11}$$

and doing the proper manipulations, leads to Eq. (12), which represents the Frequency Response Function (FRF) of the system.

$$[H(j\omega)] = \left(-\omega^2[M] + [K]\right)^{-1} \tag{12}$$

# 3 DYNAMIC STRUCTURAL ANALYSIS OF TRUSSES

In practice, the interest in using the FEM, is to solve complex problems involving trusses, with high number of bars. Thus, it is necessary to assemble the global mass and stiffness matrices of the system in the global coordinate, which is the linear sum of the contribution of the mass matrix $[M]_e$ and stiffness $[K]_e$ of each element in the global coordinate. To perform the sum, use the degrees of freedom between the connectivities of the nodes. This way,

$$[K] = [K]_0 + \sum_{\phi=1}^{m}[K]_{e_\phi} \tag{13}$$

$$[M] = [M]_0 + \sum_{\phi=1}^{m}[M]_{e_\phi} \tag{14}$$

where $m$ is the number of elements, $[K]_0$ and $[M]_0$ are the null $n \times n$ matrices.

Mass and stiffness matrices are functions of the vector $\{a\}$, which contains the geometric and structural parameters, in this way

$$[K] = [K(a)] \tag{15}$$

$$[M] = [M(a)] \tag{16}$$

with

$$\{\underline{a}\} \leq \{a\} \leq \{\overline{a}\}, \qquad a_\phi^I = [\underline{a}_\phi, \overline{a}_\phi] \qquad \text{ou} \qquad \underline{a}_\phi \leq a_\phi \leq \overline{a}_\phi \qquad \phi = 1, 2, ..., m$$

where $\{\underline{a}\} = (\underline{a}_\phi)$ and $\{\overline{a}\} = (\overline{a}_\phi)$ are, respectively, the lower and upper limits of the structural parameter $\{a\}$, with $\{a\} \in \{a\}^I$.

Thus, Eq. (13) and Eq. (14) can be rewritten as functions of the uncertain structural parameters.

$$[K]^I = [\underline{K}, \overline{K}] = [K]_0 + \sum_{\phi=1}^{m} a_\phi^I [K]_{e_\phi} \tag{17}$$

$$[M]^I = [\underline{M}, \overline{M}] = [M]_0 + \sum_{\phi=1}^{m} a_\phi^I [M]_{e_\phi} \tag{18}$$

where

$$[\underline{K}] = [K]_0 + \sum_{\phi=1}^{m} \underline{a}_\phi [K]_{e_\phi} \tag{19}$$

$$[\overline{K}] = [K]_0 + \sum_{\phi=1}^{m} \overline{a}_\phi [K]_{e_\phi} \tag{20}$$

$$[\underline{M}] = [M]_0 + \sum_{\phi=1}^{m} \underline{a}_\phi [M]_{e_\phi} \tag{21}$$

$$[\overline{M}] = [M]_0 + \sum_{\phi=1}^{m} \overline{a}_\phi [M]_{e_\phi} \tag{22}$$

Thus, with uncertainties present in the matrices of mass and stiffness of the system, it leads to the generalized interval eigenvalue problem represented by Eq. (23).

$$[K]^I \{x\} = \lambda [M]^I \{x\} \tag{23}$$

The exact bounds for eigenvalues $\lambda^I = [\underline{\lambda}, \overline{\lambda}]$ can be computed through Eq. (24) and Eq. (25).

$$\underline{\lambda} = \min \left( [\underline{K}]\{\underline{x}\} - \underline{\lambda}[\underline{M}]\{\underline{x}\} = 0, [\overline{K}]\{\overline{x}\} - \overline{\lambda}[\overline{M}]\{\overline{x}\} = 0 \right) \tag{24}$$

$$\overline{\lambda} = \max \left( [\underline{K}]\{\underline{x}\} - \underline{\lambda}[\underline{M}]\{\underline{x}\} = 0, [\overline{K}]\{\overline{x}\} - \overline{\lambda}[\overline{M}]\{\overline{x}\} = 0 \right) \tag{25}$$

where $\{\underline{x}\}$ is the lower bound and $\{\overline{x}\}$ the upper bound for eigenvectors, with $\{x\}^I = [\underline{x}, \overline{x}]$.

Because the eigenvalues are uncertain, logically the natural frequency also tends to be uncertain, with $\underline{\lambda} = \underline{\omega}_n^2$ and $\overline{\lambda} = \overline{\omega}_n^2$.

# 4 NUMERICAL EXAMPLES

Two examples are used to demonstrate the possibility of applying the proposed method. The first is a particular case of a flat truss with 24 bars, and the second is a space truss with 60 bars. For both examples there are uncertainties present in the areas of the cross sections of the bars, which makes the matrices of mass and stiffness of the structures have uncertainties. Also, for the second example, simulations using the Monte Carlo method are performed for comparison of results.

