



## Fast BEM multidomain approach for short fiber composites

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**Abstract.** Composite materials are usually treated as homogeneous when carrying out structural design. However, failure in these materials often originated at their heterogeneous microstructure or constituents; hence, the different materials should be considered in the analysis. The use of composite materials has increased considerably over the years due to their relative superior properties. The accurate determination of their mechanical properties and behavior is thus of great practical significance. The Boundary Element Method (BEM) has demonstrated to be a powerful computational technique for the analysis of many physical and engineering problems. The present work deals with the use of the multi-domain BEM to obtain a more appropriate characterization of fiber-matrix composites. The generally anisotropic fundamental solution based on a double-Fourier series is employed together with a fast BEM approach, namely, the Adaptive Cross Approximation (ACA) technique. The ACA technique is aimed at speeding up the process required to generate the BEM matrices. Some numerical examples are presented to demonstrate its applicability. The present work is a precursor to treating more complex problems involving anisotropic inclusions in general composites.

**Keywords:** Boundary Element Method, Composites, Anisotropy, Fourier series, Adaptive Cross Approximation.

## 1 INTRODUCTION

The use of composite materials in engineering has increased due to their relative superior properties such as lightness, flexibility, durability, adaptability or mechanical resistance. Hence, accurately predicting the behavior and determining its mechanical properties is of great importance. Fiber-matrix composites are one of the most common types of composites due to its particular composition and properties. For example, the addition of high strength fibers to a polymer matrix can greatly improve some of its mechanical properties such as the ultimate tensile strength, impact resistance, temperature resistance, among others.

The BEM is a well established and powerful computational technique for the analysis of many physical and engineering problems. Its implementation for elastostatics involves the evaluation of the displacement fundamental solution and its derivatives. For the case of isotropic materials it can be represented by simple explicit forms. However, the corresponding expressions for the case of generally anisotropic solids are significantly more complex and mathematically involved.

The anisotropic fundamental solution for displacements was first presented by Lifshitz and Rozenzweig (1947), but not as a closed form. It was expressed as a contour integral around a unit circle on a oblique plane at the field point and in terms of the Christoffel tensor. The first implementation of this solution into a 3D BEM code was by Wilson and Cruse (1978). There were significant challenges in the development of efficient algorithms for the accurate numerical evaluation of this quantity and its derivatives, see, e.g., Sales and Gray (1998), Tonon et al. (2001), Phan et al. (2004) and Wang and Denda (2007). Explicit, closed form expressions of the Green's function and its derivatives were derived by Ting and Lee (1997) and Lee (2003) in terms of the Stroh's eigenvalues. However, they were left unnoticed by the BEM community until Tavares et al. (2008) implemented it in BEM for the special case of transverse isotropy, and Shiah et al. (2008) and Tan et al. (2009) for general anisotropy. Taking advantage of the periodic nature of Ting and Lee (1997) Green's function, when expressed in spherical coordinates, Shiah et al. (2012), reformulated this fundamental solution and its derivatives by representing them by double Fourier series. Recently, Tan et al. (2013) further implemented it into a 3D BEM code. This development reduced significantly the computational effort for the evaluation of these quantities.

The final matrix system in BEM analysis is generally fully populated and unsymmetric. For very large numerical problems, such as when solving multi-scale problems, the memory requirements and solution times may be less than desirable. The issue of speeding up the solution process of large system of equations has been addressed by Bebendorf (2000); Bebendorf and Rjasanow (2003); Rokhlin (1985). As stated in Bebendorf and Rjasanow (2003), it is possible to use purely algebraic algorithms to generate the approximation of suitable blocks of the collocation matrix, using only relatively few entries of the original blocks. This technique is referred to as the Adaptive Cross Approximation (ACA). The ACA uses matrix hierarchization to reduce the storage requirement and the computational complexity arising in the numerical analysis.

In this paper, the application of hierarchical matrices and ACA to short fiber-matrix problems using the 3D multi-domain BEM for elastostatics with the anisotropic fundamental solution based on double Fourier series is illustrated. First, this anisotropic fundamental solution and its derivatives are reviewed. Then, the use of hierarchical matrices and ACA is discussed.

This will be followed by some numerical examples that will show the applicability of the proposed methodology.

## 2 3D anisotropic fundamental solutions and its derivatives

The displacement fundamental solution or Green's function,  $\mathbf{U}(\mathbf{x}) = U_{ij}(P, Q)$ , is defined as the displacement response in the  $x_i$  direction at the field point  $Q$  due to a unit load applied in the  $x_j$  direction at the source point  $P$  in an homogeneous infinite body. This solution for the case of 3D generally anisotropic materials can be written as, Lifshitz and Rozenzweig (1947),

$$U_{ij} = \frac{1}{8\pi^2 r} \int_0^{2\pi} \mathbf{Z}^{-1} d\psi, \quad (1)$$

where  $r$  is the distance between the source and the field point. The integrand  $\mathbf{Z}^{-1}$  is the inverse of the matrix  $\mathbf{Z}$ , that can be expressed in terms of the elastic stiffness tensor of the anisotropic material  $C_{ijks} \equiv \mathbf{C}$ , Shiah et al. (2012). Ting and Lee (1997) showed that the Green's function can be also expressed in terms of the Barnett-Lothe tensor,  $\mathbf{H}[\mathbf{x}]$ . Considering a spherical coordinate system, the explicit form of the Green's function can be expressed as follows,

$$\mathbf{U}(r, \theta, \phi) = \frac{1}{4\pi r} \mathbf{H}(\theta, \phi), \quad (2)$$

