CILAMCE 2017
Florianópolis - Santa Catarina - Brazil

A CONTINUATION ALGORITHM-LONGMAN'S METHOD SCHEME FOR THE INTEGRATION OF TRANSCENDENTAL FUNCTIONS

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Abstract. *This paper presents a method for numerical integration of indefinite integrals with oscillatory-decaying integrands. Integrands with oscillatory-decaying behavior are at the core of harmonic analysis. Problems approached through Hankel transforms, which involve Bessel functions, will invariably result in an indefinite integral of this type to be solved. The integral of Bessel functions and other such integrands between two of their consecutive roots varies in signal and decreases in amplitude in a way that matches the requirements for which Longman's integration method was designed. The difficulty in applying Longman's method for the present case of Bessel function is finding these roots, since Bessel functions do not present a constant period of oscillation. The present integration scheme consists of locating the roots of the integrand, and integrating between these roots according to Longman's technique. For the first part, the roots are located through a homotopy continuation procedure. The integration between the roots, called "volumes", can be performed with ordinary Gaussian quadratures. Longman's method is then used to approximate the value of the integral from the integral of the volumes. The combination of continuation algorithm and Longman's method (CALM) is illustrated with functions with or without constant periods of oscillation. The method is then used to integrate the Green's function for a transversely isotropic half-space, which is a type of problem that commonly arises in the solution of dynamic soil-interaction problems with the boundary element method.*

Keywords: *Numerical integration, Continuation algorithms, Longman's method, Bessel functions*

1 INTRODUCTION

Models of dynamic soil-foundation interaction such as raft foundations and embedded pile groups often require the derivation of Greens functions corresponding to surface and embedded circular or cylindrical shell loads within transversely isotropic, layered half-spaces. This derivation can be approached through Hankel transforms (Rajapakse and Wang, 1993), which must be integrated numerically in order to obtain the corresponding displacements in the physical domain. Hankel transforms are characterized by indefinite integrals containing an infinite number of singularities and oscillating-decaying components whose integrals only converge at infinity (Labaki, Mesquita and Rajapakse, 2014; Barros, 2006). There are currently no numerical methods to deal with both characteristics of these integrands accurately (Lee et al., 1980; Mosig and Melcon, 2003; Hisada, 1994). The integrand of these Green's functions, however, can typically be divided into two regions: one characterized by the presence of singularities, and one in which only the oscillatory-decaying behavior remains. This work is concerned with the latter.

The oscillatory-decaying nature of the present integrands in the second region, and the fact that they converge only at infinity, make them difficult to integrate accurately with methods that require a truncation of the integration interval, such as Gaussian quadratures. One solution to this problem was proposed by Longman (1956). Longman's method is based on Euler's transformatin of slowly convergent alternating series, and can be applied to functions whose integral between two of its consecutive roots alternate in signal and decrease in amplitude to infinity (Mesquita, Labaki and Ferreira, 2009). Such is the case of Bessel functions. Mesquita, Labaki and Ferreira (2009) presented an efficient implementation of the Longman's method in general-purpose programmable graphics hardware (GPGPU). Their implementation, however, took as a representative example of the method an oscillating-decaying function whose roots were trivial to find. The difficulty in applying this technique for the present integration of Bessel function is finding these roots, since Bessel functions do not present a constant period of oscillation. It is, therefore, necessary to locate the roots of the integrand before Longman's method can be applied.

Root-finding numerical methods such as Newton-Raphson depend on the appropriate choice of a start point, which may at times be almost as difficult to locate as the roots themselves. These methods also require the function to be derived, which, albeit being possible to compute numerically, is computationally expensive and prone to errors. In this work, finding the roots of the integrand function is approached by the homotopy continuation method presented by Rahimian et al. (2011).

The present paper introduces a method for numerical integration of indefinite integrals with oscillatory-decaying integrands. The method is a combination of Ramihian et al.'s homotopy continuation algorithm to find the roots of the integrand, and Longman's integration method to extrapolate the result of the integration from the integral between the roots. The presented method is illustrated through the integration of representative functions with or without constant periods of oscillation. The method is then used to integrate the Green's function for the vertical time-harmonic displacement response of a transversely isotropic elastic half-space with an axisymmetric disc load on its surface.

