DYNAMIC ANALYSIS OF STRUCTURES UNDER SENSITIVITY OF THE TIME STEPS BY THE BOUNDARY ELEMENT METHOD

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Abstract. *In this paper, a boundary element dynamic formulation is developed for the analysis of structures formed by two-dimensional association of plates. Initially, the boundary element formulations developed for plane elasticity isotropic problems and bending of isotropic thin plate problems are associated in one plane structure with four degrees of freedom per node, given by normal, tangential and transverse displacements and normal rotations. The time integration is carried out using the Houbolt method. The technique of transient analysis is used to determine the dynamic response of a structure under the loading how much the size of the time steps. The final system is obtained by assuming each individual plane structural element as a sub-domain. After the necessary transformation of these equations, they can be combined in order to take into account displacement compatibilities and traction equilibrium conditions. Numerical examples are presented and their results are compared to results available in literature.*

Keywords: *Boundary Element Method, DRM, Time Steps, Plates.*

1 INTRODUCTION

The Boundary Element Method (BEM) is an advanced numerical computational method developed for the solution of engineering and science problems. Its main characteristic is the boundary only discretization that reduces the modelling in one dimension. Because of this, when compared with the Finite Element Method (FEM) and Sladek (2006), the size of influence matrices is very reduced. However, different from FEM where only polynomials are integrated, in BEM, integration is not a trivial task. Some techniques have been proposed to integrate the influence terms of the BEM, especially when the integral is singular or near-singular. The aim of numerical techniques used to integrate functions is always to obtain a value that is very near the exact or analytical value of the integral. In fact, analytical integration has been purposed in literature for 2D problems.

The development of BEM for dynamic problems has been a subject of research since the 1970s. In the early 1980s, there was a major development in the BEM known as the dual reciprocity method (DRM) developed by Nardini (1982). The DRM has been used to solve potential, fluid mechanics and heat transfer problems, but there has been very few publications applying the method to dynamic problems. Among these publications are the work of Bridges (1994) where spline functions is used to model free vibration problems. Agnantiaris (1996) compared the behaviour of polynomial RBF against thin plate splines. Additional work for soil-structure interaction and fracture mechanics can be found in the textbook of Dominguez (1993).

This paper presents a dynamic formulation of the boundary element method to calculate the displacement of isotropic thin plates. The purpose is to reduce the processing time to solve plate bending problems as well as to avoid drawbacks that come from near singular integrals. The results obtained by numerical integration are compared to results obtained with the Sladek (2006).

2 Theory of bending of thin plates

A plate is a structural element defined by two flat parallel surfaces (Fig. 1), where loads are applied transversely. In this work, formulations will be developed for isotropic thin plates.

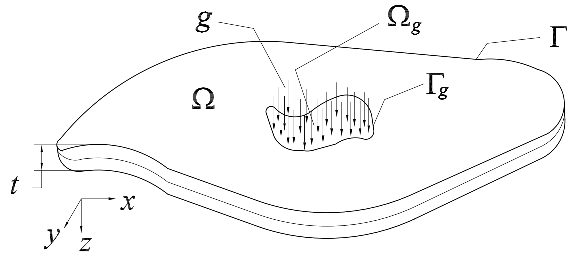

Figure 1: Thin plate.

3 Governing equation for deflection of plates

Consider equilibrium of an element $dx \times dy$ of the plate subject to a vertical distributed load of intensity $q(x, y)$ applied to an upper surface of the plate, as shown in Fig. 2.