This expression depends only on the spherical angles  $(\theta, \phi)$  and can be further expressed in terms of the Stroh's eigenvalues as

$$\mathbf{H}(\theta, \phi) = \frac{1}{|\mathbf{T}|} \sum_{n=0}^4 q_n \hat{\Gamma}^{(n)}, \quad (3)$$

where  $\hat{\Gamma}$  is the adjoint of  $\Gamma$ . The Stroh's eigenvalues are obtained solving a sextic equation in  $p$ , which is obtained from setting the determinant  $|\Gamma(p)|$  to zero. The explicit expressions for  $|\mathbf{T}|$ ,  $\Gamma$  and  $q_n$  can be found in Ting and Lee (1997) and Tan et al. (2013). Moreover, due to its periodic nature,  $\mathbf{H}(\theta, \phi)$  can be represented by double Fourier series around  $\theta$  and  $\phi$ , as follows,

$$\begin{aligned} H_{uv}(\theta, \phi) &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \lambda_{uv}^{(m,n)} e^{i(m\theta+n\phi)}, \quad (u, v = 1, 2, 3), \\ \lambda_{uv}^{(m,n)} &= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} H_{uv}(\theta, \phi) e^{-i(m\theta+n\phi)} d\theta d\phi, \end{aligned} \quad (4)$$

where  $\lambda_{uv}^{(m,n)}$  are the Fourier coefficients, which can be numerically integrated by, e.g., Gaussian quadrature. The  $k$  abscissa points,  $\lambda_{uv}^{(m,n)}$ , may be re-written as

$$\lambda_{uv}^{(m,n)} = \frac{1}{4} \sum_{p=1}^k \sum_{q=1}^k w_p w_q f_{uv}^{(m,n)}(\pi \xi_p, \pi \xi_q), \quad (5)$$

where  $w$  and  $\xi$  are the weights and Gauss points, respectively and  $f_{uv}^{(m,n)}(\theta, \phi)$  represents the integrand of  $\lambda_{uv}^{(m,n)}$ . In short, the fundamental displacement solution can also be written as

$$U_{uv}(r, \theta, \phi) = \frac{1}{4\pi r} \sum_{m=-\alpha}^{\alpha} \sum_{n=-\alpha}^{\alpha} \lambda_{uv}^{(m,n)} e^{i(m\theta+n\phi)}, \quad (6)$$

where  $\alpha$  is an integer number, large enough to yield the desired accuracy. Numerical experiments by Tan et al. (2013) have shown that values of  $k = 64$  and  $\alpha = 16$  will be adequate to evaluate even the most highly anisotropic materials. Lee (2009), using partial differentiations of the Green's function in spherical coordinates, eliminated the necessity of working with high order tensors, as those seen in Ting and Lee (1997) and Lee (2003). The displacement derivatives can be obtained by working in spherical coordinates and applying the chain rule. All explicit forms can be observed in Shiah et al. (2010, 2012) and Tan et al. (2013). The most significant advantage of using this Fourier series representation of the Green's function and its derivatives is that the Fourier series coefficients,  $\lambda_{uv}^{(m,n)}$ , are only evaluated once. This further reduces the computational efforts, besides the relatively simplicity of the implementation into a BEM code.

### 3 Acceleration scheme for the BEM

The proposed acceleration scheme has as its main objective the reduction of the memory requirements as well as to speed up the total solution time. The scheme takes advantage of the matrix hierarchization and the use of low rank approximations by the use of the Adaptive Cross Approximation (ACA) technique. The matrix is represented as a collection of blocks, some of which admit a particular approximated representation that can be obtained by computing only few entries from the original blocks. These special blocks are called admissible or low rank blocks and are obtained by the ACA. Blocks that cannot be represented in this way must be computed and stored entirely (Benedetti et al., 2009, 2011). The existence of low rank approximants is based on the asymptotic smoothness of the kernel functions, Bebendorf (2000); Bebendorf and Rjasanow (2003) and Grasedyck (2005), e.g, on the fact that kernels  $U_{ij}$  and  $T_{ij}$  are singular only when the source and field points are coincident. This fact represents a sufficient condition for the existence of low rank approximants. A low rank block  $M$  of size  $m \times n$  has the following representation

$$M_k = \sum_{i=1}^k a_i \cdot b_i^T = A \cdot B^T, \quad (7)$$

where  $A$  and  $B$  are matrices of size  $m \times k$  and  $n \times k$ , respectively. If  $k$  is low, the representation shown above requires the storage of  $(m+n)k$  real numbers instead of  $m \times n$ , speeding up the

matrix-vector product of the corresponding block. For further reading about memory savings and the speed-up allowed refer to Borm et al. (2003) and Grasedyck and Hackbusch (2003).

Large dense matrices with diagonal singularity can be hierarchical approximated following the three basic steps (Kurz et al., 2007):

1. construction of clusters,
2. finding possible admissible blocks and,
3. low rank approximation of admissible blocks.

Kurz et al. (2007) adapted the ACA algorithm proposed by Bebendorf (2000) and applied it with BEM. An implementation of this algorithm in conjunction with the double Fourier fundamental solutions is presented in this work. First, the mass and centre of each cluster are stored, then, the covariance matrix of the cluster is obtained by

$$C = \sum_{k=1}^n g_k (x_k - X) (x_k - X)^T, \quad (8)$$

where  $n$  is the number of elements of the cluster,  $g_k$  is the element area (length for 2D problems) and  $X$  is the centre of the cluster.