2 STATEMENT OF THE PROBLEM

Consider the indefinite integral I of a function $f(x)$, where a is a constant and $f(x)$ oscillates around zero in such a manner that the integral over each half-cycle is smaller in absolute magnitude than (and opposite in sign to) that over the preceding half-cycle (Fig. 1).

$$I = \int_a^{\infty} f(x)dx \tag{1}$$

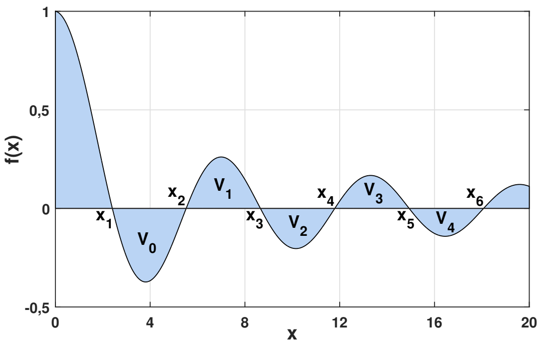

Figure 1: Curve of f(x) with illustration of the roots and volumes.

The function $f(x)$ may be such as

$$f(x) = e^{-\lambda x} cos(\omega x) \tag{2}$$

which has known roots x_k within the integration interval,

$$x_k = \frac{1}{\omega}\left[\frac{\pi}{2} + (k-1)\pi\right], k \in Z_+^* \tag{3}$$

or such as

$$f(x) = e^{-\lambda x} J_0(\beta x), \tag{4}$$

which has no closed-form expression for its roots (Abramowitz and Stegun, 1965).

This work proposes a numerical integration scheme to approximate I, regardless of the roots of the integrand being known or not.

3 INTEGRATION METHOD

The integration method proposed in this work comprises two parts: a Continuation Algorithm and Longman's Method (CALM).

3.1 Continuation Algorithm

The continuation algorithm used in this work to locate the roots of the integrand is the homotopy presented by Rahimian et al. (2011). The algorithm consists in replacing the problem of finding the roots of $f(x)$ by the problem of finding the roots of $H(x,t)$, written as the linear combination of $f(x)$ and $G(x) = (x - x^0)$, an arbitrary function for which a zero is known or readily obtained. That is,

$$H(x,t) = (x - x^0)[1 + f(x) - t] = 0, \tag{5}$$

where t, the homotopy parameter, allows tracking of a solution path that connects the arbitrary starting point, x^0, at $t = 0$, to all roots, which satisfy $f(x) = 0$. Using numerical continuation, the parameter t is gradually varied from 0 to 1, leading to a series of solutions to $H(x,t) = 0$. Whenever the homotopy path intersects $t = 1$, a solution to $f(x) = 0$ is found (Rahimian et al., 2011).

3.2 Longman's Method

Longman (1956) proposed a method to obtain I in Eq. (1) through an approximation in terms of volumes to be integrated independently:

$$I = \int_a^{\infty} f(x)dx = \int_a^{x_1} f(x)dx + \left(\frac{1}{2}V_0 - \frac{1}{4}\Delta V_0 + \frac{1}{8}\Delta^2 V_0 - ...\right), \tag{6}$$

in which

$$\Delta V_n = V_{n+1} - V_n, \quad \Delta^{r+1} V_n = \Delta^r V_{n+1} - \Delta^r V_n$$

and

$$V_n = \int_{x_n}^{x_{n+1}} f(x)dx, \tag{7}$$

The integral of $f(x)$ between two consecutive roots (Eq. 7) is called "volume" in this work. Equation 7 can be easily evaluated by classical Gaussian quadratures.

3.3 CALM Algorithm

The CALM scheme presented in this article can be summarized in three steps:

Step 1: Find the roots of $f(x)$ using the continuation algorithm (Eq. 5);
Step 2: Integrate the volumes between the roots (Eq. 7); and
Step 3: Extrapolate the value of I through Longman's method (Eq. 6).

3.4 Illustration of the method

Location of roots

Consider the function $f(x)$ given in Eq. (2), whose closed-form solution for the roots is given by Eq. 3. The continuation algorithm shown in Eq. 5 was used to locate the roots of $f(x)$ for the case in which $\lambda = 0.5$ and $\omega = 10$. The results are shown in Table 1 for the fifty first roots of the function.

