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A VARIATIONAL CONSTITUTIVE MODEL FOR ELASTIC-VISCOPLASTIC MATERIALS SUBJECTED TO PLASTIC DAMAGE AND HYDROLYTIC DEGRADATION

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Abstract. *In this article, a variational constitutive model is presented in order to propose a suitable description for the elastic-viscoplastic behavior of materials subjected to coupled plastic-hydrolytic damage. Such characteristics are usually associated with polymers, namely bioabsorbable ones, which are materials with distinguished properties among those currently used in medical applications. A remarkable feature of bioabsorbable materials is related to hydrolytic degradation. The study and modeling of such a phenomenon is far to be completely understood, being open to investigation and development of new constitutive models. Aiming at the formulation of a constitutive model, an approach based on variational principles may present attractive characteristics like its mathematical structure and its natural association with the description of physical problems. Thus, in this work, the authors present a model based on the framework of the so-called variational constitutive updates in which incremental stress-strain relations can be derived from a pseudo-elastic strain-energy density. The general structure of this method is outlined and then the model formulation is described. Preliminary results obtained by means of standard tests show the viability and the potential of this model to predict the typical mechanical behavior of bioabsorbable materials. Advantages, difficulties and proposals for future work are then discussed.*

Keywords: *variational constitutive updates, viscoplasticity, plastic-hydrolytic damage, finite element*

1 INTRODUCTION

In this paper, a variational constitutive model aimed at representing the elastic-viscoplastic behavior of materials subjected to coupled plastic-hydrolytic damage is presented. Such characteristics are frequently found in an important group of thermoplastics, namely bioabsorbable ones, with distinguished properties for medical applications. Several medical implants, like screws, suture anchors and stents, which are employed in trauma, orthopedic and vascular procedures, have been made of bioabsorbable polymers (Nho et al., 2009; Moore et al., 2010; Suzuki and Ikada, 2011). It is also common knowledge that polymeric materials are highly sensitive to variables, such as strain rate, hydrostatic stress, temperature, as well as the chemical conditions they may be subjected to. For this reason, researchers have put great effort into the development of suitable formulations to describe the thermo-mechanical behavior of such materials.

The bioabsorbable property of polymers is related to the chemical process called hydrolysis, which results in the material degradation when it is in contact with water. A controlled hydrolytic process leads a component made of such materials to degrade progressively once it has been implanted into the human body, being absorbed and/or eliminated by the organism after it has accomplished its function. Despite its importance, the study and modeling of such a phenomenon is far to be completely understood. As a consequence, a wide range of possibilities for the development of constitutive models is still open to investigation, especially those taking into account the coupling of plastic damage and hydrolytic degradation.

Several works found in the literature have suggested approaches for modeling hydrolytic damaging and other analogous processes. For instance, Soares et al. (2010) propose a formulation taking into account the hydrolysis phenomenon in order to describe the response of materials undergoing deformation-induced degradation. In such a model, material parameters are taken as function of the degradation state instead of being material constants. Similar approach appears in Vieira et al. (2011) and Vieira et al. (2014). Muliana and Rajagopal (2012) formulate a constitutive model for hydrolytic degradation accounting for concentration and water diffusion effects. Additionally, Khan and El-Sayed (2013) present a variational model for bioabsorbable materials wherein viscoelastic behavior is considered. Also, in a recently published paper, Fancello et al. (2014) present a constitutive formulation that associates hydrolytic damage effects with the classic Lemaitre's damage model (Lemaitre, 1996; Lemaitre and Desmorat, 2005). The coupled plastic-hydrolytic effect presented in the latter is a distinct characteristic once compared with the previous models, being one of the major objectives of the current proposal.

Among several alternatives to formulate a constitutive model, an approach based on variational principles may present attractive characteristics like its mathematical structure and broad association with the description of physical problems. Thus, in this work, the authors propose a model based on the framework of the so-called variational constitutive updates. This constitutive framework was initially established by Ortiz and Stainier (1999) and Radovitzky and Ortiz (1999), where it is shown that incremental stress-strain relations can be derived from a pseudo-elastic strain-energy density. A considerable amount of publications based on such an approach has arisen since then, covering a vast range of subject areas, from living tissues (Vassoler et al., 2012), thermo-mechanical problems (Fancello et al., 2006; Fancello et al., 2008; Yang et al., 2006; Stainier and Ortiz, 2010; Stainier, 2011; Kintzel and Mosler, 2010; Kintzel et al., 2010; Kintzel and Mosler, 2011; Brassart and Stainier, 2012) up to error estimation and mesh refining procedures (Mosler and Ortiz, 2007; Mosler and Bruhns, 2009).