Then, the largest eigenvalue of  $C$  defines the eigenvector,  $v_1$ , which gives the direction of the longest extension of the cluster. The separation plane (line in 2D)  $\{\mathbf{x} \in \mathbb{R}^3 : (\mathbf{x} - X, v_1) = 0\}$  goes through the centre  $X$  of the cluster and is orthogonal to  $v_1$ . The algorithm is applied recursively to the sons until they contain less than or equal to some prescribed number  $n_{min}$  of elements. Next, cluster pairs which are geometrically well separated are identified. They will be regarded as admissible cluster pairs. An appropriate admissibility criterion is the following simple geometrical condition. A pair of clusters  $(Cl_x, Cl_y)$  with  $n_x > n_{min}$  and  $m_y > n_{min}$  elements is admissible if

$$\min(\text{diam}(Cl_x), \text{diam}(Cl_y)) \leq \eta \text{dist}(Cl_x, Cl_y), \quad (9)$$

where  $\text{diam}(Cl_x)$  is the diameter of the circumscribed sphere evolving the cluster  $Cl_x$ ,  $\text{dist}(Cl_x, Cl_y)$  is the distance between clusters and  $\eta$  is called the admissibility parameter. This parameter influences the number of admissible blocks on one hand and the convergence speed of the adaptive approximation of low rank blocks on the other hand (Borm et al., 2003). A full study of this parameter was assessed by Benedetti et al. (2011) and Rodríguez et al. (2013). They showed that the choice of  $\eta$  directly affects the quality of the ACA-generated matrix and a good choice of this parameter results in a matrix closer to the optimal matrix produced by the coarsening procedure. This fact was also justified by the reduction in the number of blocks. Once the clusters were defined and all admissible blocks were detected, the Adaptive Cross Approximation (ACA) was used to approximate these blocks by low rank.

## 4 Multi-domain applied to 3D fiber-matrix problems

The multi-domain BEM approach is a natural choice when solving fiber-matrix problems, due to the possibility of analysing each region individually. After the BEM is applied in each

region, or sub-domain, the system coupling can be accounted by considering displacement compatibility and traction equilibrium of the interfaces. Further details on the multi-domain approach can be found in Kane (1994) and Aliabadi (2002). As an illustration of the current implementation, consider the simple example with four fibers shown in Figure 1.

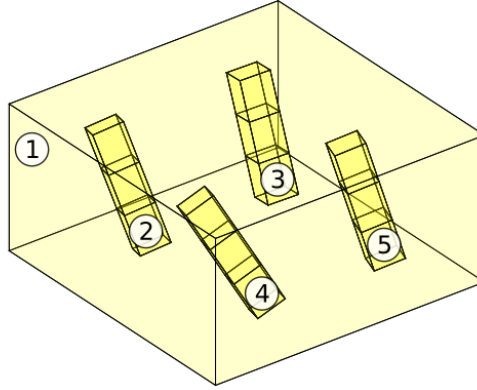


Figure 1: Multi-domain example with four fibers.

Five regions were created from the above example; the matrix and the four fibers. The multi-domain collocation matrix,  $[H_{coll}]$ , generated in this example is shown below.

$$\begin{bmatrix} [H_{11}]^1 & [H_{12}]^1 & [H_{13}]^1 & [H_{14}]^1 & [H_{15}]^1 & \cdots & -[G_{15}]^1 & [0] & [0] & [0] & [0] \\ [0] & [H_{12}]^2 & [0] & [0] & [0] & \cdots & [0] & -[G_{25}]^2 & [0] & [0] & [0] \\ [0] & [0] & [H_{13}]^3 & [0] & [0] & \cdots & [0] & [0] & -[G_{35}]^3 & [0] & [0] \\ [0] & [0] & [0] & [H_{14}]^4 & [0] & \cdots & [0] & [0] & [0] & -[G_{45}]^4 & [0] \\ [0] & [0] & [0] & [0] & [H_{15}]^5 & \cdots & [G_{15}]^5 & [G_{25}]^5 & [G_{35}]^5 & [G_{45}]^5 & [H_{55}]^5 \end{bmatrix} \quad (10)$$

where  $[H_{ij}]^e$  and  $[G_{ij}]^e$  represent the incidences in matrices  $[H]$  and  $[G]$  between sub-regions  $i$  and  $j$  in the case where quantities are associated to a specific region  $e$ . The order of the sub-region incidences are determined by first simply listing all the permutations, Kane (1994), for this example,

$$11 \ 12 \ 13 \ 14 \ 15 \ 21^* \ 22 \ 23 \ 24 \ 25 \ \cdots \ 51^* \ 52^* \ 53^* \ 54^* \ 55 \quad (11)$$

where the digits represent the subregions being analysed. For the permutation where the first digit is higher than the second digit (\*), blocks of traction components,  $[G_{ij}]^e$ , are created; otherwise, blocks of displacement components,  $[H_{ij}]^e$ , are created. The negative sign is due to the traction having opposite signs when analysing different regions at the interface. Moreover, as there is no contact between fibers, matrices with both indices  $i$  and  $j$  corresponding to fibers are equal to zero. Hence, the system can be further compressed as,