Table 1: Accuracy results between analytic and numerical roots for Eq. (2)

n, k	x_n	x_k	Relative error
1	0,157079632679487	0,157079632679490	-1,6962E-14
2	0,471238898038468	0,471238898038469	-2,0025E-15
5	1,41371669411541	1,41371669411541	2,1989E-15
10	2,98451302091029	2,98451302091030	-4,4639E-15
15	4,55530934770522	4,55530934770520	4,2894E-15
50	15,5508836352782	15,5508836352695	5,6097E-14

CALM

Consider the indefinite integral of the same $f(x)$, from $a = 0$, whose closed-form solution is given by

$$I = \int_0^\infty e^{-\lambda x} cos(\omega x) dx = \frac{\lambda}{\lambda^2 + \omega^2}. \tag{8}$$

After finding the roots of $f(x)$ shown in Table 1, Eq. 7 is used to obtain the volumes of the function between the roots. Longman's method is used to obtain the value of the integral from these volumes. The results of integration of Eq. (8) by CALM are shown in Tab. 2 for selected values of λ and ω.

Table 2: Accuracy results for integration Eq. (8) by CALM.

λ	ω	I_{CALM}	I_{Exact}	Relative Error
0.5	10	0.00498753117206979	0.00498753117206983	-6.6084E-15
0.01	1	0.00999900009998955	0.00999900009999000	4.4933E-14
1	1	0.5000000000000160	0.5000000000000000	3.2418E-14
1	10	0.00990099009900991	0.00990099009900990	8.7603E-16

4 NUMERICAL RESULTS AND DISCUSSION

This section investigates the influence of parameters of CALM, such as number of integration volumes, in the accuracy of its integration. The studies are compared with the severity with which the integrand oscillates and/or decays. Then the method is applied to integrate a representative Green's function.

Figure 2 shows the influence of the number of integration volumes in the accuracy of integration of Eq. (8) by CALM. Figure 2a considers functions with the same decaying severity $\lambda = 0.5$ and different oscillatory frequencies ω. Conversely, Fig. 2b considers functions with the same oscillatory frequency $\omega = 100$ and different decaying severities λ.

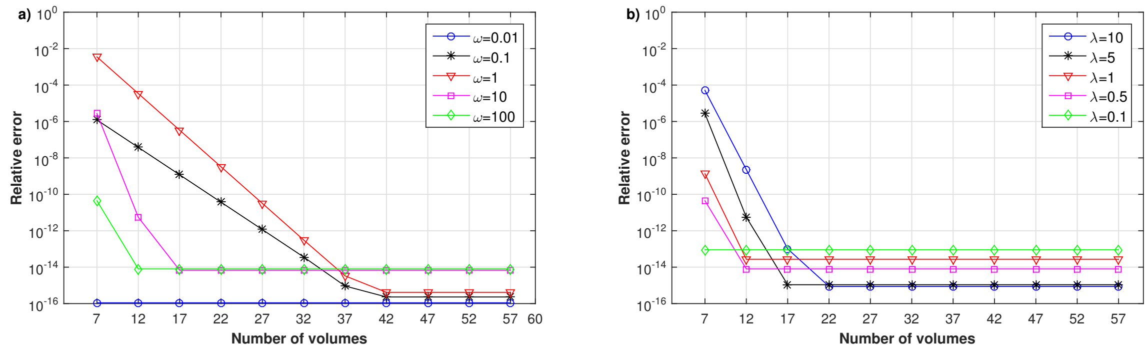

Figure 2: Accuracy of CALM in solving Eq. (8) for (a) decaying severity $\lambda = 0.5$ and different values of ω and (b) oscillatory severity $\omega = 100$ and different values of λ.

An analysis of Fig. 2 shows that the number of volumes required to obtain an integration error below a desired value depends on how severely the integrand oscillates and/or decays. For this set of data, the relation between oscillation/decay and number of volumes is not monotonic. However, for larger number of volumes ($N \geq 42$), an integration error below double-precision machine epsilon can be obtained for all oscillation and decay severities.

