CILAMCE 2017
Florianópolis - Santa Catarina - Brazil

QUANTIFICATION OF UNCERTAIN PARAMETERS IN STATIC STRUCTURES USING THE INTERVAL ANALYSIS AND THE FINITE ELEMENT METHOD

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Abstract. *In this paper, a method was presented that uses the Interval Analysis and the Finite Element Method (FEM). The aim is to determine some uncertain structural parameters, important both in the design phase and in the rehabilitation of structures. Examples of such parameters that have been determined are: nodal displacements, internal forces in the bars, internal stresses in the bars, the deformations of the bars and the support reactions. At first, to support the solution of the proposed problems and make possible the understanding, the necessary foundations of FEM and the Interval Arithmetic were presented. Subsequently, numerical examples of flat and space trusses were presented, when there are uncertainties in the overall stiffness matrix of the structure and in the vector of loads. A computational routine was implemented to solve the proposed problems. For the validation of the proposed method, numerical results and simulation graphs were presented using the Monte Carlo Method. The proposed method proved to be efficient for the solution of the presented problems, being in compliance with the Monte Carlo simulations.*

Keywords: *Interval Analysis, Static Structures, Uncertain Parameters, Finite Elements.*

1 INTRODUCTION

Errors are inherent in all known processes. They can be measurement errors, approximations, rounding, among others. Thus, the purpose of an interval analysis is to provide an interval with lower and upper limits where the exact value of the variable under analysis is contained, that is, interval in which the errors present in the system are computed. In Moore et al. (2009) can be seen several examples of the use of interval mathematics in the treatment of uncertainties.

The approach is important both in the design and in the rehabilitation of truss structures, as they are widely used in stadium coverings, transmission towers and general application sheds. The uncertainties must be quantified to ensure a longer life of the structure, avoiding sudden failures, and thus ensuring the safety of facilities and especially of people.

To ensure robustness and agility in structural analysis, computational techniques are increasingly employed. A widely used technique and the Finite Element Method FEM, which allows good implementation and malleability in solving often complex problems. In conjunction with interval mathematics, FEM can be applied to quantify uncertainties in structural systems. A fairly complete approach to FEM can be seen in Hutton (2004).

2 REVIEW OF FEM FOR BAR ELEMENT

The bar element is commonly used in solving structural problems involving trusses. Firstly, for the finite element formulation of a bar element, with nodes i e j, according to the discretization illustrated by means of Fig. 1, some considerations are necessary:

1. Material obeys Hooke's Law;
2. The bar is geometrically straight;
3. The forces are applied only at the ends of the bar in your barycenter;
4. The bar only supports axial loading.

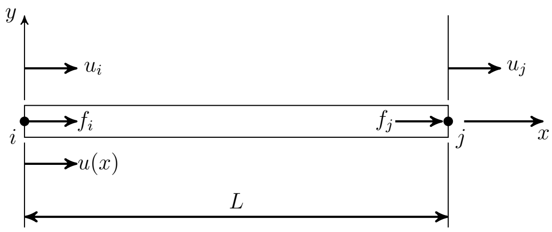

Figure 1: Bar element in local coordinate.

Using the direct approach and the linear interpolation functions N_1 e N_2, with the characteristics presented by means of the Fig. 2, and with the aid of material resistance, is found in Eq. (1) the stiffness matrix of the bar element in the local coordinate.

$$\left[k\right]_e = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{1}$$

where E is the Young's modulus, A is the cross-sectional area and L is the length of the bar member.

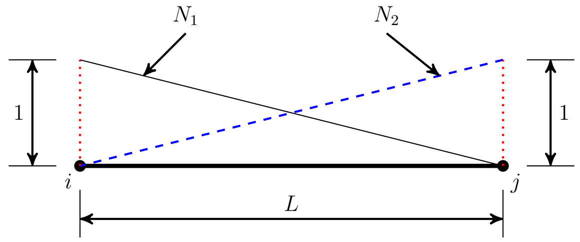

Figure 2: Linear shape functions.

2.1 Bar Element in Plane

For the solution of two-dimensional and three-dimensional problems it is necessary to carry out the transformation of the local to global coordinate, common to all elements.