In order to give a concise description of the model here proposed, without loss of generality, this paper is organized as follows. In Sec. 2, a brief description of the framework of the variational constitutive updates is presented. The core of the model formulation is then shown in Sec. 3, followed by a set of examples and results (Sec. 4), which point out some characteristics of the model and its viability. Then, in Sec. 5, final remarks and suggestions for future work are made.

2 VARIATIONAL CONSTITUTIVE UPDATE

2.1 Continuous formulation

Let a set of state variables be defined as

$$\mathcal{E} = \{\mathbf{F}, Z\} \tag{1}$$

where $\mathbf{F}$ is the deformation gradient of a material point and Z represents internal variables associated with dissipative processes. Assume also that the first Piola-Kirchhoff stress $\mathbf{P}$ and the Helmholtz free-energy W are state variables both functionally dependent on $\mathcal{E}$, that is

$$W = W(\mathcal{E}), \qquad \mathbf{P} = \mathrm{P}(\mathcal{E}). \tag{2}$$

The power $\mathcal{D}$ dissipated by a mechanical process, disregarding thermal effects, is given by the difference between the stress power and the rate of the free-energy (Ortiz and Stainier, 1999),

$$\mathcal{D} = \mathbf{P} : \dot{\mathbf{F}} - \dot{W} \geq 0 \tag{3}$$

Taking the derivative of W with respect to its arguments, substituting it into (4) and by using the Coleman-Noll procedure (Gurtin et al., 2010), the following state equations are obtained:

$$\mathbf{P} = \frac{\partial W}{\partial \mathbf{F}}, \tag{4}$$

$$Q = \frac{\partial W}{\partial Z}, \tag{5}$$

as well as

$$\mathcal{D} = -Q * \dot{Z} \geq 0, \tag{6}$$

where $*$ represents an appropriate product operation.

Expression (6) provides the power dissipation for a change of the system state and must be nonnegative in any thermodynamically consistent model. An effective way of satisfying this inequality is assuming the existence of a potential $\phi = \phi\,(Q; \mathcal{E})$ convex, nonnegative, zero value at the origin (Ortiz and Stainier, 1999) such that

$$-\dot{Z} \in \partial\phi\,(Q)\ , \tag{7}$$

where $\partial\phi\,(Q)$ defines the set of subdifferentials related to $\phi\,(Q)$. By convexity properties, there exist a convex conjugate function ϕ^*such that

$$Q\in\partial\phi^*(-\dot{Z}),\qquad \phi^* = \sup\,\{-\dot{Z}*Q-\phi\,(Q)\},\tag{8}$$

where ϕ^* is the Legendre transform of ϕ. Provided that ϕ^* is Fréchet differentiable, the set $\partial\phi^*(-\dot{Z})$ is reduced to a unique derivative element,

$$Q=-\frac{\partial\phi^*(\dot{Z})}{\partial\dot{Z}}\,.\tag{9}$$

Replacing (9) into the definition (5) for Q, the following evolution equation for the internal variable is obtained:

$$\frac{\partial W(Z)}{\partial Z}+\frac{\partial\phi^*(\dot{Z})}{\partial\dot{Z}}=0.\tag{10}$$

An alternative way of setting state equations (4), (5) and evolution equation (10) that satisfy inequality (6) is defining a potential $\mathcal{P}$ and an effective potential $\tilde{\mathcal{P}}$ as follows:

$$\mathcal{P}(\dot{\mathbf{F}},\dot{Z};\mathcal{E})=\dot{W}(\dot{\mathbf{F}},\dot{Z};\mathcal{E})+\phi^*(\dot{Z};\mathcal{E})=\frac{\partial W}{\partial\mathbf{F}}:\dot{\mathbf{F}}+\frac{\partial W}{\partial Z}*\dot{Z}+\phi^*(\dot{Z};\mathcal{E}),\tag{11}$$

$$\tilde{\mathcal{P}}(\dot{\mathbf{F}};\mathcal{E})=\inf_{\dot{Z}}\mathcal{P}(\dot{\mathbf{F}},\dot{Z};\mathcal{E})\,.\tag{12}$$

