



ROBUST TOPOLOGY OPTIMIZATION UNDER UNCERTAIN LOADS: A SPECTRAL STOCHASTIC APPROACH

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Abstract. *A spectral stochastic approach for structural topology optimization in the presence of uncertainties in the magnitude and direction of the applied loads is proposed. The application of this approach in the representation and propagation of uncertainties presents a low computational cost compared to classical techniques, such as Monte Carlo simulation. A recent development of spectral representation methods, known as generalized polynomial chaos (gPC), has become one of the most widely used methods by exhibiting fast convergence when the solution depends smoothly on the random parameters. Therefore, in this work, gPC is applied to estimate the statistical measures of the compliance of 2D continuum structures, which we call Robust Topology Optimization. To demonstrate the accuracy and applicability of the proposed method, we solve robust topology optimization where we minimize the influence of stochastic variability on the mean design. Representative examples of topology optimization of continuum structures under load uncertainties are presented. The results demonstrate that load uncertainties play an important role in the optimal design. It is also shown that results obtained from the gPC method are in excellent agreement with those obtained from Monte Carlo simulation.*

Keywords: *Topology Optimization, Spectral Stochastic Approach, Generalized Polynomial Chaos, Uncertainty Quantification, Robust Optimization*

1 INTRODUCTION

This work investigates novel robust topology optimization for the 2D continuum structures problem, where the compliance of a structure submitted to uncertain loading is minimized using stochastic spectral methods to propagate the uncertainties. Structural optimization in conjunction with stochastic computation have been used as of late to predict the impact of uncertainty sources in both numerical modeling and simulation results (Banichuk & Neittaanmäki, 2010). It has become the main tool in many fields to understand complex systems with greater accuracy. Currently, there exists an increasing variety of mechanical structures that require more critical and complex designs. For this reason, several engineers have integrated uncertainties into their simulations to find critical values in the initial stage of design.

Topology Optimization (TO) seeks to find the best layout for a structure by optimizing the material distribution in a predefined design domain (Michell, 1904; Bendsøe & Sigmund, 2003). The popularity of TO is demonstrated by its wide application in many different fields of engineering (Talischi et al., 2012b; Schousboe & Sigmund, 2013; Romero & Silva, 2014; Duan et al., 2015). Moreover, most of the applications of TO are limited to deterministic conditions, i.e., the sources of uncertainties are not taken into account. The algorithm of TO is implemented with any type of finite element used in the discretization of the domain. However, numerical instabilities, such as checkerboards and mesh dependence, can be found. The checkerboard is due to the formation of alternating regions of solid and void in the design domain. The mesh dependence refers to obtaining qualitatively different solutions for different mesh sizes or discretizations. To avoid both checkerboard and mesh dependence problems without the need for special treatment, we will use polygonal finite elements because they have shown good results (Talischi et al., 2009, 2012a).

In previous works, such as (Shikui et al., 2010; Tootkaboni et al., 2012; Junpeng & Chunjie, 2015), few stochastic approaches have been considered in the modeling or optimization process as a way of obtaining better designs. Most are based on applications or extensions of the Monte Carlo Simulation method (Malvin & Whitlock, 2008; Rubinstein & Krose, 2007). This method is widely used as a very powerful technique to propagate randomness in problems without limitations concerning the structure of the uncertainty or the properties of the probability distribution. However, this method has the major disadvantage of being extremely expensive in terms of computational cost due to its very slow convergence rate, despite being flexible and of easy implementation. Therefore, the Monte Carlo method is sometimes avoided for large real-time applications.

Several methods have recently been developed, such as Stochastic Galerkin, Polynomial Chaos, and Karhunen Loève expansion, with significantly reduced computational cost compared to the Monte Carlo method. These approaches are known as stochastic spectral methods, and their application to quantify and propagate uncertainties in model-based computations is an open and relevant research topic (Ghanem & Spanos, 1991; Le Maître & Knio, 2010; Xiu, 2010).

In this work, we will develop a methodology to quantify and propagate uncertainties into TO by using Polynomial Chaos expansion. For the Topology optimization process, we will use the `PolyTop` framework, developed in MATLAB®. (Talischi et al., 2012b). For the Polynomial Chaos method, we implemented our own code within the `PolyTop` framework. Starting from these considerations, the remainder of the paper is organized as follows: Stochastic spectral methods are introduced in section 2, presenting the mathematical

formulation and basic steps to obtain the polynomial chaos expansion. In section 3, we briefly describe the deterministic TO problem and present the proposed procedure to consider uncertainties within TO, which is called robust topology optimization. In section 4, numerical examples are presented, and the proposed methodology is compared with the Monte Carlo method. Finally, a discussion and suggestions for further work are presented in section 5.

2 STOCHASTIC SPECTRAL METHOD

Let $(\Omega, \mathfrak{F}, P)$ be a probability space, w a random event belonging to space Ω , and P a probability measure defined on the σ -field $\mathfrak{F}$ of subsets of Ω . Let $\xi = \{\xi_i(w)\}_{i=1}^N$ be a set of N independent and identically distributed random variables for w .

We denote as $L_2(\Omega, P)$ the space of the second-order random variable defined on $(\Omega, \mathfrak{F}, P)$ with the inner product $\langle \cdot, \cdot \rangle$ and associated norm $\|\cdot\|_{L_2(\Omega, P)}$.

Let $U(w)$ be a second-order random variable (finite second moment) belonging to the space:

$$L_2(\Omega, P) = \left\{ U : \Omega \rightarrow R \mid E[U^2] = \int_{\Omega} U^2 dP(w) < +\infty \right\}, \quad (1)$$

where $E[\cdot]$ is the expected operator. This is a Hilbert space with respect to the inner product of two random variables $U(w)$ and $V(w)$ belonging to $L_2(\Omega, P)$:

$$\langle U(w), V(w) \rangle \equiv E[U(w)V(w)] = \int_{\Omega} U(w)V(w) dP(w) \quad (2)$$

This inner product induces the norm:

$$\|U\|_{L_2(\Omega, P)}^2 = \langle U^2 \rangle \quad (3)$$

Generally, a spectral representation of random functionals aims to find a series expansion of the form:

$$U(w) = \sum_{j=0}^{\infty} u_j \Psi_j(\xi(w)) \quad (4)$$

where $\{\Psi_j(\xi)\}_{j=0}^{\infty}$ is the set of basis functions and $\{u_j\}_{j=0}^{\infty}$ is the set of coefficients to be determined.

2.1 Polynomial Chaos Expansion

Polynomial Chaos Expansion (PCE) theory was first introduced in the form of Wiener's Hermite-chaos (Wiener, 1938), and it is also called Homogeneous Chaos (Ghanem & Spanos, 1991). This method is used to propagate uncertainties and can efficiently reduce computational effort in highly nonlinear engineering design applications.

This expansion technique employs multi-dimensional Hermite polynomials based on standard Gaussian random variables. A general second-order Gaussian random response $U(w) \in L_2(\Omega, P)$ presents a polynomial chaos expansion represented in the form:

$$\begin{aligned}
 U(w) = & u_0 \Gamma_0 + \sum_{i_1=1}^{\infty} u_{i_1} \Gamma_1(\xi_{i_1}(w)) \\
 & + \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} u_{i_1 i_2} \Gamma_2(\xi_{i_1}(w), \xi_{i_2}(w)) \\
 & + \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} \sum_{i_3=1}^{i_2} u_{i_1 i_2 i_3} \Gamma_3(\xi_{i_1}(w), \xi_{i_2}(w), \xi_{i_3}(w)) \\
 & + \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} \sum_{i_3=1}^{i_2} \sum_{i_4=1}^{i_3} u_{i_1 i_2 i_3 i_4} \Gamma_4(\xi_{i_1}(w), \xi_{i_2}(w), \xi_{i_3}(w), \xi_{i_4}(w)) + \dots,
 \end{aligned} \tag{5}$$

where $\Gamma_p(\xi_{i_1}, \dots, \xi_{i_p})$ denotes the Hermite-Chaos of order p in the variables $(\xi_{i_1}, \dots, \xi_{i_p})$, which constitutes a complete basis in the Hilbert space $L_2(\Omega, P)$; $u_{i_1}, \dots, u_{i_p}$ are deterministic constants and denote the random character of the quantities involved.