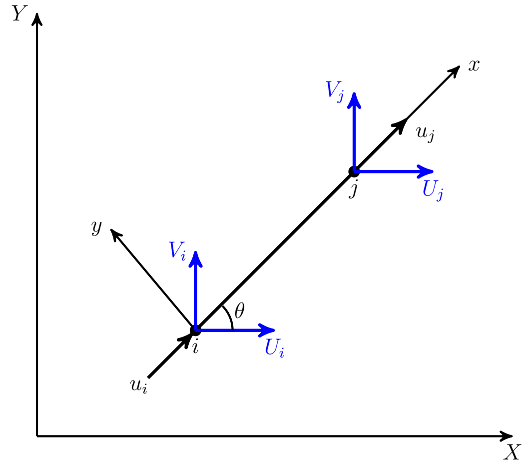

Figure 3: Slope element in the plane.

Analyzing Fig. 3, which has 2 degrees of freedom per node, U and V, and performing some substitutions, arrives at the matrix

$$\begin{bmatrix} R_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix} \tag{2}$$

which is known as the Transformation Matrix of the local coordinates for the global coordinates, for the two-dimensional case.

Where

$$\cos\theta = \frac{X_j - X_i}{L} \tag{3}$$

$$\sin\theta = \frac{Y_j - Y_i}{L} \tag{4}$$

$$L = \sqrt{(X_j - X_i)^2 + (Y_j - Y_i)^2} \tag{5}$$

Using Eq. (2) and Eq. (1), it leads to the stiffness matrix of element in the global coordinate.

$$[K_2]_e = [R_2]^T [k]_e [R_2] \tag{6}$$

2.2 Bar element in space

For this case, coordinate transformation is also necessary. The same approach described above is used for the three-dimensional case, however, here are 3 degrees of freedom per node, namely U, V and W.

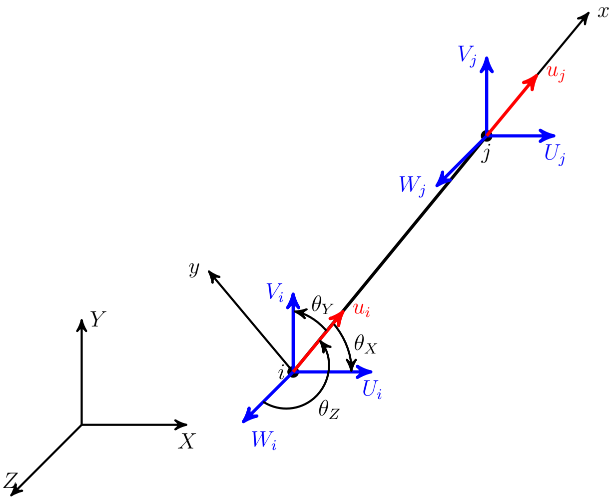

Figure 4: Bar element in three-dimensional space.

Analyzing Fig. 4 and realizing the necessary substitutions, is found in Eq. (7) the Matrix of Transformation of the local coordinate to the global coordinate, in the three-dimensional case.

$$\left[R_3\right] = \begin{bmatrix} \cos\theta_X & \cos\theta_Y & \cos\theta_Z & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta_X & \cos\theta_Y & \cos\theta_Z \end{bmatrix} \tag{7}$$

where

$$\cos\theta_X = \frac{X_j - X_i}{L} \tag{8}$$

$$\cos\theta_Y = \frac{Y_j - Y_i}{L} \tag{9}$$

$$\cos\theta_Z = \frac{Z_j - Z_i}{L} \tag{10}$$

$$L = \sqrt{(X_j - X_i)^2 + (Y_j - Y_i)^2 + (Z_j - Z_i)^2} \tag{11}$$

Using Eq. (1) and Eq. (7), it leads to the stiffness matrix in the global coordinate for the three-dimensional case.

$$[K_3]_e = [R_3]^T[k]_e[R_3] \tag{12}$$

3 STRUCTURAL ANALYSIS OF TRUSSES

In practice, the interest in using the FEM is to solve complex problems involving trusses, with high number of bars. Thus, it is necessary to perform the assembly of the global stiffness matrix of the system in the global coordinate, which is the linear sum of the contribution of the stiffness matrix of each element. To perform the sum, use the degrees of freedom between the connectivity of the nodes. This way,

$$[K] = [K]_0 + \sum_{\phi=1}^{m}[K]_{e_\phi} \tag{13}$$

where m is the number of elements, $[K]_0$ is null matrix $n \times n$ and $[K]$ is the matrix of stiffness of the system.