It can be easily verified that the optimality condition for the minimization of $\mathcal{P}$ with respect to $\dot{Z}$ provides the evolution equation (10), while the partial derivative of $\tilde{\mathcal{P}}$ (that is, the potential $\mathcal{P}$ evaluated at the minimizing variable $\dot{Z}$) with respect to $\dot{\mathbf{F}}$ results in the state equation (4),

$$\mathbf{P}=\frac{\partial W}{\partial\mathbf{F}}=\frac{\partial\tilde{\mathcal{P}}(\dot{\mathbf{F}};\mathcal{E})}{\partial\dot{\mathbf{F}}}.\tag{13}$$

In other words, $\tilde{\mathcal{P}}$ is a potential for $\mathbf{P}$ in terms of the argument $\dot{\mathbf{F}}$.

2.2 Incremental formulation

Since numerical techniques are to be employed, it becomes convenient to define incremental constitutive laws in the form

$$\mathbf{P}_{n+1}=\mathcal{F}(\mathbf{F}_{n+1},\mathbf{F}_n,Z_n,\Delta t)\,,\tag{14}$$

which may be obtained by means of discrete time integration of Eq. (13). Thus, we define the incremental potential

$$\mathcal{W}(\mathbf{F}_{n+1};\mathcal{E}_n)=\inf_{\dot{Z}}\{W(\mathbf{F}_{n+1};Z_{n+1})-W(\mathbf{F}_n;Z_n)+\Delta t\phi^*(Z_{n+1};Z_n)\}\simeq\int_{t_n}^{t_{n+1}}\tilde{\mathcal{P}}\,dt\,.\tag{15}$$

with the following properties (Ortiz and Stainier, 1999):

1. its minimization with respect to Z_{n+1} yields the incremental evolution equation for Z

$$\frac{\partial W_{n+1}}{\partial Z_{n+1}} + \Delta t \frac{\partial \phi^*(Z_{n+1})}{\partial Z_{n+1}} = 0\ ; \tag{16}$$

2. the partial derivative of $\mathcal{W}$ with respect to $\mathbf{F}_{n+1}$ results in the expression for the stress tensor update

$$\mathbf{P}_{n+1} = \frac{\partial \mathcal{W}(\mathbf{F}_{n+1}; \mathcal{E}_n)}{\partial \mathbf{F}_{n+1}}\ . \tag{17}$$

In the next section, this variational approach is employed in order to propose an elastic-viscoplastic model that couples mechanical and hydrolytic damage effects.

3 VISCOPLASTIC MODEL WITH PLASTIC-HYDROLYTIC DAMAGE

3.1 Initial considerations

The model presented here follows the viscoplastic behavior shown in Brassart and Stainier (2012), being the form of the dissipation function similar to Perzyna-type. The plastic damaging process resembles the classic Lemaitre's model (Lemaitre, 1996; Lemaitre and Desmorat, 2005), with a scalar damage internal variable and the strain equivalence hypothesis. It is assumed that the diffusion of the fluid that degrades the material within the entire domain occurs in a time scale much shorter than the degradation itself, consequently, it is not taken into account in this model. The hydrolytic damage evolution follows the model proposed by Fancello et al. (2014). All the characteristics previously mentioned are then set in the variational framework established by Ortiz and Stainier (1999) and Radovitzky and Ortiz (1999).

3.2 Model description

The model is set in the kinematic framework of finite deformations. Thus, the classic Kröner-Lee multiplicative decomposition of $\mathbf{F}$ is employed:

$$\mathbf{F} = \mathbf{F}^e\mathbf{F}^p. \tag{18}$$

Consequently, the velocity gradient $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ is split into its elastic and plastic contributions

$$\mathbf{L} = \mathbf{L}^e + \mathbf{F}^e\mathbf{L}^p(\mathbf{F}^e)^{-1}, \quad \mathbf{L}^e \equiv \dot{\mathbf{F}}^e(\mathbf{F}^e)^{-1}, \quad \mathbf{L}^p \equiv \dot{\mathbf{F}}^p(\mathbf{F}^p)^{-1}. \tag{19}$$