PCE can approximate any functionals, and as a consequence, it converges in the mean-square sense (Cameron & Martin, 1947), i.e.,

$$\lim_{p \rightarrow \infty} E \left[u_0 \Gamma_0 + \dots + \sum_{i_1=1}^{\infty} \dots \sum_{i_p=1}^{i_{p-1}} u_{i_1 \dots i_p} \Gamma_p(\xi_{i_1}, \dots, \xi_{i_p}) - U^2 \right] = 0 \tag{6}$$

The p^{th} order of PCE consists of all orthogonal polynomials of order p , involving all possible combinations of the independent random variables $\{\xi_i\}_{i=0}^{\infty}$; furthermore, $\Gamma_p \perp \Gamma_q$ for $p \neq q$. This orthogonality greatly simplifies the procedure of statistical calculations, such as moments. By construction, the chaos polynomial whose orders are greater than one have a vanishing expectation:

$$E[\Gamma_{p>0}] = 0 \tag{7}$$

The general expression to obtain the multi-dimensional Hermite polynomials of order p is given by:

$$\Gamma_p(\xi_{i_1}, \dots, \xi_{i_p}) = e^{\frac{1}{2} \xi^T \xi} (-1)^p \frac{\partial^p}{\partial \xi_{i_1} \dots \partial \xi_{i_p}} e^{-\frac{1}{2} \xi^T \xi} \tag{8}$$

where the vector ξ consists of p Gaussian random variables.

For notational convenience and to facilitate the manipulation of PCE, we rely on a univocal relation between the $\Gamma(\cdot)$ and a new functional $\Psi(\cdot)$. Then, expression (5) can be rewritten as:

$$U(w) = \sum_{j=0}^{\infty} \hat{u}_j \Psi_j(\xi) \tag{9}$$

Thus, this result is a more compact expression where there is a one-to-one correspondence between the functionals $\Gamma_p(\xi_{i_1}, \dots, \xi_{i_p})$ and $\Psi_j(\xi)$ and also between the coefficients $u_{i_1}, \dots, u_{i_p}$ and $\hat{u}_j$. The deterministic expansion coefficients $\hat{u}_j$ are simply called PCE coefficients.

The expansion above involves an infinite collection of ξ_i . In practice, it is necessary to restrict the representation to an infinite number of random variables. Specifically, the PCE of dimension N and order p is truncated to an $(M+1)$ finite number of polynomials terms defined by the following equation:

$$(M+1) = \binom{N+p}{p} = \frac{(N+p)!}{N!p!} \quad (10)$$

while for the tensor polynomial basis we have:

$$(M+1) = (p+1)^N \quad (11)$$

Therefore, the truncated expression of the random variable U can be expressed as:

$$U(w) = \sum_{j=0}^M \hat{u}_j \Psi_j(\xi) + \varepsilon(N, p) \quad (12)$$

In Eq.(12), each $\Psi_j(\xi)$ is a multi-dimensional polynomial that involves products of the one-dimensional Hermite polynomials, and the PCE representation will be computationally efficient when small values of N and p are sufficient for an accurate representation of U , or in other words, when $\langle \varepsilon^2(N, p) \rangle \rightarrow 0$ rapidly with N and p .

Let $\{\psi_{i_k}(\xi_k)\}_{k=1}^N$ be the one-dimensional Hermite polynomial of order $i_k \leq p$, with $\{\psi_0(\xi_k)\}_{k=1}^N \equiv 1$, and let $|\lambda| = i_1 + \dots + i_N$ be a multi-index such that $0 \leq |\lambda| \leq p$.

Then, Γ_p can be represented as:

$$\Gamma_p(\xi_1, \dots, \xi_{i_p}) = \Psi_j(\xi) = \prod_{k=1}^N \psi_{\lambda}(\xi_k) \quad (13)$$

Furthermore, the polynomial basis $\{\Psi_j\}$ of Hermite-Chaos forms a complete orthogonal basis of $L_2(\Omega, \mathfrak{F}, P)$, i.e.,

$$\langle \Psi_i, \Psi_j \rangle = E[\Psi_i \Psi_j] = \langle \Psi_i^2 \rangle \delta_{ij} \quad (14)$$

where δ_{ij} is the Kronecker delta and the inner product in the Hilbert space is determined by the support of the Gaussian variables.

In the one-dimensional case, we can expand the random response U using orthogonal polynomials in ξ , which present a known probability distribution such as standard normal, $N(0,1)$. If U is a function of a normally distributed random variable X , with known mean μ_X and variance σ_X^2 , ξ is a normalized variable:

$$\xi = \frac{X - \mu_X}{\sigma_X} \quad (15)$$

Generally, the one-dimensional Hermite polynomial is defined as:

$$\psi_p(\xi) = (-1)^p \frac{\varphi^{(p)}(\xi)}{\varphi(\xi)}, \quad (16)$$

where $\varphi^{(p)}(\xi)$ is the p^{th} derivative of the normal density function, $\varphi(\xi) = 1/\sqrt{2\pi} e^{-\xi^2/2}$, which represents the single-variable version of Eq. (8).

For example, the one-dimensional Hermite polynomials are:

$$\Psi_0(\xi) = \psi_0(\xi_1) = 1$$

$$\Psi_1(\xi) = \psi_1(\xi_1) = \xi$$

$$\Psi_2(\xi) = \psi_2(\xi_1) = \xi^2 - 1$$

$$\Psi_3(\xi) = \psi_3(\xi_1) = \xi^3 - 3\xi, \dots$$

In the case of a two-dimensional expansion, we can write:

$$\begin{aligned} U(w) = & u_0 \Gamma_0 + \sum_{i_1=1}^2 u_{i_1} \Gamma_1(\xi_{i_1}) + \sum_{i_1=1}^2 \sum_{i_2=1}^{i_1} u_{i_1 i_2} \Gamma_2(\xi_{i_1}, \xi_{i_2}) \\ & + \sum_{i_1=1}^2 \sum_{i_2=1}^{i_1} \sum_{i_3=1}^{i_2} u_{i_1 i_2 i_3} \Gamma_3(\xi_{i_1}, \xi_{i_2}, \xi_{i_3}) \\ & + \sum_{i_1=1}^2 \sum_{i_2=1}^{i_1} \sum_{i_3=1}^{i_2} \sum_{i_4=1}^{i_3} u_{i_1 i_2 i_3 i_4} \Gamma_4(\xi_{i_1}, \xi_{i_2}, \xi_{i_3}, \xi_{i_4}) + \dots, \end{aligned}$$

or alternatively:

$$\begin{aligned} U = & u_0 \Gamma_0 + u_1 \Gamma_1(\xi_1) + u_2 \Gamma_1(\xi_2) \\ & + u_{11} \Gamma_2(\xi_1, \xi_1) + u_{21} \Gamma_2(\xi_2, \xi_1) + u_{22} \Gamma_2(\xi_2, \xi_2) \\ & + u_{111} \Gamma_3(\xi_1, \xi_1, \xi_1) + u_{211} \Gamma_3(\xi_2, \xi_1, \xi_1) + u_{221} \Gamma_3(\xi_2, \xi_2, \xi_1) \\ & + u_{222} \Gamma_3(\xi_2, \xi_2, \xi_2) + u_{1111} \Gamma_4(\xi_1, \xi_1, \xi_1, \xi_1) + \dots, \end{aligned}$$

and using the simplified form of Eq. (9):

$$\begin{aligned} U = & \hat{u}_0 \Psi_0(\xi) + \hat{u}_1 \Psi_1(\xi) + \hat{u}_2 \Psi_2(\xi) + \hat{u}_3 \Psi_3(\xi) + \hat{u}_4 \Psi_4(\xi) + \hat{u}_5 \Psi_5(\xi) \\ & + \hat{u}_6 \Psi_6(\xi) + \hat{u}_7 \Psi_7(\xi) + \hat{u}_8 \Psi_8(\xi) + \hat{u}_9 \Psi_9(\xi) + \dots \end{aligned}$$