With the load vector $\{F\}$, already in the global coordinate, because it is transformed during obtaining the stiffness matrix in the global coordinate, and $[K]$, one can construct the equilibrium equation 14.

$$[K]\{d\} = \{F\} \tag{14}$$

where $\{d\}$ is the vector with the nodal displacements.

With calculated $\{d\}$, one can calculate other important structural parameters, like the internal force, the strain and stress in the element, as well as the support reactions.

Some basic definitions of Interval Mathematics are described below. Being $B^I = [\underline{b}, \overline{b}]$ e $C^I = [\underline{c}, \overline{c}]$, arithmetic operations are.

- $B^I + C^I = \left[\underline{b} + \underline{c}, \overline{b} + \overline{c}\right]$
- $B^I - C^I = \left[\underline{b} - \overline{c}, \overline{b} - \underline{c}\right]$
- $B^I \times C^I = \left[min\left(\underline{bc}, \underline{b}\overline{c}, \overline{b}\underline{c}, \overline{bc}\right), max\left(\underline{bc}, \underline{b}\overline{c}, \overline{b}\underline{c}, \overline{bc}\right)\right]$

- $B^I \div C^I = B^I \times \dfrac{1}{C^I} = [\underline{b}, \overline{b}] \times \left[\dfrac{1}{\overline{c}}, \dfrac{1}{\underline{c}}\right] = \left[\dfrac{\underline{b}}{\overline{c}}, \dfrac{\overline{b}}{\underline{c}}\right]$ with $[\underline{c}, \overline{c}] \neq 0$.

The key point in these definitions is that computing with intervals is computing with sets Moore et al. (2009).

The global stiffness matrix is a function of the vector $\{a\}$, which contains the geometric and structural parameters, in this way

$$[K] = [K(a)] \tag{15}$$

with

$$\{\underline{a}\} \leq \{a\} \leq \{\overline{a}\}, \qquad a_\phi^I = [\underline{a}_\phi, \overline{a}_\phi] \qquad \text{ou} \qquad \underline{a}_\phi \leq a_\phi \leq \overline{a}_\phi \qquad \phi = 1, 2, ..., m$$

where $\{\underline{a}\} = (\underline{a}_\phi)$ and $\{\overline{a}\} = (\overline{a}_\phi)$ are, respectively, the lower and upper limits of the structural parameter $\{a\}$, with $\{a\} \in \{a\}^I$.

Thus, a Eq. (13) can be rewritten as a function of the uncertain structural parameters.

$$[K]^I = [\underline{K}, \overline{K}] = [K]_0 + \sum_{\phi=1}^{m} a_\phi^I [K]_{e_\phi} \tag{16}$$

$$[\underline{K}] = [K]_0 + \sum_{\phi=1}^{m} \underline{a}_\phi [K]_{e_\phi} \tag{17}$$

$$[\overline{K}] = [K]_0 + \sum_{\phi=1}^{m} \overline{a}_\phi [K]_{e_\phi} \tag{18}$$

In the situation where the load vector $\{F\}$ has uncertainties, thus Eq. (14) can be rewritten as follows.

$$[\underline{K}, \overline{K}]\{\underline{d}, \overline{d}\} = \{\underline{F}, \overline{F}\} \tag{19}$$

There are some methods to solve Eq. (19), which can be seen in Sága et al. (2014), in Hansen and Walster (2003) and Deif (1986). In this paper, it is proposed to solve two deterministic problems, as follows

$$\{\underline{d}\} = \min[\underline{K}^{-1}\underline{F}, \overline{K}^{-1}\overline{F}] \tag{20}$$

$$\{\overline{d}\} = \max[\underline{K}^{-1}\underline{F}, \overline{K}^{-1}\overline{F}] \tag{21}$$

The exact solutions to the proposed problems are given by Eq. (20) and Eq. (21).