While $\mathbf{F}^p$ is assumed to be isochoric (i.e. $\det \mathbf{F}^p \equiv 1$), the elastic term $\mathbf{F}^e$ is decomposed into isochoric and volumetric parts:

$$\mathbf{F}^e = \hat{\mathbf{F}}^e\mathbf{F}^{\mathrm{v}}, \qquad \hat{\mathbf{F}}^e = J^{-\frac{1}{3}}\mathbf{F}, \qquad \mathbf{F}^{\mathrm{v}} = J^{\frac{1}{3}}\mathbf{I}, \tag{20}$$

where $\mathbf{I}$ is the second-order identity tensor. The Cauchy-Green isochoric deformation tensor is then

$$\hat{\mathbf{C}}^e = (\hat{\mathbf{F}}^e)^T\hat{\mathbf{F}}^e = \sum_{j=1}^{3} c_j^e \mathbf{E}_j,$$

where the last term represents the spectral decomposition of $\hat{\mathbf{C}}^e$.

The set of state variables $\mathcal{E}$ assumed for the present model is composed by the deformation gradient $\mathbf{F}$, the plastic deformation gradient $\mathbf{F}^p$, the accumulated plastic strain α, the plastic damage d^p and the hydrolytic damage d^h:

$$\mathcal{E} = \left\{\mathbf{F}, \mathbf{F}^p, \alpha, d^h, d^p\right\}.$$

The free-energy potential W is considered to be linearly dependent on the total damage $d = d^p + d^h$, that is

$$W = (1 - d)\,(W^e + W^p)\,, \tag{21}$$

where $W^e$and W^p are the elastic and plastic contributions of the free-energy. One may notice that both terms of W are affected by the total damage d, which is a major difference between the current approach and the classic Lemaitre's model. Such a proposition is reported in papers such as Kintzel and Mosler (2010); Kintzel et al. (2010); Kintzel and Mosler (2011), bringing about a convenient simplification regarding the model and its mathematical formulation as well. The elastic contribution of the free-energy is additively split into volumetric and isochoric terms,

$$W^e(\hat{\mathbf{C}}^e, J) = U(J) + \hat{W}^e(\hat{\mathbf{C}}^e). \tag{22}$$

Once defined the free-energy arguments, its time derivative takes the particular form

$$\dot{W} = \frac{\partial W}{\partial \mathbf{F}} : \dot{\mathbf{F}} + \frac{\partial W}{\partial \mathbf{F}^p} : \dot{\mathbf{F}}^p + \frac{\partial W}{\partial \alpha}\dot{\alpha} + \frac{\partial W}{\partial d^p}\dot{d}^p + \frac{\partial W}{\partial d^h}\dot{d}^h, \tag{23}$$

and from inequality (3), and by applying the Coleman-Noll procedure, the following conjugate forces are obtained:

$$\mathbf{P} = \frac{\partial W}{\partial \mathbf{F}}, \qquad \chi = \frac{\partial W}{\partial \mathbf{F}^p}, \qquad \kappa = \frac{\partial W}{\partial \alpha}, \tag{24}$$

$$-Y^p = \frac{\partial W}{\partial d^p}, \qquad -Y^h = \frac{\partial W}{\partial d^h}. \tag{25}$$

Since $d = d^p + d^h$, assumption (21) leads to the equality $Y = Y^p = Y^h$.

The evolution of the internal variable $\dot{\mathbf{F}}^p$ comes from the definition of the plastic stretching tensor $\mathbf{D}^p$:

$$\mathbf{L}^p = \mathbf{D}^p = \dot{\mathbf{F}}^p\,(\mathbf{F}^p)^{-1}\,, \quad \mathbf{D}^p{=}\dot{\alpha}\mathbf{M}, \quad \mathbf{M} = \sum_{j=1}^{3} q_j \mathbf{M}_j, \tag{26}$$

where the skew part of the plastic velocity gradient $\mathbf{L}^p$ is assumed to be zero, $\mathbf{W}^p = 0$. The scalar $\dot{\alpha} \geq 0$ is the stretching amplitude while $\mathbf{M}$ is the stretching direction. This direction is conveniently decomposed into spectral terms, preserving amplitude and isochoricity properties (Fancello et al., 2006; Fancello et al., 2008). This means eigenvalues q_j and eigenprojections $\mathbf{M}j$ must belong to the following sets:

$$q_j \in \mathcal{K}_Q = \left\{ p_j \in \Re : \sum_{j=1}^{3} p_j = 0;\ \sum_{j=1}^{3} p_j^2 = \frac{3}{2} \right\}, \tag{27}$$

$$\mathbf{M}j \in \mathcal{K}_M = \{\mathbf{N}_j \in \texttt{Sym.} : \mathbf{N}_j : \mathbf{N}_j = 1;\ \mathbf{N}_i : \mathbf{N}_j = 0, i \neq j\} \ . \tag{28}$$