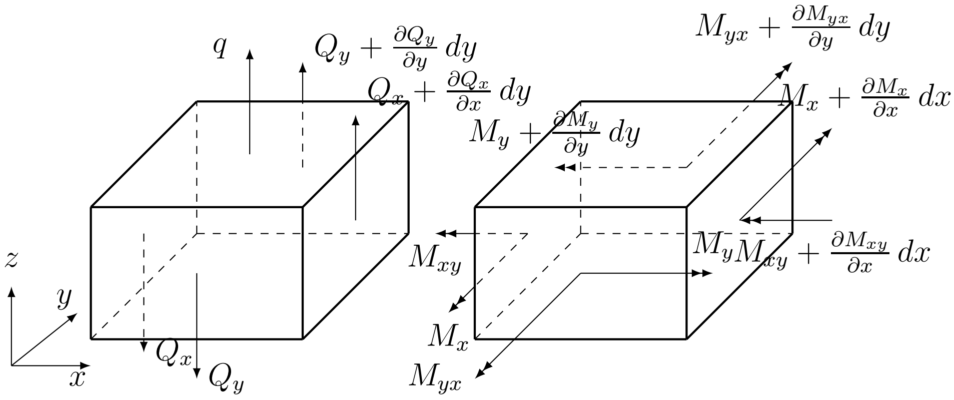

Figure 2: In-plane normal forces, bending moments, shear force, twisting moment and out of plane shearing forces.

$$
\begin{aligned}
Q_x &= \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}, \\
Q_y &= \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y}.
\end{aligned} \tag{1}
$$

4 Basic relations for isotropic plates

This is the *governing differential equation for the deflections* of thin plate bending analysis based on Kirchhoff assumptions. This equation was obtained by Lagrange (1811).

$$
\nabla^4(\,) = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}. \tag{2}
$$

is commonly called the *biharmonic operator*.

5 Boundary integral equation for bending problems of isotropic plates

The equilibrium Eq. 2 can be transformed into a boundary integral equation given by

$$
\begin{aligned}
Kw(Q) + \int_\Gamma \left[V_n^*(Q,P) w(P) - m_n^*(Q,P) \frac{\partial w(P)}{\partial n} \right] d\Gamma(P) \\
+ \sum_{i=1}^{N_c} R_{c_i}^*(Q,P) w_{c_i}(P) \\
= \sum_{i=1}^{N_c} R_{c_i}(P) w_{c_i}^*(Q,P) + \int_\Omega q(P) w^*(Q,P)\, d\Omega \\
+ \int_\Gamma \left[V_n(P) w^*(Q,P) - m_n(P) \frac{\partial w^*}{\partial n}(Q,P) \right] d\Gamma(P),
\end{aligned} \tag{3}
$$

where $\frac{\partial()}{\partial n}$ is the derivative in the direction of the outward vector $\mathbf{n}$ that is normal to the boundary Γ; m_n and V_n are, respectively, the normal bending moment and the Kirchhoff equivalent shear force on the boundary Γ; R_c is the thin-plate reaction of the corners; w_c is the transverse displacement of the corners; P is the field point; Q is the source point; and an asterisk denotes a fundamental solution. The constant K is introduced in order to take account of the possibilities that the point Q can be placed in the domain, on the boundary, or outside the domain. If the point Q is on a smooth boundary, then $K = 1/2$.

$$\begin{aligned}
&\frac{1}{2}\frac{\partial w(Q)}{\partial n_1} + \int_\Gamma \left[\frac{\partial V^*}{\partial n_1}(Q,P)w(P) - \frac{\partial m_n^*}{\partial n_1}(Q,P)\frac{\partial w}{\partial n}(P)\right] d\Gamma(P) \\
&+\sum_{i=1}^{N_c}\frac{\partial R_{c_i}^*}{\partial n_1}(Q,P)w_{c_i}(P) \\
&\qquad = \sum_{i=1}^{N_c} R_{c_i}(P)\frac{\partial w_{c_i}^*}{\partial n_1}(Q,P) + \int_\Omega q(P)\frac{\partial w^*}{\partial n_1}(Q,P)\,d\Omega \\
&\qquad + \int_\Gamma \left\{V_n(P)\frac{\partial w^*}{\partial n_1}(Q,P) - m_n(P)\frac{\partial}{\partial n_1}\left[\frac{\partial w^*}{\partial n}(Q,P)\right]\right\} d\Gamma(P), \qquad (4)
\end{aligned}$$

where $\frac{\partial()}{\partial n_1}$ is the derivative in the direction of the outward vector $\mathbf{n}_1$ normal to the boundary Γ at the source point Q.

6 Fundamental solutions for bending problems of isotropic plates

The fundamental solution for the transverse displacement in plate bending is computed by setting the non-homogeneous term of the differential Eq. 2 as equal to a concentrated force given by a Dirac delta function $\delta(Q, P)$), i.e.,

$$w^* = \frac{1}{8\pi D_{22}} r^2[\ln r], \qquad (5)$$

where $r^2 = [(x - x_0)^2 + (y - y_0)^2]$ is the distance between source and field point.