Then, calculating the multi-dimensional basis polynomials over two random dimensions, we obtain:

$$\Psi_0(\xi) = \psi_0(\xi_1) \psi_0(\xi_2) = 1$$

$$\Psi_1(\xi) = \psi_1(\xi_1) \psi_0(\xi_2) = \xi_1$$

$$\Psi_2(\xi) = \psi_2(\xi_1) \psi_0(\xi_2) = \xi_1^2 - 1$$

$$\Psi_3(\xi) = \psi_2(\xi_1) \psi_1(\xi_2) = \xi_1^2 \xi_2 - \xi_2$$

$$\Psi_4(\xi) = \psi_1(\xi_1) \psi_1(\xi_2) = \xi_1 \xi_2$$

$$\Psi_5(\xi) = \psi_1(\xi_1) \psi_2(\xi_2) = \xi_1 \xi_2^2 - \xi_1$$

$$\Psi_6(\xi) = \psi_2(\xi_1) \psi_2(\xi_2) = \xi_1^2 \xi_2^2 - \xi_1^2 - \xi_2^2 + 1$$

$$\Psi_7(\xi) = \psi_2(\xi_1) \psi_3(\xi_2) = \xi_1^2 \xi_2^3 - 3\xi_1^2 \xi_2 + \xi_1^2$$

$$\Psi_8(\xi) = \psi_3(\xi_1) \psi_2(\xi_2) = \xi_1^3 \xi_2^2 - 3\xi_1 \xi_2^2 + 3\xi_1^2 \xi_2 - \xi_1^2$$

$$\Psi_9(\xi) = \psi_3(\xi_1) \psi_3(\xi_2) = \xi_1^3 \xi_2^3 - 3\xi_1^3 \xi_2 + 3\xi_1^2 \xi_2^2 - 3\xi_1^2 \xi_2 + \xi_1^2$$

and, finally, the expansion of U is given by:

$$U = \hat{u}_0 + \hat{u}_1 \xi_1 + \hat{u}_2 \xi_2 + \hat{u}_3 (\xi_1^2 - 1) + \hat{u}_4 (\xi_1 \xi_2) + \hat{u}_5 (\xi_2^2 - 1) + \hat{u}_6 (\xi_1^3 - 3\xi_1) \\ + \hat{u}_7 (\xi_1^2 \xi_2 - \xi_2) + \hat{u}_8 (\xi_2^2 \xi_1 - \xi_1) + \hat{u}_9 (\xi_2^3 - 3\xi_2) + \dots,$$

2.2 Generalized Polynomial Chaos

The Wiener's Hermite-chaos expansion has been quite effective in solving stochastic differential equations for the case of Gaussian input. However, many stochastic problems involve certain types of non-Gaussian inputs, e.g., lognormal distributions; this can be justified by the Cameron-Martin theorem (Cameron & Martin, 1947). Moreover, when general non-Gaussian random inputs are approximated by Wiener's Hermite-chaos expansions, the convergence may be slow and, in some cases, severely deteriorated (Xiu, 2004).

Xiu and Karniadakis (Xiu & Karniadakis, 2002a, 2002b) introduced the term generalized polynomial chaos (gPC) expansions and have shown that, for a large number of common probabilities laws, any random functions can be represented by the corresponding families of the set of hypergeometric polynomials from the Askey-scheme (Wiener-Askey polynomial chaos) as a generalization of the original Wiener Hermite-chaos expansion.

Then, we observe that some types of orthogonal polynomials from the Askey-scheme have weighting functions of the same form as the probability density function of certain types of random distributions. This establishes a relationship between the distributions of the independent random variable ξ and the type of orthogonal polynomial $\{\Psi_j(\xi)\}$ of the Askey-scheme (gPC basis), as shown in Table 1.

Table 1. Correspondence between the type of gPC and the underlying random variable

Distribution of ξ	gPC basis polynomials	Weight function	Support
Uniform	Legendre	$\frac{1}{2}$	$[-1,1]$
Gaussian	Hermite	$\frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}}$	$\langle -\infty, +\infty \rangle$
Gamma	Laguerre	$\frac{\xi^\alpha e^{-\xi}}{\Gamma(\alpha+1)}$	$[0, +\infty)$
Beta	Jacobi	$\frac{(1+\xi)^\alpha (1-\xi)^\beta}{2^{\alpha+\beta+1} B(\alpha+1, \beta+1)}$	$[-1,1]$

It is clear that the original Wiener polynomial chaos corresponds to the Hermite-chaos and is a subset of Wiener-Askey polynomial chaos.

Furthermore, in the case of the probability functions for which one does not readily dispose of an orthogonal family of polynomials, it is generally possible to rely on a numerical

construction of the PCE basis, following a Gram-Schmidt orthogonalization process (Stoer & Bulirsch, 1993).

There are several techniques to compute the deterministic coefficients $\hat{u}_j$ of the PCE, such as NISP (Non-intrusive spectral method), which aims to compute the coefficients through its orthogonal projection. With NISP, we can find different strategies and integration methods, e.g., pseudo-random sampling, quadrature formulas and, for multi-dimensional cases of the tensor product, the sparse grid method.

In addition, we can use the Regression Method as an alternative formulation of the problem of estimating the expansion coefficients. Another related technique is known as the Probabilistic Collocation approach and employs a selected subset of Gaussian quadrature points (those with highest tensor-product weighting) to provide a more optimal collocation location and preserve interpolation properties (Eldred, 2009). This technique will be applied in this work.

The PCE can be easily extended to the second-order stochastic processes $U : \Xi \times \Omega \mapsto R$ and $U(x, \cdot) \in L_2(\Omega, P)$ by letting the deterministic coefficients depend on the index $x \in \Xi$, namely:

$$U(x, \xi) \approx \sum_{j=0}^M \hat{u}_j(x) \Psi_j(\xi) \quad (17)$$

where the deterministic function $\hat{u}_j(x)$ represents the stochastic modes of the process and corresponds to the value of $U(x, \xi)$ at the collocation point $\xi^{(i)}$ and m_c corresponds to the total collocation points $\xi^{(i)}$ (these points are chosen such that they correspond to the Gaussian quadrature points).

This non-intrusive method always achieves an exponential convergence; however, the number of collocation points increases rapidly with the number of uncertain parameters, i.e., $m_c = (p+1)^N$.

Letting the unknown coefficients be a vector $\mathbf{u} = [\hat{u}_0, \dots, \hat{u}_{m_c}]^T$, the polynomial chaos expansion a matrix $\Psi = (\Psi_{ij})$ and the response output a vector $\mathbf{U}$, the system from Eq. (17) can be rewritten as:

$$\mathbf{U} = \begin{bmatrix} U(x, \xi^{(1)}) \\ U(x, \xi^{(2)}) \\ \vdots \\ U(x, \xi^{(m_c)}) \end{bmatrix} = \begin{bmatrix} \Psi_0(\xi^{(1)}) & \Psi_1(\xi^{(1)}) & \dots & \Psi_{m_c}(\xi^{(1)}) \\ \Psi_0(\xi^{(2)}) & \Psi_1(\xi^{(2)}) & \dots & \Psi_{m_c}(\xi^{(2)}) \\ \vdots & \vdots & \ddots & \vdots \\ \Psi_0(\xi^{(m_c)}) & \Psi_1(\xi^{(m_c)}) & \dots & \Psi_{m_c}(\xi^{(m_c)}) \end{bmatrix} \begin{bmatrix} \hat{u}_0(x) \\ \hat{u}_1(x) \\ \vdots \\ \hat{u}_{m_c}(x) \end{bmatrix} = \Psi \mathbf{u} \quad (18)$$

Therefore, the polynomial expansion coefficients are obtained by solving the following linear system:

$$\mathbf{u} = \Psi^{-1} \mathbf{U} \quad (19)$$

3 ROBUST TOPOLOGY OPTIMIZATION

The main objective of Topology Optimization is to find the optimal distribution of materials in a given domain $\Omega \in R^n$ subjected to traction and displacement boundary conditions. This process involves the determination of new topologies or layouts of the structure when the optimal performance is reached. This work considers the minimum compliance problem as the objective function, i.e., the stiffness of the structure is maximized while satisfying a constraint on the total volume.