$$[H_{coll}] = \begin{bmatrix} [A], & [B], & [C] \end{bmatrix}, \quad (12)$$

where,

$$[A] = \begin{bmatrix} [H_{11}]^1 & [H_{12}]^1 & [H_{13}]^1 & [H_{14}]^1 & [H_{15}]^1 \\ [0] & [H_{12}]^2 & [0] & [0] & [0] \\ [0] & [0] & [H_{13}]^3 & [0] & [0] \\ [0] & [0] & [0] & [H_{14}]^4 & [0] \\ [0] & [0] & [0] & [0] & [H_{15}]^5 \end{bmatrix}, \quad (13)$$

$$[B] = \begin{bmatrix} [0] & [0] & [0] & [0] \\ [H_{22}]^2 & [0] & [0] & [0] \\ [0] & [H_{33}]^3 & [0] & [0] \\ [0] & [0] & [H_{44}]^4 & [0] \\ [0] & [0] & [0] & [H_{55}]^5 \end{bmatrix}, \quad (14)$$

$$[C] = \begin{bmatrix} -[G_{12}]^1 & -[G_{13}]^1 & -[G_{14}]^1 & -[G_{15}]^1 \\ [G_{12}]^2 & [0] & [0] & [0] \\ [0] & [G_{13}]^3 & [0] & [0] \\ [0] & [0] & [G_{14}]^4 & [0] \\ [0] & [0] & [0] & [G_{15}]^5 \end{bmatrix}. \quad (15)$$

The sub-matrices,  $[H_{ij}]^e$  and  $[G_{ij}]^e$ , are obtained applying the BEM to each  $e$  region individually. The Adaptive Cross Approximation (ACA) technique can also be used to this end. It is worth to point out that, usually, the fibers have a very small number of elements and consequently the ACA would not present any advantage on the approximation. Thus, the ACA will be applied to the region which contains the matrix medium. For regular multi-domain problems no compression can be made and the system of Eq. (10) should be used instead.

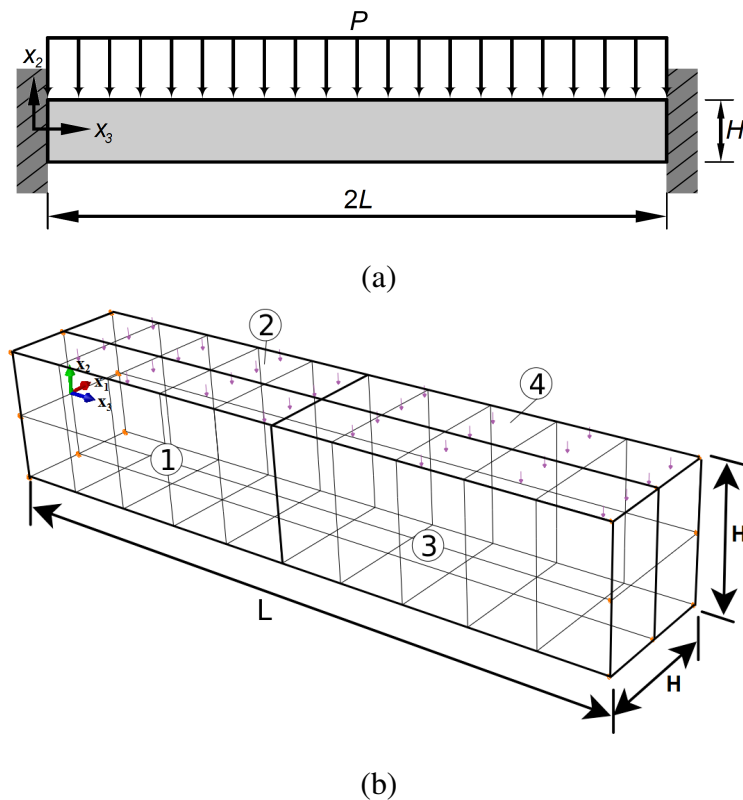
## 5 Numerical examples

Three numerical examples are presented to demonstrate the successful implementation of the proposed BEM approach with the ACA acceleration scheme described above. In the first

example, a square cross-section beam under transverse pressure is analysed. Four regions are defined with the same material in order to test the multi-domain approach. Normalized displacements are compared with the continuous one-region BEM. In the second example, the fiber-matrix model is tested with an isotropic material. Results are compared with the analytical solution. Finally, the fiber-matrix model is tested with a more complex geometric combination of materials. The main objective is to test the acceleration scheme. Displacements are compared between conventional BEM and results obtained after applying the fast BEM approach. Moreover, the application of the ACA is tested by comparing the solution CPU times. It should be noted that materials for the problems treated are isotropic and transversely isotropic even though they are analysed using the BEM program for general anisotropy. The present work serves as a precursor to treating more complex problems involving anisotropy inclusion in a matrix medium.

### 5.1 Example A: Multidomain with 4 regions

In this example the multidomain approach for more than two regions is tested with quadratic discontinuous element. A short beam of length  $2L$ , square cross-section of side  $H$  and  $L = 5H$ , is subjected to a uniformly distributed pressure load on its top surface, as shown in Fig. 2(a). Fig. 2(b) shows the BEM mesh for half of the beam that was modelled, taking advantage of symmetry. It consists of four sub-regions.