## 4.1 Flat truss with 24 bars

All joints are pinned, as can be seen by means of Fig. 1. The areas of the cross sections of bars 1, 2, 3, 6, 9, e 11 are considered uncertain, according $A^I_\phi = [A^c - \alpha A^c, A^c + \alpha A^c]$, $\phi = 1, 2, 3, 6, 9, 11$, where $A^c = 7.0 \times 10^{-4}\,\mathrm{m}^2$ is the deterministic value of the area for all bars and $\alpha$ is the percentage factor that will cause variation of the uncertain parameters to occur, with $\alpha$ ranging from $0$ a $5\%$. The Young's modulus of the material is $E = 210\,\mathrm{GPa}$ and the specific mass is $\rho = 7.8 \times 10^3\,\mathrm{kg/\,m^3}$.

An important fact to be observed in Fig. 2 is the monotonic behavior of the structural parameters, either increasing or the inverse, starting from the deterministic value. Similar fact, related to this behavior of eigenvalues, can be seen in Modarreszadeh (2005) and Qiu et al. (2005). One can see more about this monotonic inclusion behavior in Moore (1979).

## 4.2 Space truss with 60 bars

The Fig. 3, shows a space truss with 60 bars, where all joints are pinned, with fixed supports on nodes 1, 2, 3, and 4, which allow only rotation. The uncertain parameters are due to the cross-sectional area, according to $A^I_\phi = [0.95 \times 10^{-3}\,, 1.05 \times 10^{-3}\,]\ \mathrm{m}^2$ and Young's module, with $E^I_\phi = [199.5, 220.5]$ GPa. The specific mass is $\rho = 7.8 \times 10^3\,\mathrm{kg/\,m^3}$.

By means of Fig. 4 the interval frequency response for the first $4$ modes of vibration can be observed. In order to assure the reliability of the obtained results are shown by means of Fig. 5 graphical results of simulations using the Monte Carlo method, with number of samples equal to 2000. In table 1 it is possible to observe comparison between the results obtained through the proposed method and the simulations with the Monte Carlo method. Note that the results converge at least to the second decimal place.

# 5 CONCLUSIONS

In this paper, it was demonstrated, through the analyzed examples, that the Finite Element Method, together with the interval analysis, are important tools for the quantification of parametric uncertainties in structures, being of great relevance to guarantee structural reliability, avoiding failures. For this, a computational program was developed to solve the interval eigenvalue problem using the proposed method, which demonstrated a rapid solution of the problems, and provided reliable solutions, with the possibility of application in practical situations. The Monte Carlo method was used to compare the results, demonstrating the reliability of the obtained results.