The Topology Optimization problem may be formulated as:

$$\begin{aligned} \min_{\rho} \quad & C(\rho) = \mathbf{f}^T \mathbf{u}(\rho) \\ \text{s.t.:} \quad & V(\rho) = \int_{\Omega} \rho dV \leq V_s \\ \text{with} \quad & \mathbf{K}(\rho) \mathbf{u} = \mathbf{f} \\ & 0 < \rho_{\min} \leq \rho(\Omega) \leq 1 \end{aligned} \quad (20)$$

Here, $C(\rho)$ is the objective function (i.e., the compliance of the continuum structure) and $\mathbf{u}$ and $\mathbf{f}$ are the global displacement and loading vector, respectively. $\mathbf{K}$ represents the global stiffness matrix, which is dependent on the design variable ρ (density), and the parameter V_s is the specified maximum volume of the structural material.

We are interested in determining the optimal placement of a given isotropic material in design domain Ω , i.e., determining which region should not present material (a void region), and obtaining the final topology of the structure. By convention, points where material exists are represented by a density value of 1, and the density value is 0 otherwise. Note that, by using this convention, the problem becomes an integer programming problem:

$$\rho_e(x) = \begin{cases} 1 & \text{if } x \in \Omega \\ 0 & \text{if } x \notin \Omega \end{cases} \quad (21)$$

Although there are several methods to solve these types of problems, the high number of design variables can make it very difficult to handle.

An alternative to this problem is to consider a typical approach to TO known as SIMP (Solid Isotropic Material with Penalization). This approach defines the stiffness of intermediate densities by means of a penalization, i.e.,

$$E(\rho) = E_{\min} + (E_0 + E_{\min}) \rho^p \quad (22)$$

where E_0 is the Young's modulus of the material in the solid phase and p is the penalization factor, with a value greater than 1.

When we solve the TO problem, numerical anomalies, such as checkerboards, can appear. These problems can be overcome using higher-order elements or filters (Sigmund & Peterson, 1998; Bruggi, 2008)). Moreover, (Talischi et al., 2012b) have shown that the use of polygonal elements can also address the checkerboard problem.

The optimization problem from Eq. (20) can be solved using different optimization methods, such as Optimality Criteria (OC) (Ananiev, 2005), Sequential Linear Programming (SLP) (Nocedal & Wright, 1999), and Method of Moving Asymptotes (MMA) (Svanberg,

1987). Each of these methods has advantages and disadvantages depending on the problem type it is to be applied to. In this work, we use the `PolyTop` framework, for which both the SIMP and RAMP approaches are available, together with the Optimality Criteria as the optimization method.

Figure 1 illustrates the algorithm used for obtaining the optimal design with TO.

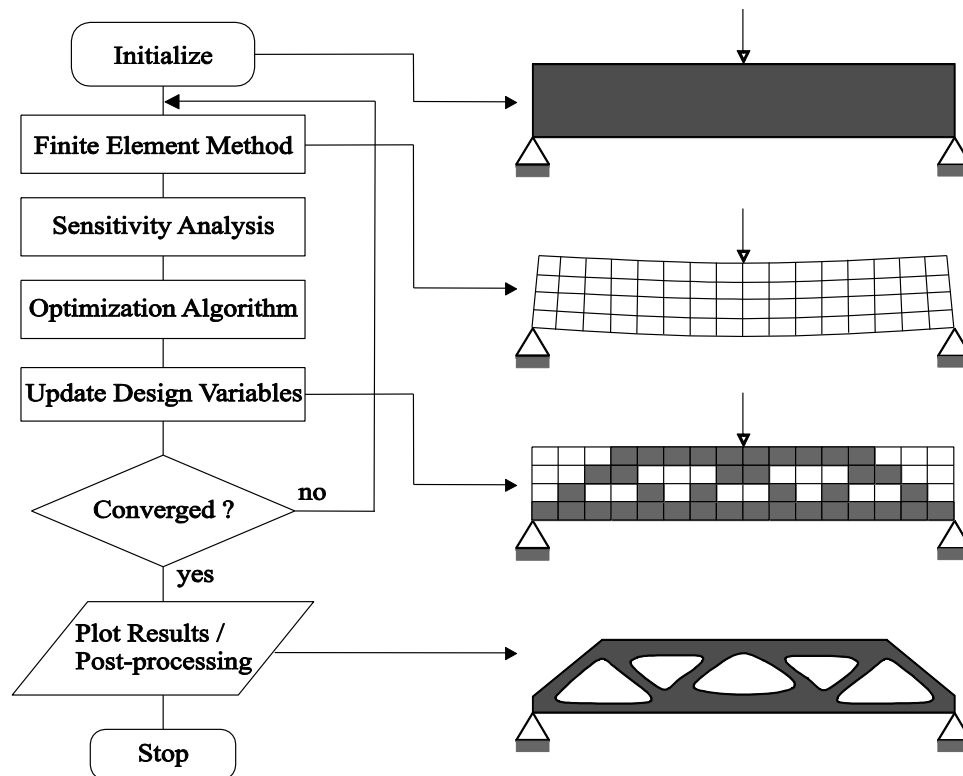


Figure 1. Procedure of Topology Optimization (Adapted from Bendsøe & Sigmund, 2003)

In the field of engineering, Robust Optimization is usually known as Robust Design Optimization (RDO). Differently from the traditional nonlinear programming, it contains non-deterministic parameters and therefore consists of a stochastic problem. RDO is a type of optimization that simultaneously addresses optimization and robustness analysis, obtaining an optimal design that is less susceptible to variability in the system parameters. Figure 2 shows a simple scheme of RDO to understand the basic idea when the response model is obtained with low variability to the small perturbations in their parameters.

Furthermore, it is very important to understand the mathematical definition of robustness, i.e., the choice of the robustness measure that is generally expressed by the combination of statistical properties of the objective function. In this manner, the stochastic problem will become a deterministic problem.

Several definitions of measures of robustness have been proposed in the literature (Beyer & Sendhoff, 2007; Birge & Louveaux, 1997; Doltsinis & Kang, 2004; Shin et al., 2011), and the weighted sum of both the mean and the standard deviation of the objective function is often considered. Then, the tradeoff between those two statistical measures could describe the final design, i.e., a design less conservative and with a much smaller range of variation.

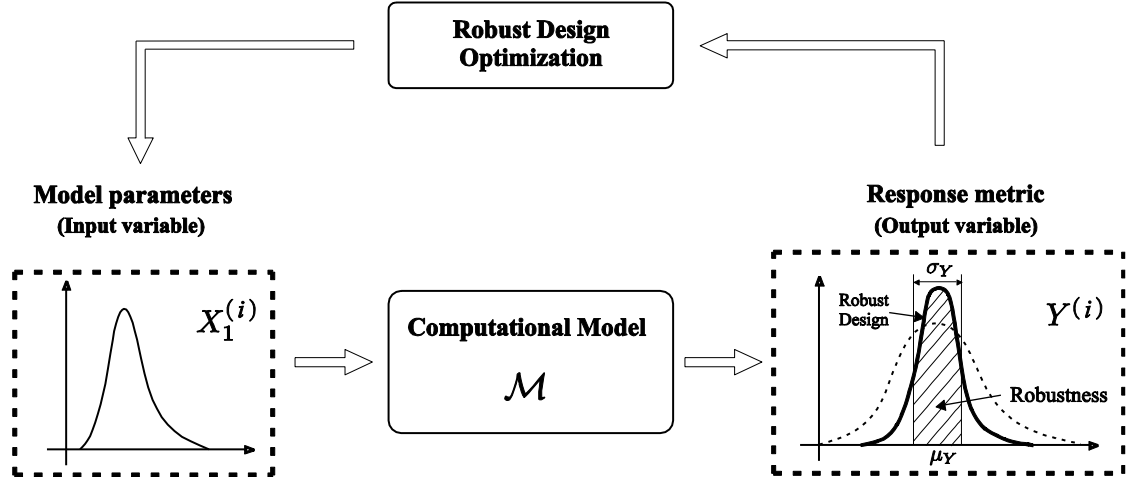


Figure 2. The idea of Robust Design Optimization

The concept of robust optimization can be applied to the topology optimization to increase the robustness of the optimal design because the compliance is a function of random variables (uncertain loading). A straightforward way to define a measure of structural performance (robustness of the objective function) is to minimize the mean of the compliance. However, the final design may still be sensitive to fluctuation of the uncertainties, and this gives rise to a need for a more robust design. Then, the standard deviation of the compliance is introduced into the structural performance to achieve the desired requirements.