The boundary quantities $\frac{\partial w^*}{\partial n^*}$, m_n^* and v_n^* at a regular point on Γ are expressed in terms of the deflection w by the relations is given by:

$$\begin{aligned}
V_n^* &= \frac{1}{4\pi\rho}(\mathbf{nr})[-3+\nu+2(1-\nu)(\mathbf{sr})^2], && (6)\\
m_n^* &= -\frac{1}{8\pi}[2(\mathbf{nr})^2(1-\nu)+2\nu+(1+\nu)(1+2\ln r)], && (7)\\
\frac{\partial w^*}{\partial n} &= \frac{1}{8\pi D_{22}}[(\mathbf{nr})r(1+2\ln r)], && (8)\\
\frac{\partial w^*}{\partial m} &= -\frac{1}{8\pi D_{22}}[(\mathbf{nr})r(1+2\ln r], && (9)\\
\frac{\partial v_n^*}{\partial m} &= -\frac{1}{4\pi r^2}[(-\mathbf{mn}+2(\mathbf{nr})(\mathbf{nr}))(3-\nu)+ \\
&\quad 2(1-\nu)\mathbf{sr}(2\mathbf{ms}(\mathbf{nr})+\mathbf{mn}(\mathbf{sr})-4(\mathbf{nr})(\mathbf{nr})(\mathbf{sr}))], && (10)
\end{aligned}$$

$$\frac{\partial m_n^*}{\partial m} = \frac{1}{4\pi r}[-2(\mathbf{nr})(-\mathbf{mn}+\mathbf{nr}(\mathbf{nr}))(1-\nu)+\mathbf{nr}(1+\nu)], \quad (11)$$

$$\frac{\partial^2 w^*}{\partial n \partial m} = -\frac{1}{4\pi D_{22}}[\mathbf{nr}(\mathbf{nr})+(\mathbf{mn})\ln r]. \quad (12)$$

In Equations above, the following relations were used:

$$\mathbf{nr} = d/r \quad (13)$$

$$\mathbf{sr} = d\tan(\theta)/r \quad (14)$$

$$\mathbf{nr} = m_1'\cos(\theta) + m_2'\sin(\theta) \quad (15)$$

where m_1' and m_2' are the coordinates of the normal vector at source point at local system (x', y').

7 Transformation of domain integrals into boundary integrals in the isotropic plate bending problem

As can be seen in Eqs. (3) and (4), there are domain integrals in the formulation due to the distributed load in domain. These integrals can be computed for:

the domain by direct integration over the area Ω_g (see Fig. 1). However, the boundary element formulation loses its main feature, i.e., the boundary only discretization. In this work, domain integrals which arise from distributed loads are transformed into boundary integrals by Dual Reciprocity Method.

Consider that the term b_i of the Equation (3) is approached throughout the domain as a set of unknown coefientes α_i^m and a class of functions denoted by $f^m(x)$. The approximated $b_i(x)$ is given by

$$b_i(x) = \sum_{m=1}^{M} \alpha_i^m f^m(x) \quad (16)$$

Thus, the integral over the area in Eq. (3) can be written as

$$\int_\Omega b_i U_{ik} d\Omega = \sum_{m=1}^{M} \alpha_i^m \int_\Omega U_{ik} f^m d\Omega \quad (17)$$

Particular solutions $\hat{u}^j$ can be found in order to satisfy equilibrium equations

$$C_{mnrs}\hat{u}^j_{rk,ns} = f^j_{mk} \quad (18)$$

The reciprocal relationship between fundamental solution and particular solution can be written as

$$\hat{u}^m_{ln} + \int_\Gamma T_{li}\hat{u}^m_{in} d\Gamma = -\int_\Omega f^m_{in} U_{li} d\Omega + \int_\Gamma U_{li}\hat{t}^m_{in} d\Gamma \quad (19)$$