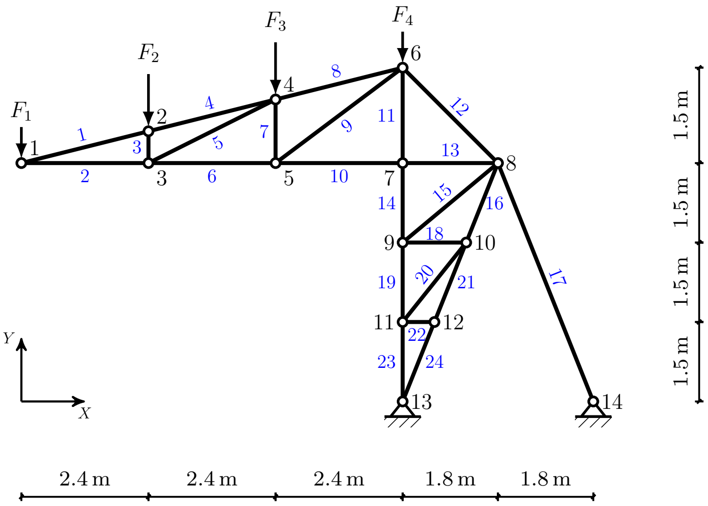

Figure 5: Flat truss with 24 bars.

4 NUMERICAL EXAMPLES

Two examples will be used to demonstrate the possibility of applying the proposed method. The first is a particular case of a flat truss with 24 bars, presenting uncertainties in the cross-sectional areas of some bars and loading. The second example is a space truss with 60 bars where the uncertainties are present in the Young's modulus.

4.1 Flat truss with 24 bars

All joints are pinned, as can be seen by means of Fig. 5. The areas of the cross sections of bars 1, 2, 3, 6, 9, e 11 are considered uncertain, according $A_{\phi}^{I} = [A^c - \alpha A^c, A^c + \alpha A^c]$, $\phi = 1, 2, 3, 6, 9, 11$, where $A^c = 7.0 \times 10^{-4} m^2$ is the deterministic value of the area for all bars and α is the percentage factor that will cause variation of the uncertain parameters to occur, with α ranging from 0 a 10%. Also, there are uncertainties in loading $F_1^I = [F_1 - \alpha F_1, F_1 + \alpha F_1]$, where $F_1 = 5\,\mathrm{kN}$, $F_2 = F_3 = 10\,\mathrm{kN}$ and $F_4 = 5\,\mathrm{kN}$. The Young's modulus of the material is $E = 210 \times 10^9 N/m^2$.

4.2 Space truss with 60 bars

The Fig. 6 shows a space truss with 60 bars, where all joints are pinned, with fixed supports on nodes 1, 2, 3, e 4, which allow only rotation. In this example, the uncertain parameter will be due to the property of the material, with Young's modulus $E_{\phi}^{I} = [199.5, 220.5]\ \mathrm{GN/\,m^2}$. The cross-sectional area for all the bars is $A_{\phi} = 1.0 \times 10^{-3}\ \mathrm{m}^2$, with loadings $F = 15.0\,\mathrm{kN}$.

5 RESULTS AND DISCUSSIONS

5.1 Flat truss

For the Figures 7, 8 and 9 the dotted line refers to the *upper bound* and the solid line to the *lower bound*. The displacements for node 1 in the x (d_x) and y (d_y) axis directions are shown by Fig. 7. Results for the internal force, stress and strain for element 11, can be observed by