The performance of CALM in integrating transcendental functions is shown in Fig. 3. The integrals considered here are shown in Eqs. 9 and 10, whose analytical solution is known. The particularity about these integrands is that, in the case of Bessel functions, the severity of oscillation and decay cannot be controlled independently, but rather are tied together in the argument of the Bessel function. In the particular case of Eq. 9, an additional decay argument λ is incorporated in the exponential term.

$$I = \int_0^\infty e^{-\lambda x} J_0(\beta x) dx = \frac{1}{\sqrt{(\lambda^2 + \beta^2)}} \tag{9}$$

$$I = \int_0^\infty x(x^2 + 1)^{-3/2} J_0(\rho x) dx = e^{-\rho} \tag{10}$$

Figure 3 shows that CALM integrates Eqs. 9 and 10 with errors below double-precision machine epsilon with $N \geq 42$ and $N \geq 47$, respectively.

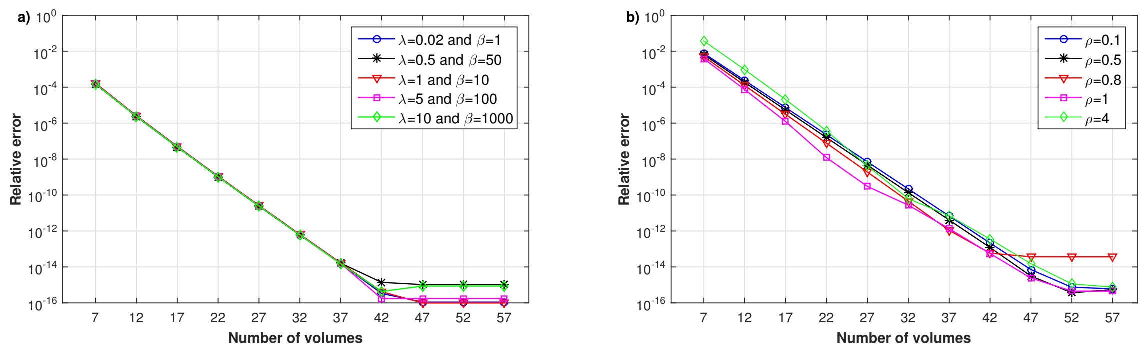

Figure 3: Accuracy of CALM for different integrand parameters in solving (a) Eq. (9) and (b) Eq. (10).

All results in this section show that CALM achieves integration errors below single - precision machine epsilon with $N \geq 27$ for all integrals considered.

4.1 Application: Green's function

Consider a three-dimensional, transversely isotropic half-space, on the surface of which a disc area of unit radius is under a unit, time-harmonic, uniformly distributed axisymmetric vertical load. A solution for the axisymmetric vertical displacement of a point of coordinates (r, z) within the half-space due to the surface load, u_{zz}, is given in terms of an indefinite integral arising from a Hankel transform (Rajapakse and Wang, 1993, see Appendix). The integrand of this Green's function, $u^*_{zz}(\zeta)$, contains singularities in the region $0 \leq \zeta < \zeta'$, and presents an oscillatory-decaying behavior for $\zeta > \zeta'$ that matches the requirements for which CALM was designed.

CALM was used to integrate $u^*_{zz}(\zeta > \zeta' = 1.2)$ for various excitation frequencies ω. The results are compared with the integration performed by adaptive Gaussian quadratures over a truncated integration interval (Piessens, Doncker-Kapenga and Überhuber, 1983). This study considers a half-space with Lamé constants $\mu = \lambda = 1$, mass density $\rho = 1$, and measuring point $(r, z) = (0, 1)$ (consistent units). The integration of the singular region, $u^*_{zz}(0 \leq \zeta < \zeta' = 1.2)$, has been addressed by Azevedo, Cavalcante and Labaki (2017) and is not shown here.