Substituting Eq. (19) in Eq. (17), gives

$$\int_\Omega U_{li} p_i d\Omega = \sum_{m=1}^{M} \alpha_n^m \left\{ -\hat{u}^m_{ln} + \int_\Gamma \hat{t}^m_{in} U_{li} d\Gamma - \int_\Gamma T_{li}\hat{u}^m_{in} d\Gamma \right\} \quad (20)$$

Now, substituting Eq. (20) in the Eq. (3), you have

$$u_l + \int_\Gamma T_{li} u_i d\Gamma = \int_\Gamma U_{li} t_i d\Gamma + \sum_{m=1}^{M} \alpha_n^m \left\{ \hat{u}_{ln}^m - \int_\Gamma \hat{t}_{in}^m U_{li} d\Gamma + \int_\Gamma T_{li} \hat{u}_{in}^m d\Gamma \right\} \tag{21}$$

Equation (21) will be used as basic equation for the dual reciprocity boundary element. It may be noted the absence of domain integrals.

In order to obtain a solution for the elastic-dynamic problem, boundary is divided into elements. Thus:

$$\begin{aligned} \mathbf{u} &= \phi \mathbf{u}^{(i)} \\ \mathbf{t} &= \phi \mathbf{t}^{(i)} \\ \hat{\mathbf{u}} &= \phi \hat{\mathbf{u}}^{(i)} \\ \hat{\mathbf{t}} &= \phi \hat{\mathbf{t}}^{(i)} \end{aligned} \tag{22}$$

where variables represented in bold letters are vectors of size $2N$ where N is the number of nodes, $\mathbf{u}^{(i)}$ and $\mathbf{t}^{(i)}$ are displacements and tractions nodal values, respectively, ϕ is the shape function matrix, $\mathbf{u}$ and $\mathbf{t}$ represent displacements and stresses along the element and $\hat{\mathbf{u}}^{(i)}$ and $\hat{\mathbf{t}}^{(i)}$ are vectors and nodal displacements and tractions, particular solutions respectively. Thus:

$$\begin{aligned} &\mathbf{c}^l \mathbf{u}^l + \sum_{j=1}^{NE} \left\{ \fint_{\Gamma_j} \mathbf{T} \phi d\Gamma \right\} \mathbf{u}^l = \sum_{j=1}^{NE} \left\{ \int_{\Gamma_j} \mathbf{U} \phi d\Omega \right\} \mathbf{t}^j + \\ &\sum_{m=1}^{M} \left[\mathbf{c}^l \hat{\mathbf{u}}^l - \sum_{j=1}^{NE} \left\{ \int_{\Gamma_j} \mathbf{U} \phi d\Gamma \right\} \hat{\mathbf{T}}^{jm} + \sum_{j=1}^{NE} \left\{ \fint_{\Gamma_j} \mathbf{T} \phi d\Gamma \right\} \hat{\mathbf{U}}^{jm} \right] \alpha^m \end{aligned} \tag{23}$$

considering that,

$$\int_\Gamma \mathbf{U} \phi d\Gamma = \mathbf{G} \tag{24}$$

and

$$\fint_{\Gamma_j} \mathbf{T} \phi d\Gamma = \mathbf{H} \tag{25}$$

it is possible towrite Eq. 23 as:

$$\sum_{j=1}^{N} H^{lj} u^j = \sum_{j=1}^{N} G^{lj} t^j + \sum_{m=1}^{M} \sum_{j=1}^{N} \left(H^{lj} \hat{u}^{jm} - G^{lj} \hat{t}^{jm} \right) \alpha^m \tag{26}$$

The M vectors α $(2\text{x}1)$ corresponding to the different approximatting functions f^m, can be grouped into a vector α $(2M\text{x}1)$ and related to the body forces vector$\mathbf{p}^m$ $(2M\text{x}1)$ containing body forces for all we outline. Thus, equation (7) can be written as

$$\mathbf{b} = \mathbf{F} \alpha \tag{27}$$

where elements of matrix $\mathbf{F}$ are values of functions $f^m(x)$ at all nodes. Reversing the matrix $\mathbf{F}$

can be written

$$\alpha = \mathbf{E}\mathbf{p} \tag{28}$$

where $\mathbf{E} = \mathbf{F^{-1}}$.