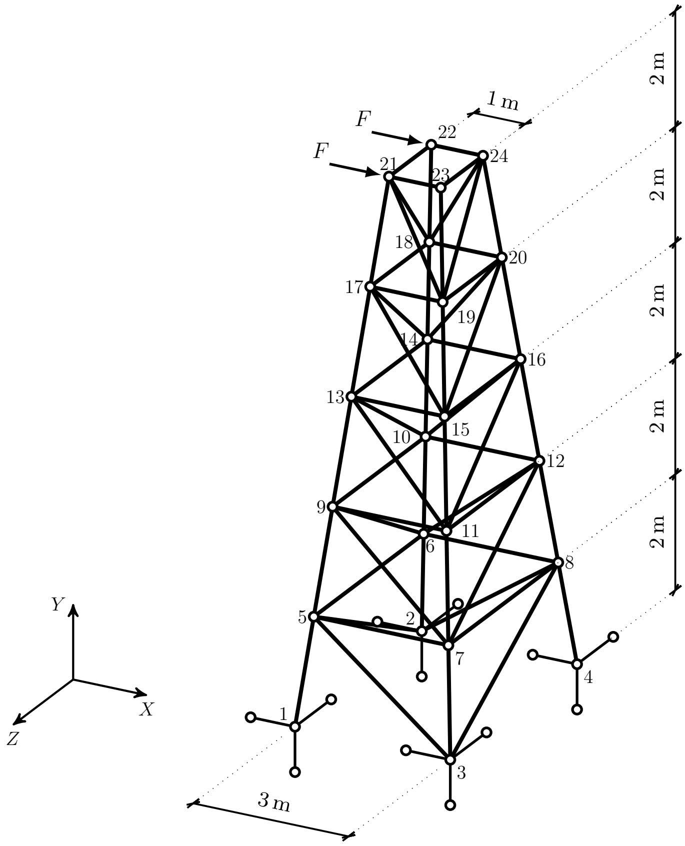

Figure 6: Space truss with 60 bars.

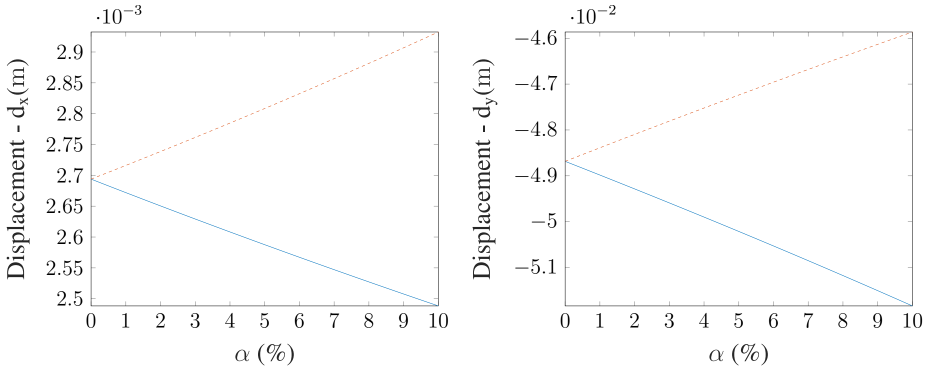

Figure 7: Nodal displacements to node 1 for truss with 24 bars.