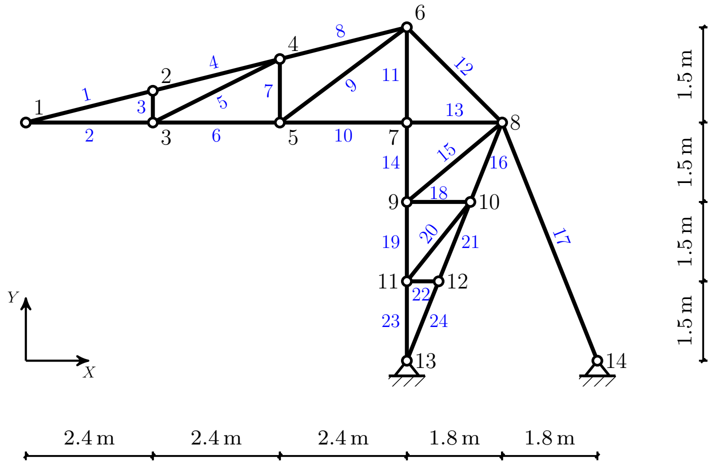


**Figure 1: Flat truss with 24 bars.**

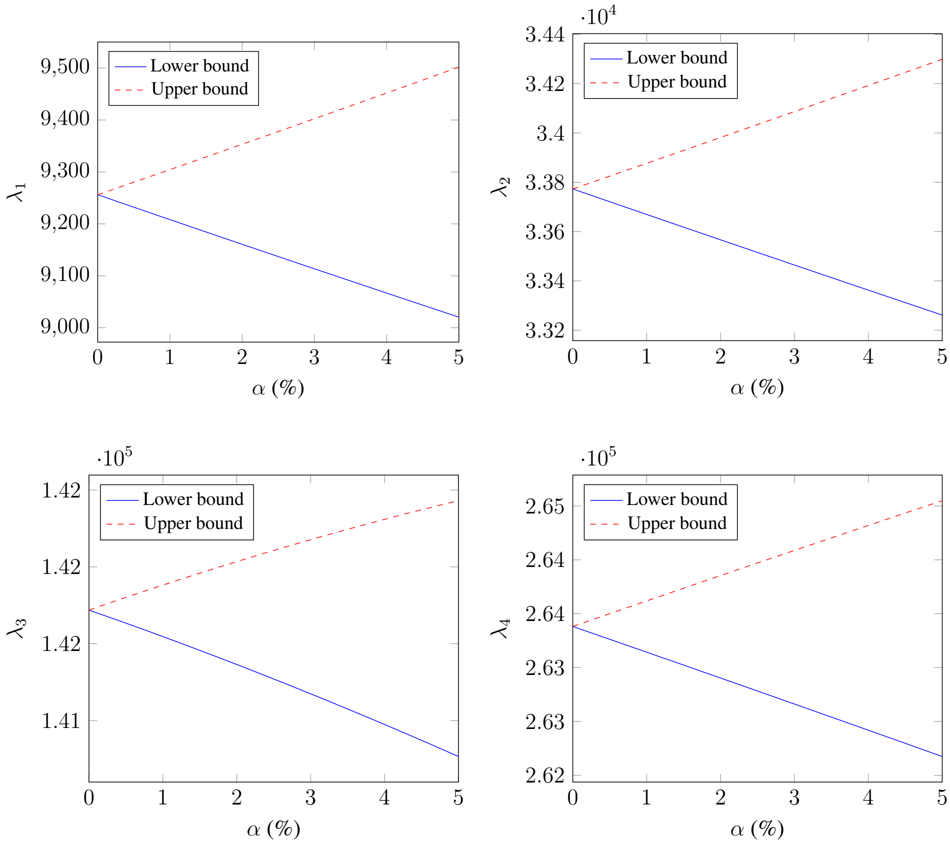


**Figure 2: Eigenvalues of the truss with 24 bars.**

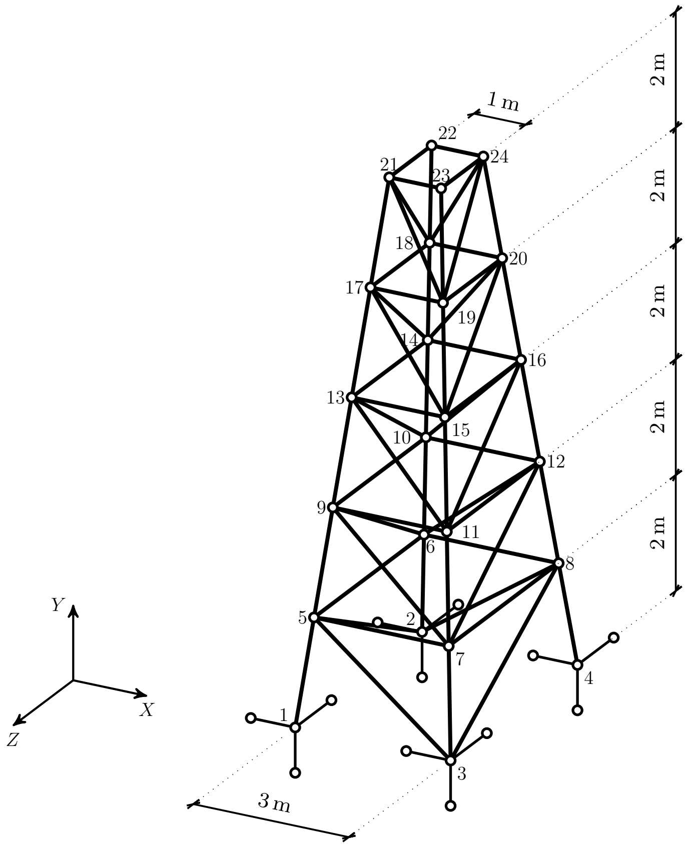


**Figure 3: Space truss with 60 bars.**

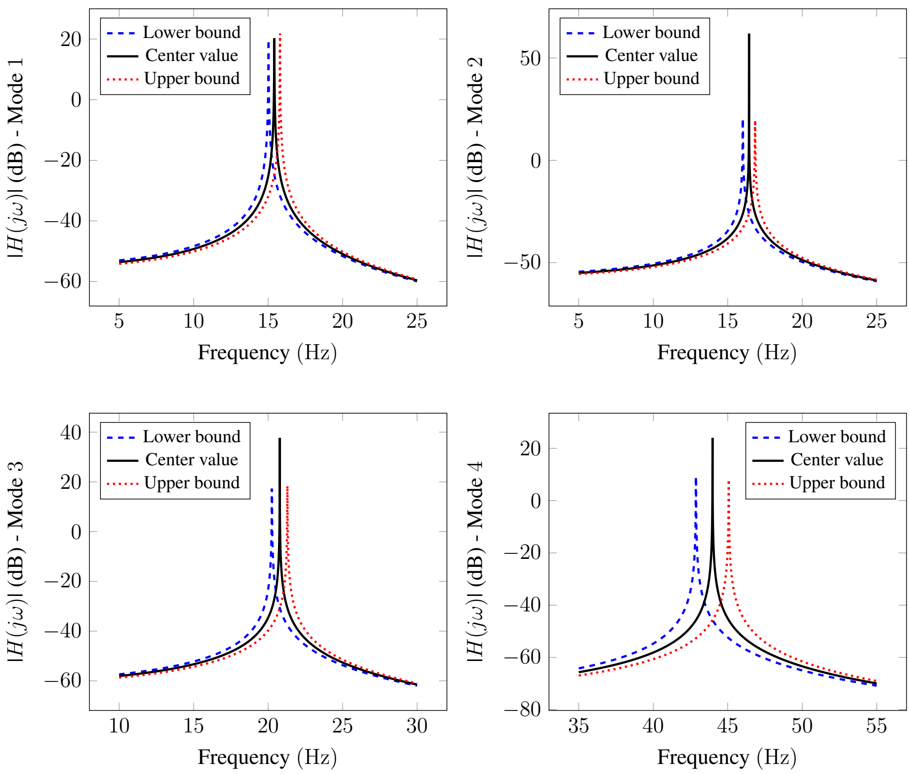


**Figure 4: Frequency response of the truss with 60 bars.**

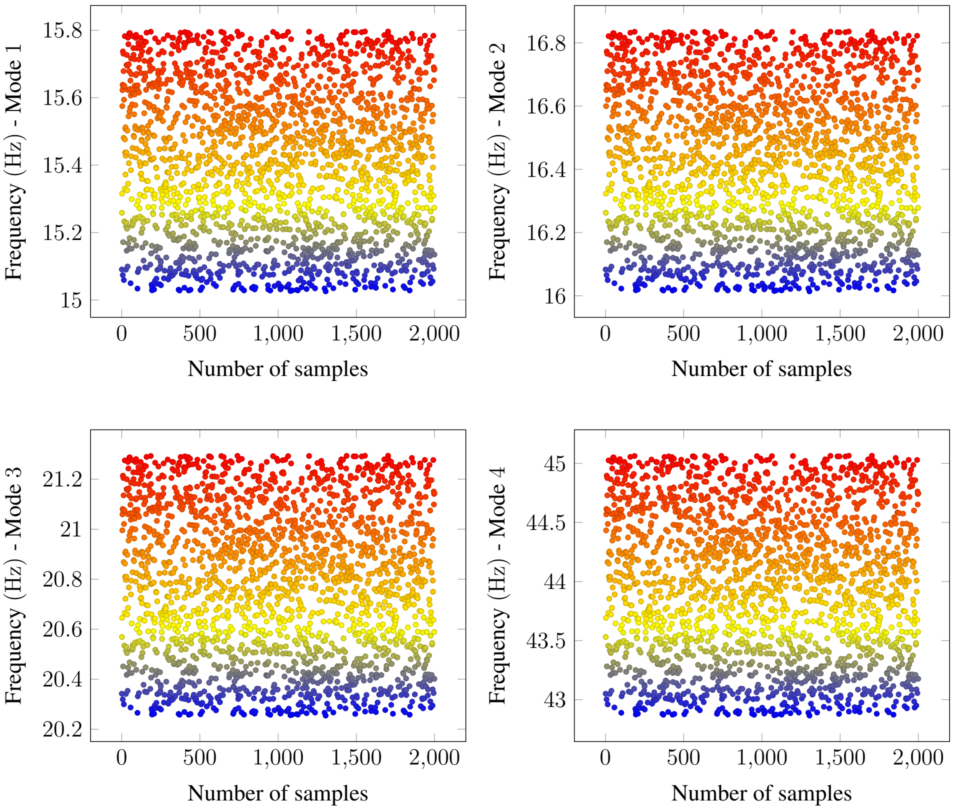


**Figure 5: Simulations Monte Carlo method for truss with 60 bars.**

**Table 1: Natural frequency truss with 60 bars.**

| **Frequency** (Hz) | | | | |
|---|---|---|---|---|
| | *Proposed Method* | | *Monte Carlo Method* | |
| Modes | Lower bound | Upper bound | Lower Bound | Upper bound |
| 1 | 15.0254 | 15.7964 | 15.0255 | 15.7963 |
| 2 | 16.0140 | 16.8358 | 16.0142 | 16.8357 |
| 3 | 20.2544 | 21.2937 | 20.2546 | 21.2936 |
| 4 | 42.8654 | 45.0650 | 42.8658 | 45.0647 |

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