The mechanical damage is assumed to be ductile and proportional to the evolution of the accumulated plastic strain:

$$\dot{d^p} = \dot{\alpha}\frac{Y^S}{N}, \tag{29}$$

where S and N are material constants. Equation (29) establishes a nonholonomic constraint between internal variables d^p and α.

In accordance with the general conditions presented in Sec. 3, a suitable dissipation function is proposed, being split into viscoplastic, mechanical damage and hydrolytic contributions

$$\phi^* = \psi^*_{vp} + \varphi^*_{dp} + \varphi^*_{dh} \ . \tag{30}$$

The potential ψ^*_{vp} is similar to Perzynas's function:

$$\psi^*_{vp} = \begin{cases} (1-d)\left[\sigma_Y \dot{\alpha} + \frac{\mathtt{c} f_A(\alpha)}{\eta+1}\left(\frac{\dot{\alpha}}{\mathtt{c}}\right)^{\eta+1}\right] & \text{se } \dot{\alpha} \geq 0 \\ +\infty & \text{se } \dot{\alpha} < 0 \end{cases} , \tag{31}$$

where d is the total damage, σ_Y is the initial yield stress, $f_A(\alpha)$ is a function associated with material viscoplastic saturation (Chaboche, 2008), and η and $\mathtt{c}$ are material constants. The plastic damage dissipative function φ^*_{dp} is given by

$$\varphi^*_{dp} = \begin{cases} Y\dot{d^p} & \text{se } \dot{d^p} \geq 0 \\ +\infty & \text{se } \dot{d^p} < 0 \end{cases} . \tag{32}$$

Finally, the hydrolytic dissipative function φ^*_{dh} has the form

$$\varphi^*_{dh} = \begin{cases} \frac{R}{2(1-d)^n (Y+g)^{m-1}}\left(\dot{d^h}\right)^2 - g\dot{d^h} & \text{se } \dot{d^h} \geq 0 \\ +\infty & \text{se } \dot{d^h} < 0 \end{cases} , \tag{33}$$

where R, m, g are material constants. The term $g\dot{d^h}$ guarantees the evolution of d^h even in absence of stresses. It is worth noticing that negative values of $\dot{\alpha}, \dot{d^p}$ and $\dot{d^h}$ are avoided due to exact penalization on the corresponding dissipation functions.

Also, let the effective thermodynamics forces be defined as

$$\tilde{\mathbf{P}} = \frac{\mathbf{P}}{(1-d)}, \qquad \tilde{\chi} = \frac{\chi}{(1-d)}, \qquad \tilde{\kappa} = \frac{\kappa}{(1-d)} \ . \tag{34}$$

Then, substituting expressions (31), (32), (33) into (30), the pseudo-potential $\mathcal{P}$ can be written, after some algebraic manipulation, as

$$\mathcal{P} = (1-d)\left[\tilde{\mathbf{P}} : \dot{\mathbf{F}} - \tilde{\chi} : \dot{\mathbf{F}}^p + \tilde{\kappa}\dot{\alpha}\right] - Y\dot{d^h} + (1-d)\left[\sigma_Y\dot{\alpha} + \frac{c f_A(\alpha)}{\eta+1}\left(\frac{\dot{\alpha}}{c}\right)^{\eta+1}\right] + \frac{R}{2(1-d)^n (Y+g)^{m-1}}\left(\dot{d^h}\right)^2 - g\dot{d} \tag{35}$$

and the effective potential as

$$\tilde{\mathcal{P}}(\dot{\mathbf{F}};\mathcal{E}) = \inf_{\dot{\alpha}\geq 0,:q_j\in\mathcal{K}_Q,:\mathbf{M}_j\in\mathcal{K}_M,:\dot{d}^h\geq 0} \mathcal{P}(\dot{\mathbf{F}},\dot{\alpha},q_j,\mathbf{M}_j,\dot{d}^h)\ . \tag{36}$$