Therefore, the goal here is to combine these two concepts to improve the design by minimizing the variability of the structural performance and by meeting the constraint conditions of volume, which will be called Robust Topology Optimization (RTO).

The RTO problem can be mathematically formulated as:

$$\begin{aligned}
 \min_{\boldsymbol{\rho}} \quad & \tilde{C} = (k-1)E[C(\boldsymbol{\rho}, \boldsymbol{\xi})] + k\sigma[C(\boldsymbol{\rho}, \boldsymbol{\xi})] \\
 \text{s.t.:} \quad & V(\boldsymbol{\rho}) = \int_{\Omega} \rho dV \leq V_s \\
 \text{with} \quad & \mathbf{K}(\boldsymbol{\rho})\mathbf{u} = \mathbf{f} \\
 & 0 < \rho_{\min} \leq \rho(\Omega) \leq 1
 \end{aligned} \tag{23}$$

where $\boldsymbol{\xi}$ is the vector of random variables, $\tilde{C}$ is the objective function of the RTO problem to be minimized, $k \in [0,1]$ is the weighting factor for the two parts of the objective, $E[\cdot]$ and $\sigma[\cdot]$ represent the mean and standard deviation of compliance, respectively, and $C(\boldsymbol{\rho}, \boldsymbol{\xi})$ is the stochastic compliance according to the uncertain loading.

The RTO problem (Eq. (3)) can be solved by considering non-intrusive uncertainties analysis techniques. The basic idea of non-intrusive methods is to use a set of deterministic model resolutions to construct approximations of the desired output response. The deterministic models are obtained from each specific value or realization of parameter $\boldsymbol{\xi}$ associated with a deterministic solver (e.g., Finite Elements) used as a black-box. Therefore, non-intrusive methods become very efficient in propagating uncertainties in complex models, such as structural optimization where only deterministic solvers are available.

In this work, we focus our attention on two non-intrusive techniques, namely, the Monte Carlo Method (MCM) and Generalized Polynomial Chaos (gPC) expansion for the

propagation and approximation of the model response that involve uncertainties in a finite set of independent random parameters.

3.1 Calculation of the mean and standard deviation of compliance

Frequently, the Monte Carlo method is one of the simplest crude simulation techniques, and it may be used to compute the mean and the standard deviation of the compliance function. However, the computational effort is extremely expensive, especially for topology optimization problems, i.e., we would need a large number of deterministic model resolutions to achieve an adequate response due to the slow convergence rate of the method. Then, the statistical measures can be calculated as:

$$E[C(\mathbf{p}, \xi)] \approx \frac{1}{m} \sum_{i=1}^m C(\mathbf{p}, \xi^{(i)}) \quad (24)$$

and

$$\sigma^2[C(\mathbf{p}, \xi)] \approx \frac{1}{m-1} \left[\sum_{i=1}^m C^2(\mathbf{p}, \xi^{(i)}) - m E^2[C(\mathbf{p}, \xi)] \right] \quad (25)$$

where m is the number of realizations. The convergence rate of MCM is well-known, and a variance of the error asymptotically approaches zero for a sufficiently large m according to the law of large numbers.

We also can compute the mean and standard deviation of $C(\mathbf{p}, \xi)$ through gPC from Eq. (17). Its expansion on the orthogonal gPC basis $\{\Psi_0, \Psi_1, \dots\}$, where we assume an indexation such that $\Psi_0(\xi) = 1$ and satisfying the property of orthogonal polynomials, i.e.,

$$\langle \Psi_i(\xi), \Psi_j(\xi) \rangle = E[\Psi_i(\xi) \Psi_j(\xi)] = E[\Psi_i^2(\xi)] \delta_{ij} \quad (26)$$

is an important property that can be obtained; such as:

$$\begin{aligned} E[\Psi_0(\xi) \Psi_j(\xi)] &= 0 \quad \forall j > 0 \\ E[\Psi_0(\xi) \Psi_0(\xi)] &= 1 \end{aligned} \quad (27)$$

Then, we see immediately the expression that calculates the mean of $C(\mathbf{p}, \xi)$ is given by:

$$E[C(\mathbf{p}, \xi)] = E[\Psi_0(\xi) C(\mathbf{p}, \xi)] = \langle \Psi_0(\xi), C(\mathbf{p}, \xi) \rangle \approx \sum_{j=0}^M \hat{u}_j \langle \Psi_0(\xi), \Psi_j(\xi) \rangle \quad (28)$$

Then:

$$E[C(\mathbf{p}, \xi)] \approx \hat{u}_0 \quad (29)$$

Therefore, we can obtain the mean of $C(\mathbf{p}, \xi)$; from the indexation convention, the coefficient $\hat{u}_0$ can be determined from Eq. (19). Let $\Phi = (\Phi_{ji})$ be the inverse matrix of Ψ , i.e., $\Phi = \Psi^{-1}$, and then Eq. (29) can be written as:

$$E[C(\mathbf{p}, \xi)] \approx \sum_{i=1}^M \Phi_{1i} C(\mathbf{p}, \xi^{(i)}) \quad (30)$$

Note that Eq. (30) is the Gaussian quadrature formula for numerical integration of the mean. Then, $W_i = \Phi_{1i}$ is the weight corresponding to the collocation points $\xi^{(i)}$:

$$E[C(\boldsymbol{\rho}, \boldsymbol{\xi})] \approx \sum_{i=1}^M W_i C(\boldsymbol{\rho}, \xi^{(i)}) \quad (31)$$

From Eq. (31), we can compute the standard deviation

$$\sigma^2[C(\boldsymbol{\rho}, \boldsymbol{\xi})] = E[(C(\boldsymbol{\rho}, \boldsymbol{\xi}) - E[C(\boldsymbol{\rho}, \boldsymbol{\xi})])^2]$$

Then,

$$\sigma^2[C(\boldsymbol{\rho}, \boldsymbol{\xi})] \approx \sum_{i=1}^M W_i (C(\boldsymbol{\rho}, \xi^{(i)}) - E[C(\boldsymbol{\rho}, \boldsymbol{\xi})])^2 \quad (32)$$

Finally, we obtain the expression for the standard deviation:

$$\sigma^2[C(\boldsymbol{\rho}, \boldsymbol{\xi})] \approx \sum_{i=1}^M W_i C^2(\boldsymbol{\rho}, \xi^{(i)}) - E^2[C(\boldsymbol{\rho}, \boldsymbol{\xi})] \quad (33)$$

In the present work, the generalized polynomial chaos is adopted to calculate, in an efficient manner, the mean and standard deviation of the compliance with great accuracy and with a small number of determinist model resolutions.

3.2 Sensitivity Analysis

Commonly, optimization algorithms use the gradients during the optimization process to find possible local minima. These gradients are also known as sensitivity analysis, and the field of structural optimization draws much attention because its calculation may be computationally expensive.