**Figure 2: (a) Geometry and boundary conditions for example A; (b) Multidomain BEM mesh of half beam.**

Each region has 34 quadratic discontinuous elements; a total of 136 elements, 305 geometrical nodes and 1088 physical nodes were used to model this problem. The anisotropic 3D

formulation based on Fourier series was used for an isotropic material with  $E = 1$  and  $\nu = 0$ . For the double Fourier fundamental solution,  $\alpha = 16$  and 64 Gauss integration points were used. Normalized transverse displacements along the  $x_2$  direction for the four regions are shown in Figure (3). Finally, displacements at nodes  $(0.5, 0.5, x_3)$  are compared with results obtained from the conventional BEM with quadratic continuous elements and the analytical solution, as shown in Figure (4).

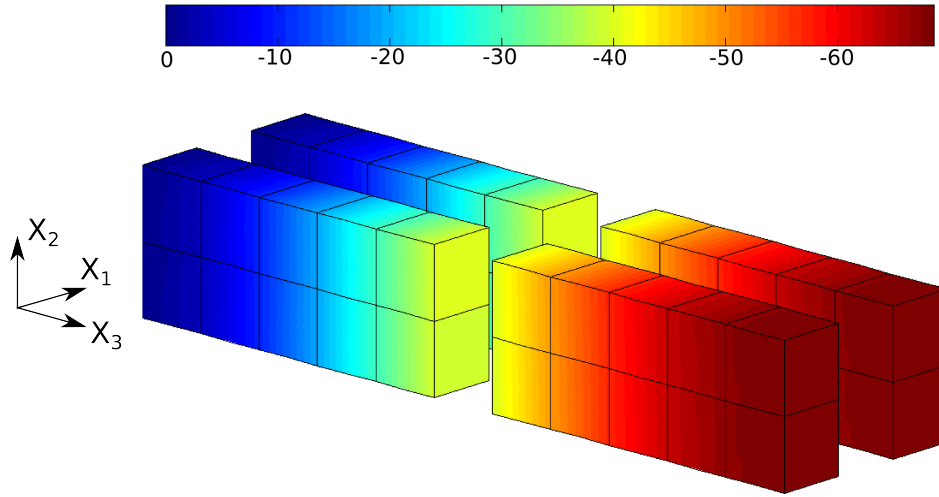


Figure 3: Normalized transverse  $x_2$  displacement for the four regions.