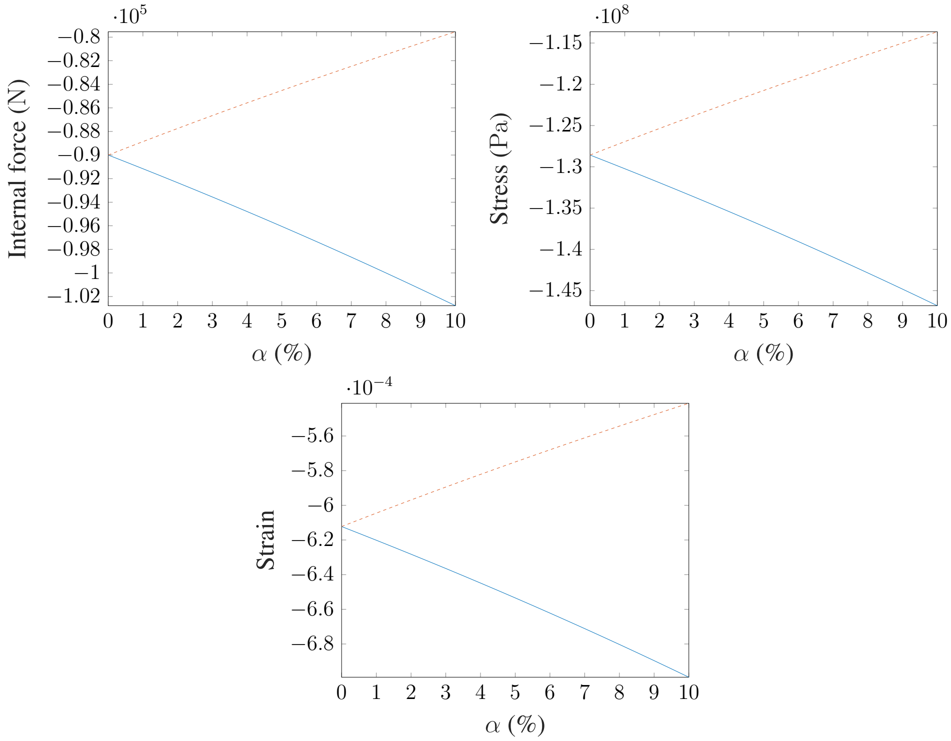

Figure 8: Force, stress and strain for element 11 for truss with 24 bars.

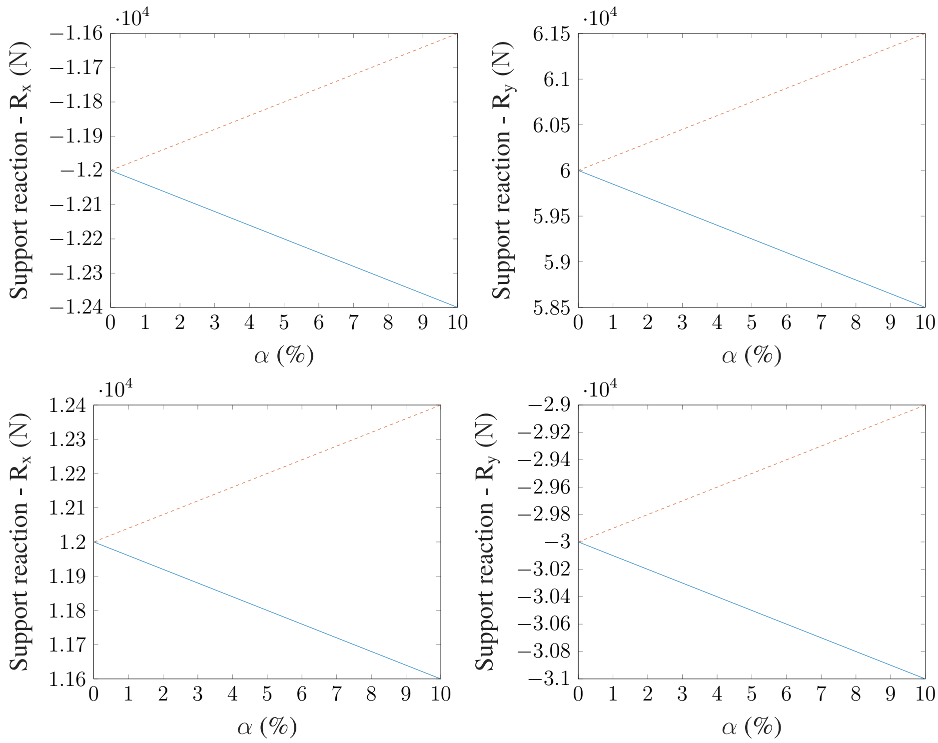

Figure 9: Reactions of support for nodes 13 (top) e 14 (bottom) for truss with 24 bars.

Table 1: Displacements for node 21 for truss with 60 bars.

	Nodal displacements (m)			
	Proposed Method		Monte Carlo Method	
	Lower bound	Upper bound	Lower Bound	Upper bound
d_x	0.00923125	0.01020296	0.00923140	0.01020276
d_y	0.00071351	0.00078861	0.00071352	0.00078860
d_z	-0.00172121	-0.00155729	-0.00172118	-0.00155731

means of Fig. 8, and Fig. 9 shows the results for the support reactions on nodes 13 and 14. These structural parameters are of great importance in the structural design phase, and with the computational implementation one can simulate several conditions and configurations for the trusses.