An effective method for calculating the sensitivity of the objective function C with respect to the element design variable is to use the adjoint method, by which we can rewrite the equation of the minimal compliance problem by adding at the right-hand side the equilibrium equation $\mathbf{K}\mathbf{u} - \mathbf{f} = 0$, i.e.,

$$C(\boldsymbol{\rho}) = \mathbf{f}^T \mathbf{u} - \boldsymbol{\lambda}^T (\mathbf{K}\mathbf{u} - \mathbf{f}) \quad (34)$$

where $\boldsymbol{\lambda}$ is any arbitrary real vector. Taking derivatives of Eq. (34) with respect to the design variable, we obtain the following expression:

$$\frac{\partial C}{\partial \rho_e} = (\mathbf{f}^T - \boldsymbol{\lambda}^T \mathbf{K}) \frac{\partial \mathbf{u}}{\partial \rho_e} - \boldsymbol{\lambda}^T \frac{\partial \mathbf{K}}{\partial \rho_e} \mathbf{u} \quad (35)$$

When the arbitrary $\boldsymbol{\lambda}$ vector is chosen to eliminate $\partial \mathbf{u} / \partial \rho_e$ from Eq. (35), i.e., $\boldsymbol{\lambda} = \mathbf{u}$, then the adjoint equation is satisfied:

$$\mathbf{f}^T - \boldsymbol{\lambda}^T \mathbf{K} = 0 \quad (36)$$

Therefore, Eq. (35) can be reduced to:

$$\frac{\partial C}{\partial \rho_e} = -\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial \rho_e} \mathbf{u} = -\mathbf{u}_e^T \frac{\partial \mathbf{K}_e}{\partial \rho_e} \mathbf{u}_e \quad (37)$$

where $\mathbf{K}_e$ denotes the stiffness matrix of the element e and $\mathbf{u}_e$ is the displacement vector. Note that Eq. (36) is in the form of an equilibrium equation, so we can directly obtain that $\lambda = \mathbf{u}$ (the minimum compliance is self-adjoint). Therefore, the partial derivative of the stiffness matrix with respect to the element design variable is straightforwardly computed:

$$\frac{\partial C}{\partial \rho_e} = -p(E_0 - E_{\min})\rho^{p-1}\mathbf{u}_e^T \mathbf{K}_e^0 \mathbf{u}_e \quad (38)$$

where $\mathbf{K}_e^0 = \mathbf{K}_e / E(\rho)$ is the element stiffness matrix of the solid material. Therefore, the sensitivity of the compliance with deterministic parameters is easily obtained. The derivative of the constraint function is given by:

$$\frac{\partial V}{\partial \rho_e} = \int_{\Omega} dV \quad (39)$$

The sensitivity of the objective function $\tilde{C}$ with respect to the design variable ρ_e is given as:

$$\frac{\partial \tilde{C}}{\partial \rho_e} = (k-1) \frac{\partial E[C(\boldsymbol{\rho}, \boldsymbol{\xi})]}{\partial \rho_e} + k \frac{\partial \sigma[C(\boldsymbol{\rho}, \boldsymbol{\xi})]}{\partial \rho_e} \quad (40)$$

We can take the direct derivative of the equations that compute the statistical measures through the MCM (Eqs. (24) and (25)):

$$\frac{\partial E[C(\boldsymbol{\rho}, \boldsymbol{\xi})]}{\partial \rho_e} \approx \frac{1}{m} \sum_{i=1}^m \frac{\partial C(\boldsymbol{\rho}, \boldsymbol{\xi}^{(i)})}{\partial \rho_e} \quad (41)$$

and

$$\frac{\partial \sigma[C(\boldsymbol{\rho}, \boldsymbol{\xi})]}{\partial \rho_e} \approx \frac{1}{(m-1)\sigma} \left[\sum_{i=1}^m \left(C(\boldsymbol{\rho}, \boldsymbol{\xi}^{(i)}) \frac{\partial C(\boldsymbol{\rho}, \boldsymbol{\xi}^{(i)})}{\partial \rho_e} \right) - mE[C(\boldsymbol{\rho}, \boldsymbol{\xi})] \frac{\partial E[C(\boldsymbol{\rho}, \boldsymbol{\xi})]}{\partial \rho_e} \right] \quad (42)$$

Similarly, with a direct derivative of the equations that compute the statistical measures through the gPC, we can obtain the sensitivity of $\tilde{C}$

$$\frac{\partial E[C(\boldsymbol{\rho}, \boldsymbol{\xi})]}{\partial \rho_e} \approx \sum_{i=1}^M w_i \frac{\partial C(\boldsymbol{\rho}, \boldsymbol{\xi}^{(i)})}{\partial \rho_e} \quad (43)$$

$$\frac{\partial \sigma[C(\boldsymbol{\rho}, \boldsymbol{\xi})]}{\partial \rho_e} \approx \frac{1}{\sigma} \left[\sum_{i=1}^M \left(w_i C(\boldsymbol{\rho}, \boldsymbol{\xi}^{(i)}) \frac{\partial C(\boldsymbol{\rho}, \boldsymbol{\xi}^{(i)})}{\partial \rho_e} \right) - E[C(\boldsymbol{\rho}, \boldsymbol{\xi})] \frac{\partial E[C(\boldsymbol{\rho}, \boldsymbol{\xi})]}{\partial \rho_e} \right] \quad (44)$$

3.3 RTO algorithm and numerical implementation

The values computed from the TO algorithm, i.e., the compliance and sensitivity information, are used to compute the statistical measures in a non-intrusive way. Therefore, the RTO algorithm for problems with uncertain loading can be formulated as follows:

1. Initialize the problem by setting the order of the gPC and choosing the roots to each random variable according to the orthogonal polynomial;

2. Set the boundary condition for the problem and generate the polygonal element mesh with PolyMesh;
3. Solve the finite element equation $\mathbf{Ku}^{(i)} = \mathbf{f}^{(i)}$ for $i = 1, \dots, M$;
4. Calculate the sensitivity of the compliance C and constraint with respect to ρ_e according to Eq. (38) and Eq. (43);
5. Calculate the coefficients $\hat{c}_j$ of the gPC from Eq. (19);
6. Calculate the statistical measures ($E[\cdot]$ and $\sigma[\cdot]$) according to Eq. (32) and Eq. (33);
7. Calculate the sensitivity of the objective function $\tilde{C}$ according to Eq. (43);
8. Update the design variables $\boldsymbol{\rho}$ by the optimizer. Repeat from step 3 until convergence;

In Fig. 3, we can see the flow-chart of the proposed RTO algorithm integrated with a TO problem.

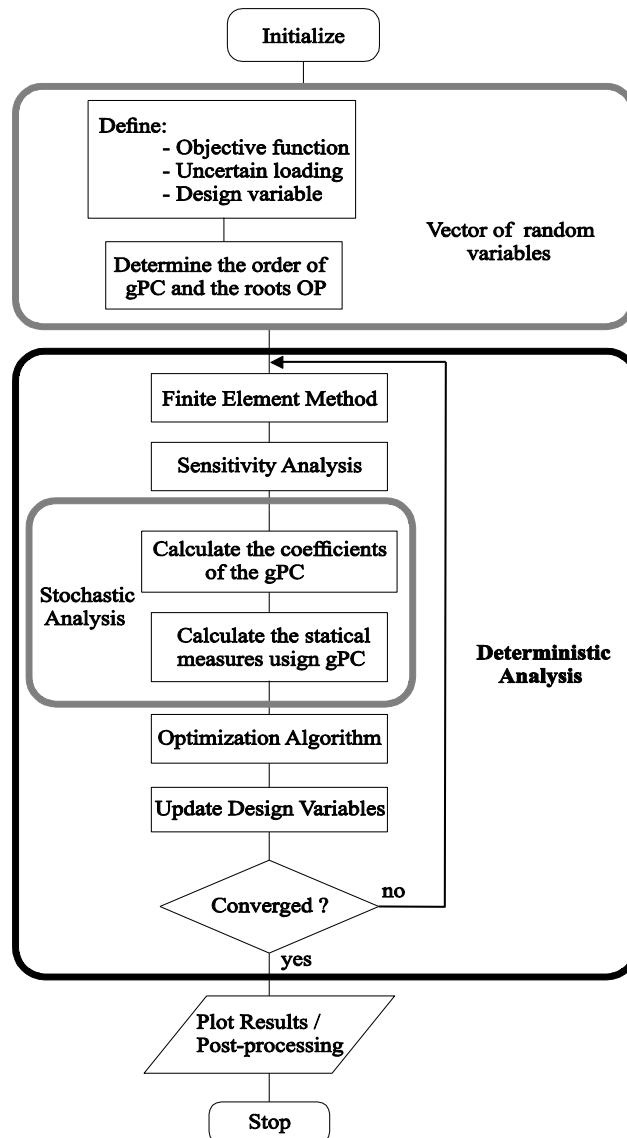


Figure 3. General flow-chart of the RTO- integrated procedure.