3.3 Incremental update formulation

In order to solve numerically the constitutive problem, an incremental potential consistent with (35) has to be defined. This can be performed by a discrete time integration. Assuming the following discrete approximations,

$$\dot{\alpha} := \frac{\Delta\alpha}{\Delta t},\qquad \dot{d} := \frac{\Delta d}{\Delta t},\qquad \Delta d^p = \Delta\alpha\frac{Y^S}{N}, \tag{37}$$

it is possible to define the incremental function

$$\mathcal{W} = \inf_{\Delta\alpha\geq 0,q_j\in\mathcal{K}_Q,\mathbf{M}_j\in\mathcal{K}_M,\Delta d^h\geq 0} \mathcal{P}_\Delta\ , \tag{38}$$

where

$$\mathcal{P}_\Delta = \Delta W + \Delta t\phi^*. \tag{39}$$

Moreover, $\dot{\mathbf{F}}^p = \mathbf{D}^p\mathbf{F}^p$ is evaluated by means of the exponential mapping

$$\mathbf{F}^p_{n+1} = \exp\left[\Delta t\mathbf{D}^p_{n+1}\right]\mathbf{F}^p_n = \exp\left[\Delta\alpha\sum_{j=1}^{3} q_j\mathbf{M}_j\right]\mathbf{F}^p_n\ . \tag{40}$$

And other useful relations between the elastic and plastic deformations are

$$\hat{\mathbf{F}}^e_{n+1} = \hat{\mathbf{F}}_{n+1}\left(\hat{\mathbf{F}}^p_{n+1}\right)^{-1} = \hat{\mathbf{F}}^{pr}_{n+1}\left[\exp\left(\Delta t\mathbf{D}^p\right)\right]^{-1},\qquad \hat{\mathbf{F}}^{pr}_{n+1} = \hat{\mathbf{F}}_{n+1}\left(\hat{\mathbf{F}}^p_n\right)^{-1}, \tag{41}$$

$$\hat{\mathbf{C}}^e_{n+1} = \left(\hat{\mathbf{F}}^e_{n+1}\right)^T\hat{\mathbf{F}}^e_{n+1} = \hat{\mathbf{C}}^{pr}_{n+1}\left[\exp\left(\Delta t\mathbf{D}^p\right)\right]^{-2},\qquad \hat{\mathbf{C}}^{pr}_{n+1} = \left(\hat{\mathbf{F}}^p_n\right)^{-T}\hat{\mathbf{C}}_{n+1}\left(\hat{\mathbf{F}}^p_n\right)^{-1}, \tag{42}$$

$$\hat{\boldsymbol{\varepsilon}}^e_{n+1} = \frac{1}{2}\ln\hat{\mathbf{C}}^e_{n+1} = \boldsymbol{\varepsilon}^{pr}_{n+1} - \Delta t\mathbf{D}^p,\qquad \boldsymbol{\varepsilon}^{pr}_{n+1} = \frac{1}{2}\hat{\mathbf{C}}^{pr}_{n+1} \tag{43}$$

where $(\bullet)^{pr}$indicates predictor quantities. Equation (42) is valid only if $\hat{\mathbf{C}}^{pr}$and $\mathbf{D}^p$are assumed colinear, allowing permutation between both tensors (Fancello et al., 2006; Fancello et al., 2008).

From this point, the constrained optimization procedure must be carried out by means of a lagrangian function defined as

$$\begin{aligned}\mathcal{L}_\Delta = \mathcal{P}_\Delta &+ \lambda_1(\mathbf{M}_a:\mathbf{M}_a-1) + \lambda_2(\mathbf{M}_a:\mathbf{M}_b) + \lambda_3(\mathbf{M}_a:\mathbf{M}_c)\\ &+ \lambda_4(\mathbf{M}_b:\mathbf{M}_b-1) + \lambda_5(\mathbf{M}_b:\mathbf{M}_c) + \lambda_6(\mathbf{M}_c:\mathbf{M}_c-1)\\ &+ (1-d_{n+1})\,\beta_1(q_a+q_b+q_c) + (1-d_{n+1})\,\beta_2(q_a^2+q_b^2+q_c^2-\frac{3}{2})\end{aligned} \tag{44}$$

where $\lambda_{j=1,...,6}$ and $\beta_{j=1,2}$ are lagrangian multipliers. The inclusion of the term $(1-d_{n+1})$ does not modify the constraint and it is convenient for further manipulation.