An important fact to be observed in these figures is the monotonic behavior of the structural parameters as a function of α, either increasing or the inverse, starting from the deterministic value. Similar fact occurs with the symmetric matrix eigenvalues using FEM, which can be seen in Modarreszadeh (2005), remembering that this is the case also of this paper, that is, symmetric stiffness matrix. One can see more about this monotonic inclusion behavior in Moore (1979).

5.2 Space truss

Table 1 shows the nodal displacements for node 21 in the directions of the axes x (d_x), y (d_y) and z (d_z), using the proposed method and the Monte Carlo method. The simulations were performed using the Monte Carlo method, which is a robust numerical method in stochastic analysis, to guarantee the reliability of the results obtained with the proposed method. By means of Fig. 10, the graphical results of the simulations with number of samples equal to 2,000 can be observed. Note that the results converge to at least the sixth decimal place.

It should be emphasized the importance of implementing the method computationally for the quantification of uncertainties, because in this case the truss has 60 bars arranged in the three-dimensional space, being able to change important parameters easily and quickly during the analysis. Of course the program solves problems involving trusses with any number of bars.

6 CONCLUSIONS

In this paper, through the examples analyzed, it can be demonstrated that the Finite Element Method, together with the interval mathematics, is an important tool for the quantification of parametric uncertainties in structures. This quantification is of great importance to guarantee the structural reliability, avoiding failures. For this, a computer program was developed, which demonstrated speed in the solution of problems, and provided reliable solutions, with possibility of application in practical situations. The Monte Carlo method was used to compare the results, demonstrating the reliability of the obtained results.

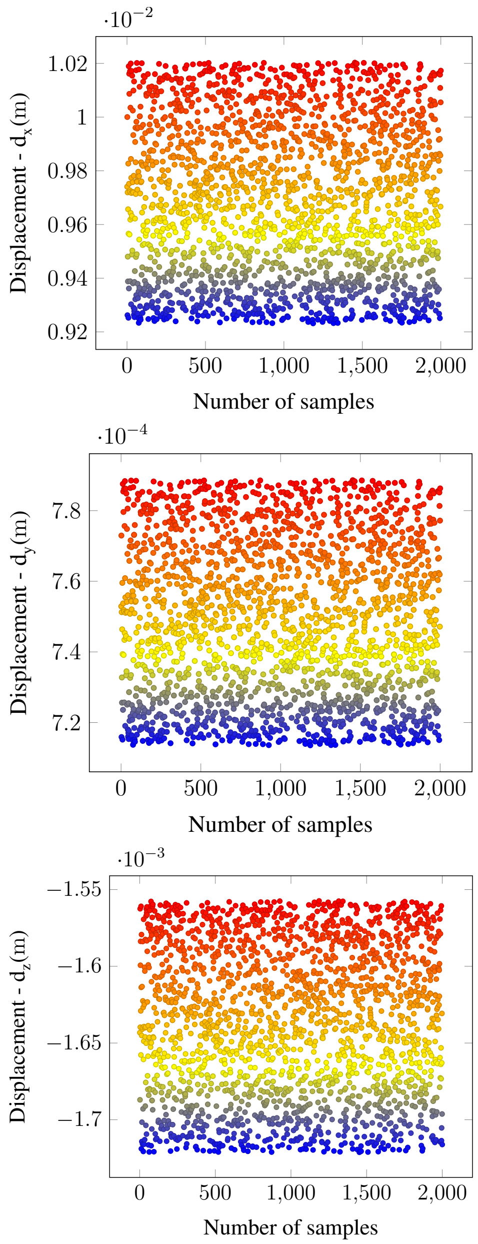

Figure 10: Simulations Monte Carlo method for truss with 60 bars.

ACKNOWLEDGEMENTS

The authors would like to thank CAPES for providing a research grant and for the Post-Graduation Program in Mechanical Engineering of the Federal University of Itajubá - UNIFEI for the opportunity to carry out the studies.

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