Substituting equation (28) in the equation (26), you can write

$$\mathbf{H}\mathbf{u} = \mathbf{G}\mathbf{t} + \left[\mathbf{H}\hat{\mathbf{U}} - \mathbf{G}\hat{\mathbf{T}}\right]\mathbf{E}\mathbf{p} \tag{29}$$

and maning

$$\mathbf{M} = \left[\mathbf{G}\hat{\mathbf{T}} - \mathbf{H}\hat{\mathbf{U}}\right]\mathbf{E} \tag{30}$$

Equation (29) becomes

$$\mathbf{M}\mathbf{p} + \mathbf{H}\mathbf{u} = \mathbf{G}\mathbf{t} \tag{31}$$

which has the same aspect of the equilibrium equations of finite elements, namely, a mass matrix $\mathbf{M}$ a stiffness matrix $\mathbf{H}$ and force vector $\mathbf{G}\mathbf{t}$.

8 Transient Problems: Formulation in the time domain

That this the method is the suitable for use in the direct time integration along with the radial integration reciprocity boundary element method.

8.1 Matrix equations

Seeing that in this case, the only body forces present in body domain Ω and contour Γ are due to the acceleration field,

$$b_i = \rho \ddot{u}_i \tag{32}$$

may be written to a time step $\tau + \Delta\tau$ as

$$\rho\mathbf{M}\ddot{\mathbf{u}}_{\tau+\Delta\tau} + \mathbf{H}\mathbf{u}_{\tau+\Delta\tau} = \mathbf{G}\mathbf{t}_{\tau+\Delta\tau} \tag{33}$$

To carry out time integration for a period $\mathcal{T}$, this time period is divided into N equal intervals $\Delta\tau$ ($\mathcal{T} = N\Delta\tau$). Assuming that the solution of equation is known $0, \Delta\tau, 2\Delta\tau, ..., \tau$ solution at time $\tau + \Delta\tau$ is found according to the approach proposed Houbolt (1950). The acceleration in $\tau + \Delta\tau$ is approximated by the expression of finite difference

$$\ddot{\mathbf{u}}_{\tau+\Delta\tau} = \frac{1}{\Delta\tau^2}\left(2\mathbf{u}_{\tau+\Delta\tau} - 5\mathbf{u}_{\tau} + 4\mathbf{u}_{\tau-\Delta\tau} - \mathbf{u}_{\tau-2\Delta\tau}\right) \tag{34}$$

An important feature of Houbolt method is that it has a high numerical damping.

Substituting equation (34) in the equation (33), you have

$$\left[\frac{2}{\Delta\tau^2}\rho\mathbf{M} + \mathbf{H}\right]\mathbf{u}_{\tau+\Delta\tau} = \mathbf{G}\mathbf{t}_{\tau+\Delta\tau} + \frac{1}{\Delta\tau^2}\rho\mathbf{M}\left(5\mathbf{u}_{\tau} - 4\mathbf{u}_{\tau-\Delta\tau} + \mathbf{u}_{\tau-2\Delta\tau}\right) \tag{35}$$

Since all unknown variables are passed to the left side, equation (35) can be written as

$$\mathbf{A}\mathbf{y}_{\tau+\Delta\tau} = \mathbf{w}_{\tau+\Delta\tau} \tag{36}$$

where $\mathbf{y}_{\tau+\Delta\tau}$ is the unknown variable vector and $\mathbf{w}_{\tau+\Delta\tau}$ is the known variables vector, and its elements are calculated from the values of $\mathbf{u}$ from the earlier times steps, and boundary conditions in time $\tau + \Delta\tau$.

9 Numerical results

The example chosen to evaluate the Boundary Element Method (BEM) performance in the sense of thin-shaped structures, as well as to analyse the effect of the size of the time step, is the elastodynamic problem. The problem corresponds to a plate under uniform load applied as a step function at time $t = 0,0127\,\mathrm{s}$, as defined in Figure 3. Dimensions of the plate are: length L = 1000 mm, thickness t = 8 mm. The Material properties are: elastic modulus $E = 6,895,\mathrm{GPa}$, Poisson coefficient $\nu = 0,3$ and density $\rho = 7,166\,10^4\,\mathrm{N/m^3}$. The plate is under a uniformly distributed load $q = 2,07\,10^6\,\mathrm{N/m^2}$.