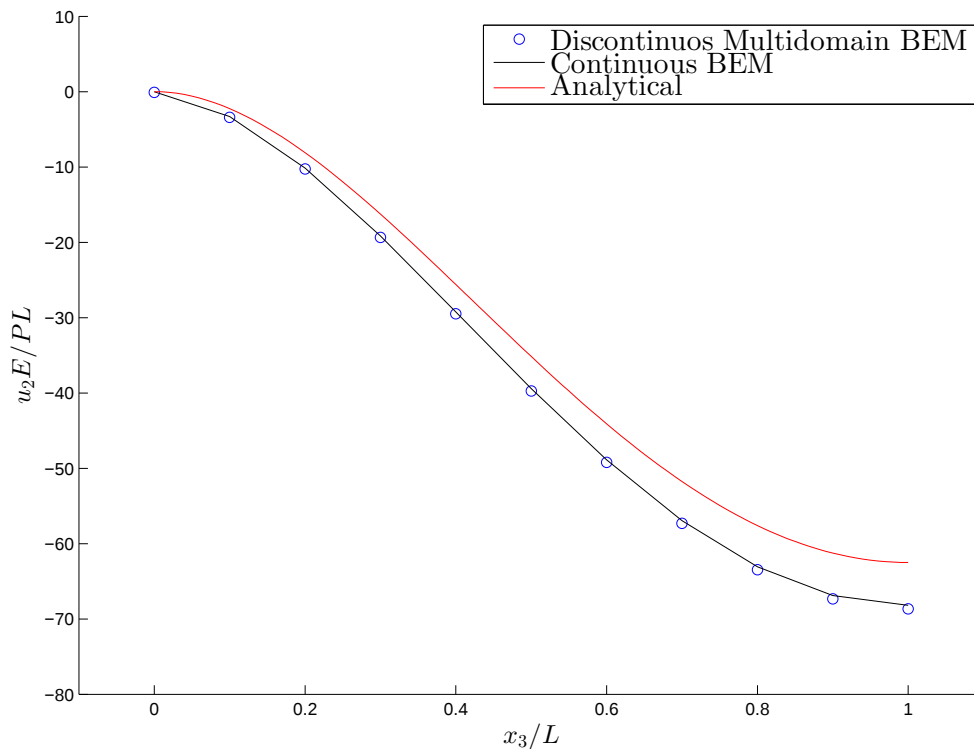


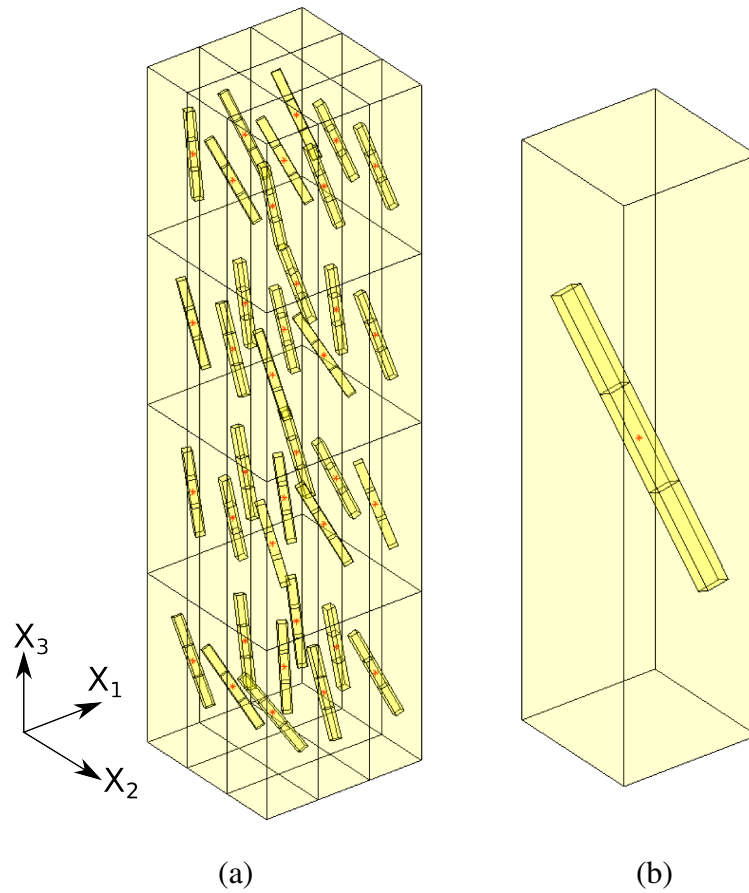
Figure 4: Normalized transverse  $x_2$  displacement comparison at  $(0.5, 0.5, x_3)$ .

Good agreement was observed when both approaches were compared. Discontinuous

quadratic elements were applied due to its simpler implementation when dealing with more than two regions in the multidomain approach and when debonding between elements occur.

## 5.2 Example B: Short Fiber-Matrix model (Isotropy check)

In this example the fiber-matrix model was implemented and tested. A general mesh generator was implemented. The fibers can be randomly oriented, as shown in Fig. 5. The length and cross-section of the short fibers are also input parameters; thus, the fraction volume can be also imposed. A fiber-matrix model with 36 short fibers is shown in Fig. 5(a), while a unit cell is shown in Fig. 5(b).



**Figure 5: (a) Fiber-Matrix model with 36 fibers; (b) unit cell with a randomly oriented short fiber.**

First, the model was tested with an isotropic material with  $E = 10000$  and  $\nu = 0.3$  for the short fibers and the matrix medium, however, the anisotropic 3D fundamental solution based on Fourier series was used. As usual,  $\alpha = 16$  and 64 Gauss integration points were used for the Fourier representation. An isotropic material was used in order to properly compare results with the analytical solution. For this first fiber-matrix example, a bar of length 4 units and square cross-section of one unit in length is subjected to a uniformly distributed pressure load,  $P = 1$ , on its top surface, while, its bottom surface is restricted in the three coordinate directions. The simple analytical solution for the  $u_3$  displacement is  $Px_3/E$ . Displacements along the  $x_3$  direction for the fiber-matrix model are shown in Fig. 6.

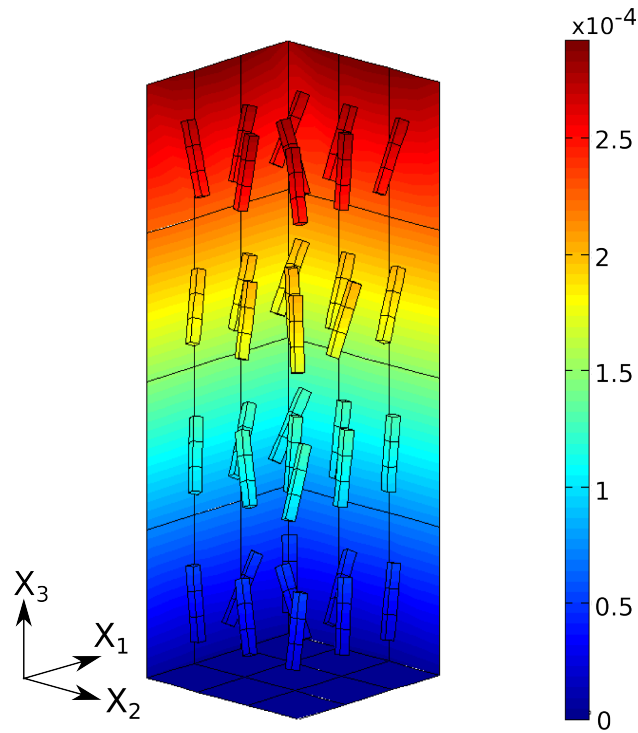


Figure 6: Fiber-matrix model with an isotropic material.

Displacements at nodes  $(0.5, 0.5, x_3)u$  are compared with the analytical solution, as shown in Fig. 7.