4 NUMERICAL EXAMPLES

In this section, we will show the effectiveness of the proposed RTO algorithm w.r.t. a 2D numerical example. This example is used to show the different topologies that can be found within the robust design and the deterministic counterparts. To validate the results, a Monte Carlo simulation with a large number of realizations is performed to compare the accuracy in the calculation of statistical measures with the proposed method.

4.1 2D Supported Beam

In this example, a simply supported beam structure is optimized considering an uncertain loading applied in the middle of the bottom edge. The dimensions of the beam structure are $l = 4m$, $h = 1m$ and $t = 1mm$ for length, height and thickness, respectively. The structure is composed by an isotropic material with a Young's modulus of $E_0 = 1MPa$ and a Poisson's ratio of $\nu = 0.03$. The settings in the robust optimization example are the same as in the deterministic optimization except that the load is the force $f = -2.5N$ in the vertical direction, as shown in Fig. 6a.

The design domain is discretized with a polygonal mesh of 4800 finite elements for elastic analysis. For the purpose of validation, we use a Monte Carlo simulation with 10^4 realizations of the load cases according to its distribution function. The magnitude and direction of the load for the robust optimization example are represented by two independent random variables.

The direction is represented by the angle θ and takes a uniform distribution with the interval from $-\frac{3\pi}{4}$ to $-\frac{\pi}{4}$.

$$f_{dp}(\theta) = \begin{cases} \frac{2}{\pi} & \theta \in \left[-\frac{3\pi}{4}, -\frac{\pi}{4}\right] \\ 0 & \text{else} \end{cases} \quad (45)$$

The magnitude F take a lognormal distribution with mean $\mu_{\log} = 2.5$ and standard deviation $\sigma_{\log} = 1$.

$$f_{dp}(F) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\frac{(\ln x - \mu)^2}{2\sigma^2}\right\} \quad (46)$$

where μ and σ are the mean and standard deviation, respectively, of the associated normal distribution such that:

$$\mu = \ln\left(\frac{\mu_{\log}^2}{\sqrt{\mu_{\log}^2 + \sigma_{\log}^2}}\right) \quad \text{and} \quad \sigma = \sqrt{\ln\left(\frac{\sigma_{\log}^2}{\mu_{\log}^2} + 1\right)} \quad (47)$$

and the joint probability density function for the load is shown in Fig. 4.

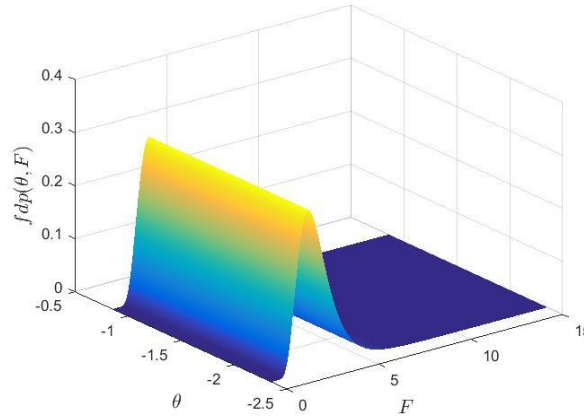


Figure 4. Joint probability density function of the load.

We will use the gPC expansion of order $p=10$ to calculate the mean and standard deviation of the compliance in each iteration. An 11-point tensor product that is used to identify the locations and corresponding weights of the quadrature nodes for the two random variables that compound the uncertain loading provides 121 collocation points, as shown in Fig. 5.

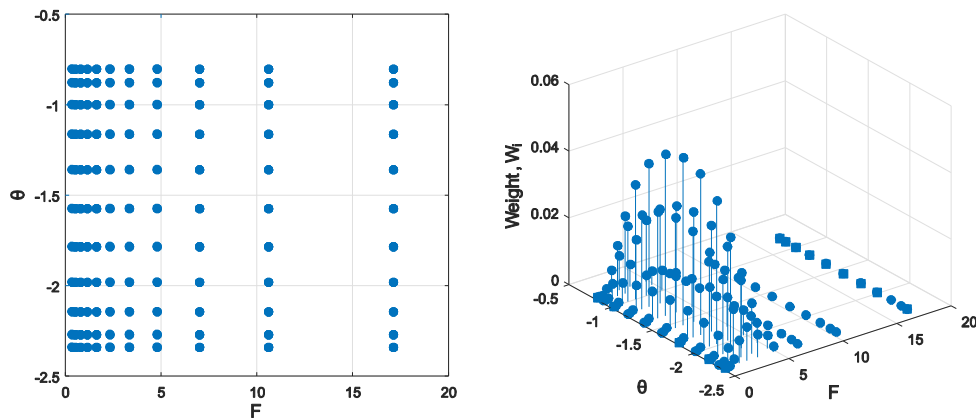


Figure 5. Location of tensor product nodes with their corresponding weights.

As uncertainties are considered during the optimization process, we use the robust formulation and set the weighting factor $k \in [0,1]$ to different values, where $k=1$ means the minimization of the standard deviation of the compliance and $k=0$ corresponds to the minimization of the mean of the compliance. The choice of other values of k will depend on the type of design required.

Considering only a vertical loading, the deterministic TO problem is solved using the conventional algorithm presented in Fig. 1 with a volume fraction $V_s = 0.35$. Then, we obtain a compliance $C = 241.6941$ and a symmetrical pattern as shown in Fig. 6a.

Similarly, for the case where the load is uncertain, we solve the robust problem using a Monte Carlo simulation with $k=0.4$ and a volume fraction $V_s = 0.35$. Then, we obtain $E[C] = 271.5211$ and $\sigma[C] = 252.5134$ with an asymmetrical configuration as shown in Fig. 6b.

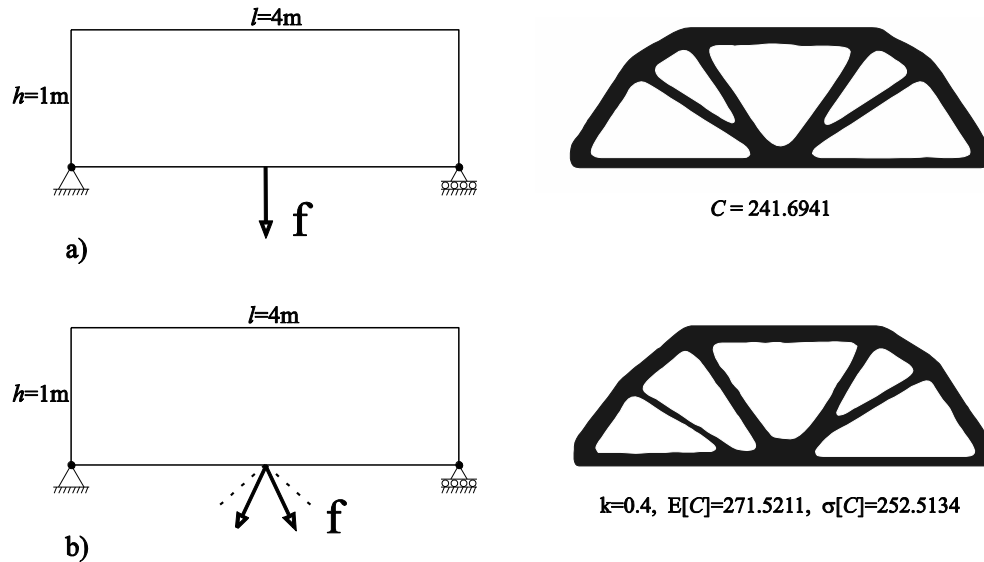


Figure 6. Deterministic (a) and Robust (b) TO of a beam structure.