Following the procedures described by Fancello et al. (2008), the minimization of $\mathcal{L}_\Delta$ with respect to the eigenprojections $\mathbf{M}_j$ is achieved when the tensors $\hat{\mathbf{C}}^{pr}$and $\mathbf{D}^p$ have the same eigenprojections: $\mathbf{E}_j^{pr} = \mathbf{E}_j^e = \mathbf{M}_j$.

Thus, the first order optimality conditions are given by

$$r_j = \frac{\partial \mathcal{L}_\Delta}{\partial q_j} = -\frac{\partial \hat{W}^e_{n+1}}{\partial \hat{\varepsilon}^e_j}\Delta\alpha + \beta_1 + 2\beta_2 q_j = 0 \qquad j = 1,2,3 \tag{45}$$

$$\begin{aligned} r_4 = \frac{\partial \mathcal{L}_\Delta}{\partial \Delta\alpha} = (1-d_{n+1})\left[-\sum_{j=1}^{3}\frac{\partial \hat{W}^e_{n+1}}{\partial \hat{\varepsilon}^e_j}q_j + \frac{\partial W^p_{n+1}}{\partial \Delta\alpha} + \sigma_Y + f_A(\alpha_{n+1})\left(\frac{\mathring{\alpha}}{\mathsf{c}}\right)^{\eta}\right] \\ - \frac{Y^S}{N}\beta_1(q_a+q_b+q_c) - \frac{Y^S}{N}\beta_2(q_a^2+q_b^2+q_c^2-\frac{3}{2}) \geq 0 \end{aligned} \tag{46}$$

$$r_5 = \frac{\partial \mathcal{L}_\Delta}{\partial \beta_1} = (1-d_{n+1})\sum_{j=1}^{3} q_j = 0 \tag{47}$$

$$r_6 = \frac{\partial \mathcal{L}_\Delta}{\partial \beta_2} = (1-d_{n+1})\left(\sum_{j=1}^{3} q_j^2 - \frac{3}{2}\right) = 0 \tag{48}$$

$$\begin{aligned} r_7 = \frac{\partial \mathcal{L}_\Delta}{\partial \Delta d^h} = -\left[Y_{n+1}+g\right] + \frac{R}{\left(1-d_{n)}\right)^n\left(Y_{n+1)}+g\right)^{m-1}}\frac{\Delta d^h}{\Delta t} \\ - \beta_1(q_a+q_b+q_c) - \beta_2(q_a^2+q_b^2+q_c^2-\frac{3}{2}) = 0 \end{aligned} \tag{49}$$

$$\Delta\alpha \geq 0 \tag{50}$$

$$r_4\Delta\alpha = 0 \tag{51}$$

An important aspect to be observed is that, if $r_4 \geq 0$ at $\Delta\alpha = 0^+$, this implies that $\Delta\alpha = 0$ and the incremental problem is updated as an elastic step. Otherwise, the stationary condition $r_4 = 0$ must be achieved for $\Delta\alpha > 0$, and therefore the incremental step is inelastic. This evaluation is comparable to the the elastic-plastic trial test that arises in traditional return-mapping algorithms (Fancello et al., 2008).

If an inelastic step is then verified, conditions (50) and (51) are no more required, and after some algebraic manipulation, the problem may be reduced to the following staggered procedure.

Firstly, find the solution of the viscoplastic problem given by the system of equations

$$-2\frac{\partial \hat{W}^e_{n+1}}{\partial \hat{\varepsilon}^e_1}+\frac{\partial \hat{W}^e_{n+1}}{\partial \hat{\varepsilon}^e_2}+\frac{\partial \hat{W}^e_{n+1}}{\partial \hat{\varepsilon}^e_3}+2Aq_1=0 \tag{52}$$

$$\frac{\partial \hat{W}^e_{n+1}}{\partial \hat{\varepsilon}^e_1}-2\frac{\partial \hat{W}^e_{n+1}}{\partial \hat{\varepsilon}^e_2}+\frac{\partial \hat{W}^e_{n+1}}{\partial \hat{\varepsilon}^e_3}+2Aq_2=0 \tag{53}$$

$$\frac{\partial \hat{W}^e_{n+1}}{\partial \hat{\varepsilon}^e_1}+\frac{\partial \hat{W}^e_{n+1}}{\partial \hat{\varepsilon}^e_2}-2\frac{\partial \hat{W}^e_{n+1}}{\partial \hat{\varepsilon}^e_3}+2Aq_3=0 \tag{54}$$