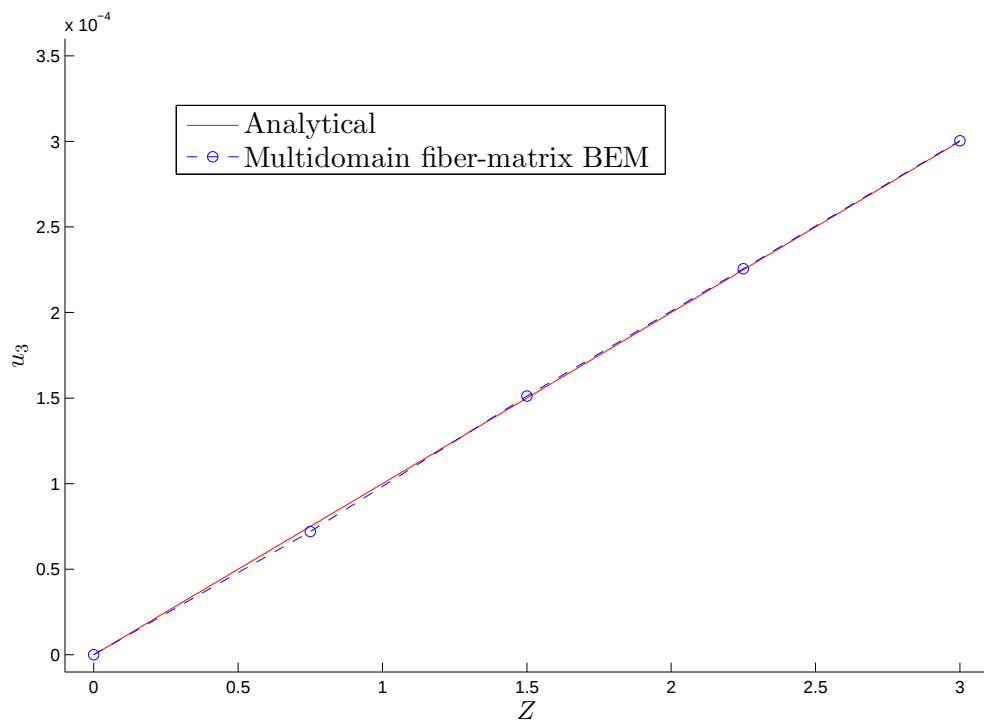


Figure 7: Numerical comparison with the analytical solution.

Good agreement was observed when the fiber-matrix multidomain model was compared with the analytical solution.

### 5.3 Example C: Short Fiber-Matrix realistic model with ACA

In this example, the short fiber-matrix model is shown with a more realistic combination of materials for both, fiber and matrix. Furthermore, the ACA method was applied in the generation of BEM matrices [H] and [G]. For the matrix medium, a 3501-6 epoxy was considered. Material properties were considered as isotropic with  $E = 4.2\text{GPa}$  and Poisson ratio  $\nu = 0.34$ . For the short fibers, a AS4 carbon material was considered. Material properties were considered as transversely isotropic with  $E_1 = 225\text{GPa}$ ,  $E_2 = 15\text{GPa}$ ,  $G_{12} = 15\text{GPa}$  and  $\nu_{12} = 0.2$ . More details on the properties of the chosen combination fiber-matrix can be seen in Soden et al. (1998).

In this example, a bar of length 3 units and square cross-section with side length of unity was subjected to a uniformly distributed pressure load,  $P = 1$ , on its top surface, while, its bottom surface is restricted in the 3 coordinate directions. In order to test the ACA method, five different cases were analysed (4, 8, 12, 18 and 27 fibers). The ACA error tolerance is set to  $\epsilon_c = 10^{-4}$ . The maximum number of elements per cluster was set to 30 and the admissibility parameter,  $\eta$ , to 0.9. More details of the choice of these parameters are available in Benedetti et al. (2009) and Rodríguez et al. (2013). For the most refined case (27 fibers, 6480 quadratic discontinuous nodes) there were 45 clusters and 487 blocks, from which 46 were admissible pairs. Results from the BEM anisotropic multidomain model using the ACA for the  $u_3$  displacement are shown in Figure (8).

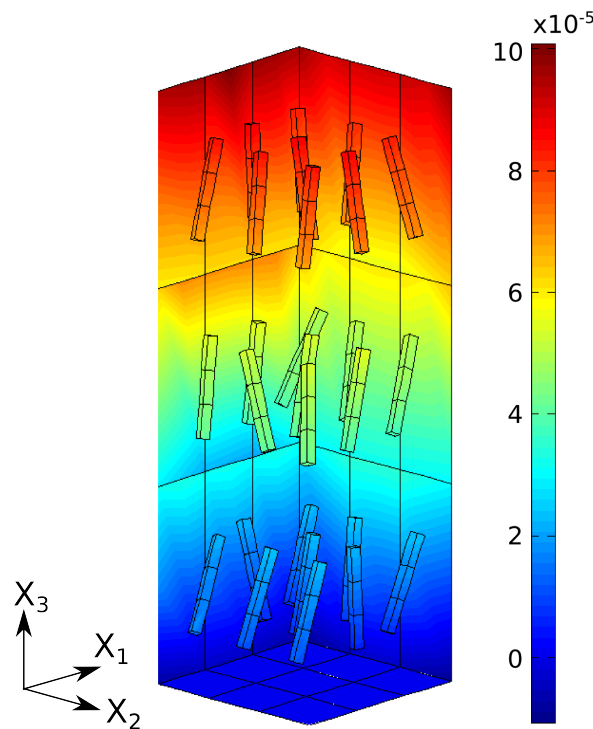


Figure 8: Displacement in the  $x_3$  direction for the most refined mesh.

Table 1 shows the numerical comparison of displacements  $u_1$ ,  $u_2$  and  $u_3$  obtained using the conventional BEM (C-BEM) and the present ACA scheme. Five randomly chosen nodes (349, 841, 1050, 3439 and 6052) were chosen for the comparison. The ACA displacements show also the error when compared with the conventional BEM.