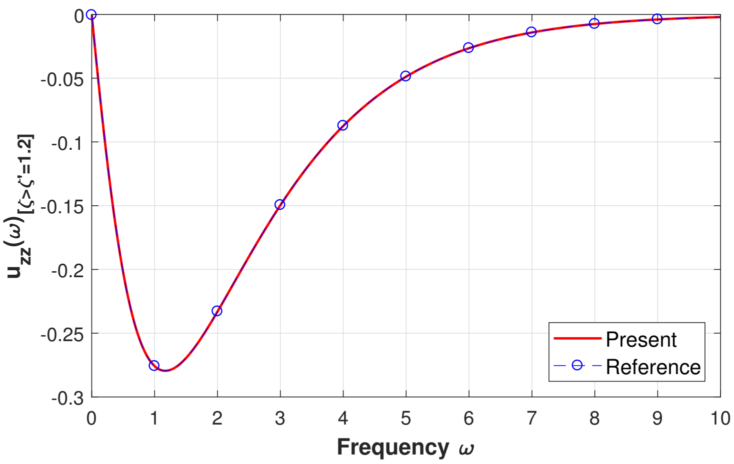

Figure 4: Displacement solutions for various excitation frequencies ω.

Figure 4 shows that the integration by CALM of the present Green's function in its oscillatory - decaying region agrees with the integration by the classical adaptive Gaussian quadrature for the entire frequency range considered.

5 CONCLUSIONS

This work proposed a numerical scheme for indefinite integration of oscillatory-decaying functions. The method is based on Longman's method, with the addition of a homotopy algorithm to find the roots of the integrand when no closed-form solution is known. The article shows that there is a robust set of parameters for the present integration scheme that guarantees the approximation of the integral to be within desirable limits. The method was used to integrate a Green's function that commonly arises in the solution of soil-foundation problems by the boundary element method, with good agreement with the integration performed by a classical adaptive Gaussian quadrature.

Appendix

This appendix describes the Green's function integrated in section 4.1. The vertical displacement of the medium described in section 4.1 is given by:

$$u_{zz} = \delta^2 \int_0^\infty u^*_{zz}(\zeta)\zeta d\zeta \tag{11}$$

where, for $i = 1, 2$,

$$u^*_{zz}(\zeta) = -a_7 A e^{-\delta\xi_1 z} - a_8 C e^{-\delta\xi_2 z}, \tag{12}$$

$$a_{7,8} = \delta\xi_{1,2}, \tag{13}$$

$$A = +\frac{b_{52}}{b_{21}b_{52} - b_{51}b_{22}}\frac{J_1(\zeta)}{\zeta\mu}, \tag{14}$$

$$C = -\frac{b_{51}}{b_{21}b_{52} - b_{51}b_{22}}\frac{J_1(\zeta)}{\zeta\mu}, \tag{15}$$

$$b_{2i} = [\alpha\delta^2\xi_i^2 - (\kappa - 1)\delta^2\zeta^2\vartheta_i], \tag{16}$$

$$b_{5i} = -(1 + \vartheta_i)\delta^2\xi_i, \tag{17}$$

in which J_1 is the Bessel function of the first kind and order 1, and

$$\vartheta_{1,2} = \frac{\alpha\xi_{1,2}^2 - \zeta^2 + 1}{\kappa\zeta^2},$$

$$\xi_{1,2} = \frac{1}{\sqrt{2\alpha}}(\gamma\zeta^2 - 1 - \alpha \pm \sqrt{\Phi})^{\frac{1}{2}},$$

$$\Phi = (\gamma\zeta^2 - 1 - \alpha)^2 - 4\alpha(\beta\zeta^4 - \beta\zeta^2 - \zeta^2 + 1),$$

$$\alpha = \beta = 2 + \frac{\lambda}{\mu}, \quad \kappa = 1 + \frac{\lambda}{\mu}, \quad \delta^2 = \frac{\rho\omega^2}{\mu} \quad \text{and} \quad \gamma = 1 + \alpha\beta - \kappa^2.$$

In Eq. (11), δ is a normalized frequency of excitation, and $\zeta = k/\delta$, in which k is the Hankel space variable. It is because of the normalization of the Hankel integration interval from k to ζ, which encapsulates all excitation frequencies into δ (Rajapakse and Wang, 1993), that it is safe to say that all the singularities in $u^*_{zz}(\zeta)$ fall within a predictable interval, after which only the oscillatory-decaying behavior of the integrand remains (see section 4.1). ζ' is found empirically to be slightly under 1.2.

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Acknowledgements

The research leading to this article has been funded in part by the São Paulo Research Foundation - Fapesp, through grant 2017/01450-0. The support of Capes, CNPq, and Faepex-Unicamp is also gratefully acknowledged.

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