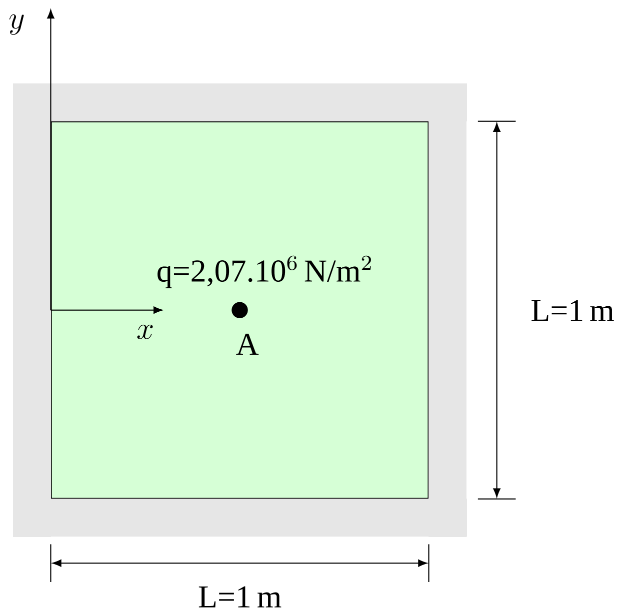

Figure 3: Clamped square plate

The plate was discretized in 12 constant boundary elements. In this paper, we are interested in evaluating how the solution for vertical displacement obtained with BEM varies with the time step used on the numerical integration and the Radial Integration Method. The result can be seen in the in Figure 4.

Table 1: Results obtained for clamped plate.

Time Step Number	Time Step ($\Delta\tau$)	Displacement
70	2,3194.10^{-4}	2,0757
90	1,8040.10^{-4}	2,1220
110	1,4760.10^{-4}	2,1344
130	1.2489.10^{-4}	2,1387

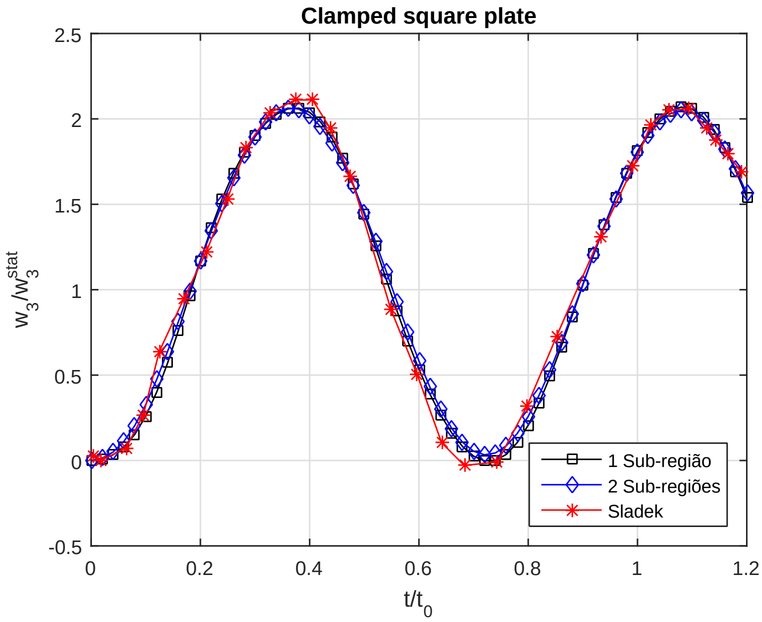

Figure 4: Variation in the displacement time at the center point of the clamped square plate subjected to a suddenly applied load.

10 Conclusions

This paper presents numerical expressions for influence matrices and dynamic analysis using the radial integration method and direct method elements for isotropic thin plates. The technique of transient analysis is used to determine the dynamic response of a structure under the loading how much the size of the time steps. The proposed formulation has been applied to a thin plate subjected to uniformly distributed loads. It is concluded that the damping has little influence on the displacement calculation. It used the RIM method to perform the transformation from domain to boudary integral. It was observed that the use of constant boundary elements has shown a good agreement when compared with Sladek results. Indeed, the use of radial integration method shows better results for this application, despite the greater computational cost.

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