**Table 1: Comparison of displacements obtained between conventional BEM (C-BEM) and the present ACA scheme for randomly chosen nodes.**

|      | C-BEM                |                              | ACA                  |                              | C-BEM                |                              | ACA |  | C-BEM |  | ACA |  |
|------|----------------------|------------------------------|----------------------|------------------------------|----------------------|------------------------------|-----|--|-------|--|-----|--|
| Node | $u_1 \times 10^{-6}$ | $u_1 \times 10^{-6}$ (Err.%) | $u_2 \times 10^{-6}$ | $u_2 \times 10^{-6}$ (Err.%) | $u_3 \times 10^{-4}$ | $u_3 \times 10^{-4}$ (Err.%) |     |  |       |  |     |  |
| 349  | -0.4341              | -0.4325 (0.37)               | 26.750               | 26.790 (0.15)                | 0.2916               | 0.2912 (0.14)                |     |  |       |  |     |  |
| 841  | 1.5440               | 1.5470 (0.19)                | -0.8381              | -0.8246 (1.61)               | 0.1564               | 0.1564 (0.00)                |     |  |       |  |     |  |
| 1050 | 1.1260               | 1.1270 (0.09)                | -0.2478              | -0.2321 (6.34)               | 0.7561               | 0.7562 (0.01)                |     |  |       |  |     |  |
| 3439 | -0.2270              | -0.2335 (2.86)               | -5.1020              | -5.0660 (0.71)               | 0.7826               | 0.7829 (0.04)                |     |  |       |  |     |  |
| 6052 | -0.5833              | -0.5849 (0.27)               | -0.5269              | -0.5118 (2.87)               | 0.8506               | 0.8508 (0.02)                |     |  |       |  |     |  |

It is worth pointing that the maximum error was 6.34% for the displacement in the  $x_2$ -direction at node 1050. For this case, the displacement was  $-0.2478 \times 10^{-6}$  mm for the conventional BEM, being substantially less than the ACA tolerance of  $\epsilon_c = 10^{-4}$ . As pointed out by Benedetti et al. (2011), the ACA tolerance should be set to be compatible with the expected smallest response value.

Figure 9 shows the comparison between CPU time for generating BEM matrices for conventional BEM and when the ACA method is applied. It can be noted that when the numerical problem size gets to be large, the ACA scheme becomes more efficient than the conventional BEM.

## CONCLUSIONS

In this work, the use of hierarchical matrices and low-rank approximations applied to 3D multi-domain short fiber-matrix problems which employs the double Fourier series representation of the Green's functions has been presented. Low rank approximations were accomplished by the use of ACA. This method is suitable for memory and time savings, especially in the case of large-scale problems. Moreover, the multi-domain approach was modified to account for short fiber-matrix problems. Results for a more realistic combination of fiber-matrix were also shown; they demonstrated that the ACA works better beyond a certain number of elements in the mesh. Beyond this level of discretization, the CPU solution time by the ACA will be less than the conventional BEM formulation. The present study serves as a precursor to treating more general anisotropic inclusions in a matrix medium.

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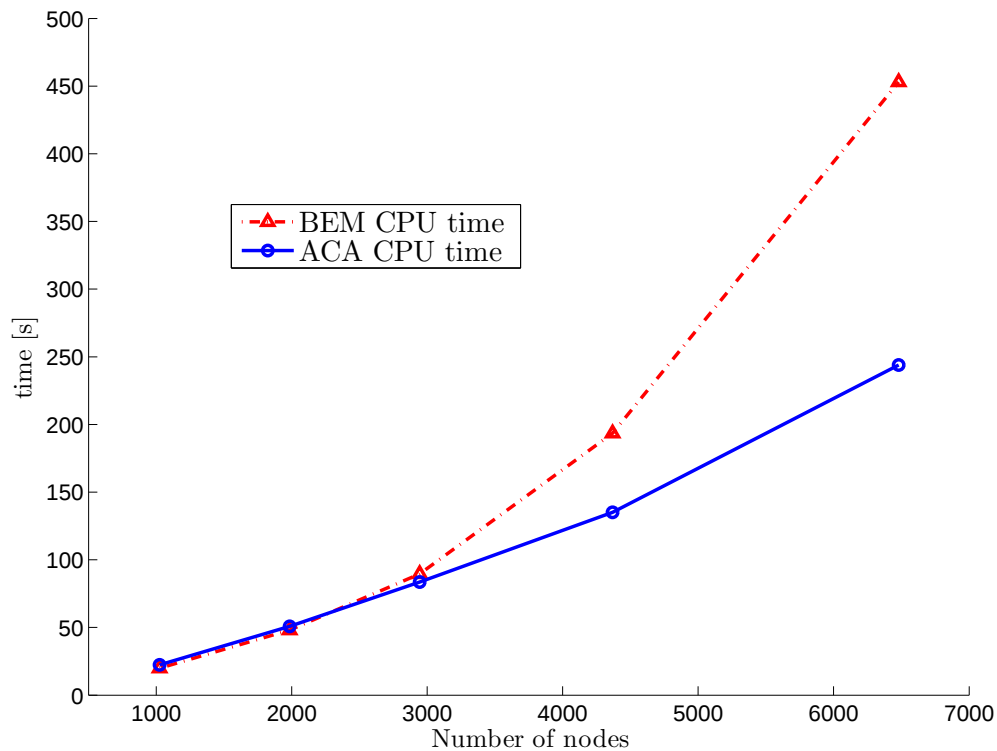


Figure 9: CPU times comparison between conventional BEM and ACA.

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