In Fig. 6, we can see different topologies for the deterministic and robust topology optimizations. The asymmetry obtained in the robust design is due to its randomness, the loading may present a horizontal component, resulting in an asymmetrical boundary condition compared with the deterministic counterpart.

In Table 2, we show the results of the robust solution using the gPC expansion together with the Monte Carlo and the deterministic solutions. Note that the computation time for the deterministic case on an Intel ® Core™ i7- 2.00 GHz CPU is 7.24 min, whereas for the Monte Carlo it is 25.5 hours and for the gPC method only 30.9 min.

Table 2. Validation of the results using gPC and Monte Carlo

$k = 0.4$	Deterministic	gPC	Monte Carlo
$E[C]$	278.7781	271.6222	271.5211
$\sigma[C]$	268.1736	252.1234	252.5134
# of samples	10^4	121	10^4
Time	7.24 min	34.9 min	1530 min

We calculate the sensitivity analysis using the two methods, i.e., the Monte Carlo simulation (Fig. 7a) and gPC expansion (Fig. 7b). Note that the results obtained with both methods are very similar.

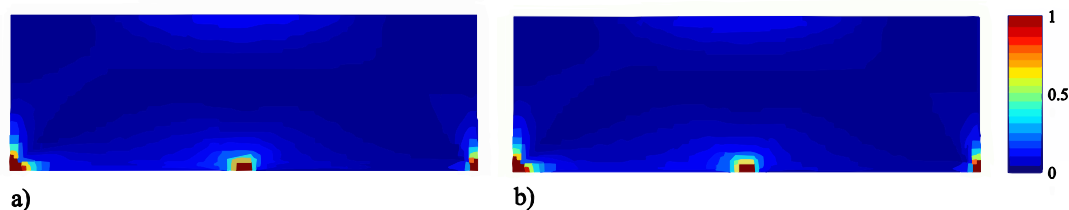


Figure 7. Sensitivity analysis using the MCS (a) and gPC (b) methods.

Next, the robust topology optimization problem is solved for a range of k values using gPC expansion. In Fig. 8, we show that all problems converge reasonably toward expected optimum solutions.

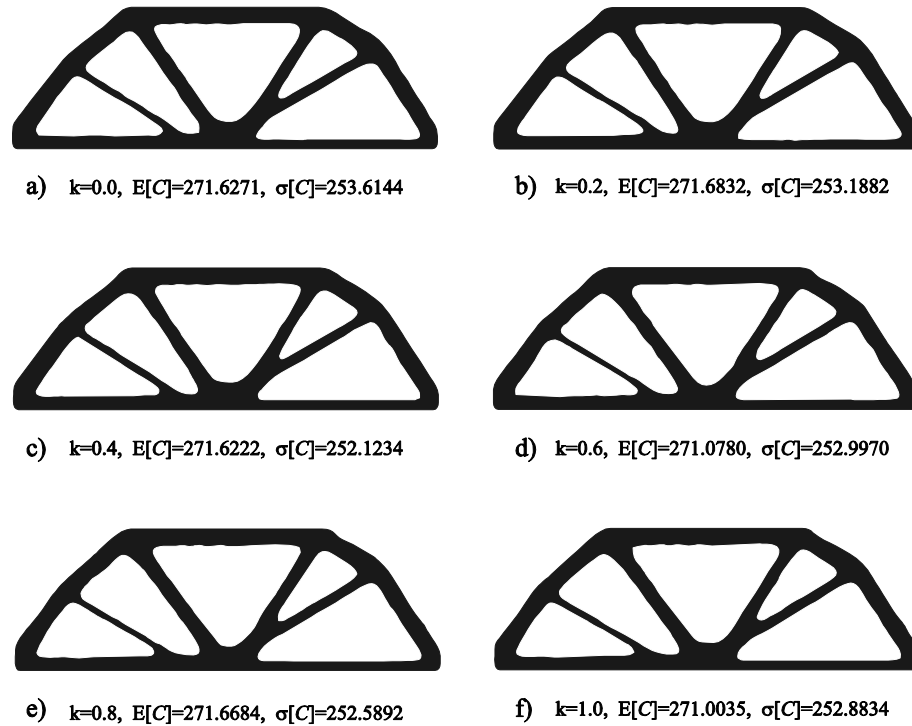


Figure 8. Robust solution of the beam structure using various k values.

Finally, we show in Fig. 9 an examination of the mean and standard deviation values of the optimum solution for the range of k . We obtain a tradeoff relationship in the robust solution, i.e., where the mean increases and the standard deviation decreases as the weight factor increases, which means a considerable improvement of the robustness of the design.

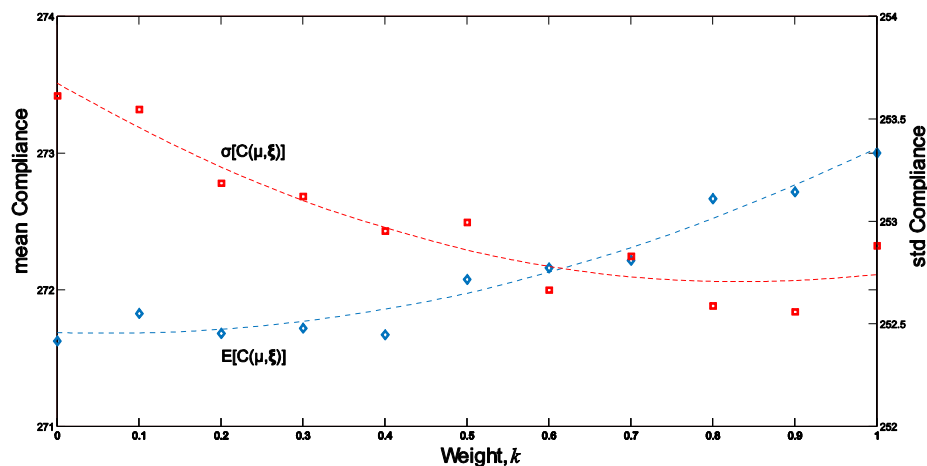


Figure 9. The mean and standard deviation of the compliance for the range of k .

This tradeoff between the two sub-objectives reveals the characteristics and specifications that are essential for the final optimal design.

5 CONCLUSIONS

In the present study, the RTO problem has been formulated and solved by optimization techniques incorporating stochastic spectral expansion based on gPC and an extensive Monte Carlo simulation to verify the accuracy and validation of the proposed approach. This approach was introduced to reduce the variability due to the uncertain loading applied to the mechanical structure. The objective function of the robust problem is represented by the weighted sum of the mean and standard deviation of the compliance, and it can be computed by considering a number of additional load cases. This makes the RTO computationally tractable and accessible by any topology optimization algorithm. Furthermore, the gPC is compatible with RTO for calculating the statistical measures of the compliance.

The numerical example presented here shows a substantial benefit and exhibits topology changes within its design domains compared with the deterministic counterpart. The optimal topology configurations confirm that the uncertainty parameters might change the deterministically obtained optimal topologies. The methodology proposed allows us to obtain approximate outcomes with a much lower computational cost than that via Monte Carlo simulation, which makes it attractive, particularly in iterative structural topology optimization. The limitation of the method is the propagation of a larger number of random variables because the computational cost increases significantly with the dimension. This is often referred to as the curse of dimensionality, and it can be reduced using other techniques, such as Sparse grid.

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