$$-\frac{\partial \hat{W}^e_{n+1}}{\partial \hat{\varepsilon}^e_1}q_1-\frac{\partial \hat{W}^e_{n+1}}{\partial \hat{\varepsilon}^e_2}q_2-\frac{\partial \hat{W}^e_{n+1}}{\partial \hat{\varepsilon}^e_3}q_3+A=0 \tag{55}$$

where

$$A=\kappa+\sigma_Y+f_A(\alpha_{n+1})\left(\frac{\mathring{\alpha}}{\mathsf{c}}\right)^{\eta},$$

Secondly, obtain the updating of the damage variables,

$$\Delta d^h=\left[(1-d_n)^n\,\frac{(Y_{n+1}+g)^m}{R}\right]\Delta t\,, \tag{56}$$

and

$$\Delta d^p=\Delta\alpha\frac{Y^S_{n+1}}{N}\,. \tag{57}$$

It is worth noting that even under elastic conditions Eq. 56 must be solved.

Having described the the general formulation, some numerical examples and results are shown in the next section in order to evaluate the main capabilities of this model.

4 NUMERICAL EXAMPLES AND RESULTS

In this section, response features of the model are evaluated by means of numerical examples, including tensile and shear tests. In order to represent the material behavior, some particular choices were made. The elastic-isochoric potential is considered to have the form of Ogden's strain energy function (de Souza Neto et al., 2008), which, in terms of the logarithm strains ε_j, is written as

$$\hat{W}^e=\sum_{k=1}^{N}\sum_{j=1}^{3}\frac{\mu_k}{\gamma_k}\left[\exp\left(\varepsilon_j\right)^{\gamma_k}-1\right], \tag{58}$$

where μ_k and γ_k are material constants, and N is the number of terms in the Odgen model. In all the examples that follow, a two terms version $(N=2)$ is employed. Furthermore, the viscoplastic saturation function $f_A(\alpha)$ is taken as a linear function of the accumulated plastic strain (Brassart and Stainier, 2012):

$$f_A(\alpha)=k_v+h\alpha. \tag{59}$$

Table 1: Constitutive material parameters.

Material parameter	Nomenclature	Value
Ogden constant (MPa)	μ_1	1.00
Ogden constant (MPa)	μ_2	142.00
Ogden constant	γ_1	0.05
Ogden constant	γ_2	0.70
Bulk modulus (MPa)	K	294.21
Isotropic hardening modulus (MPa)	H	50.00
Yield stress (MPa)	σ_Y	7.00
Viscoplasticity constant (rate sensitive)	η	0.50
Viscoplasticity constant (viscosity, 1/s)	c	7.00
Saturation function modulus (MPa)	h	0.00
Saturation function constant (MPa)	k_v	7.00
Plastic damage constant	S	1.00
Plastic damage constant	N	0.05
Hydrolytic damage constant	m	1.00
Hydrolytic damage constant	n	1.00
Hydrolytic damage constant	R	200
Hydrolytic damage constant	g	0.0001

As shown in Table 1, specific values are then defined for the set of material constants (hypothetical material).

In the uniaxial tests, a prescribed displacement is imposed to a model (Fig. 1-left) with unitary dimensions (1 mm x 1 mm x 1 mm) in such a way it is deformed up to a total logarithm strain of 0.10. Two constant strain rate conditions are then evaluated. In the first case, the final strain $(\varepsilon_x = 0.10)$ is achieved for a discrete time equivalent to 1 hour, whereas in the second case the final state is reached for a time correspondent to 2400 hours (100 days). In both situations, strain increases linearly with time. In the case of the shear tests, the model is deformed such as indicated in Fig. 1-right, where a displacement of $u_x = 0.15$ mm is prescribed. Such as in the uniaxial tests, the final deformation state is achieved for the equivalent discrete times correspondent respectively to 1 hour and 2400 hours. However, instead of constant strain rates, constant displacement rates are employed.

Figures 2 and 3 present the results for the uniaxial tests, showing respectively stress vs. strain and damage vs. strain curves. In Fig. 2, one can observe the following conditions. First, elastic-viscoplastic effects for a material that does not undergo any damage are illustrated (No

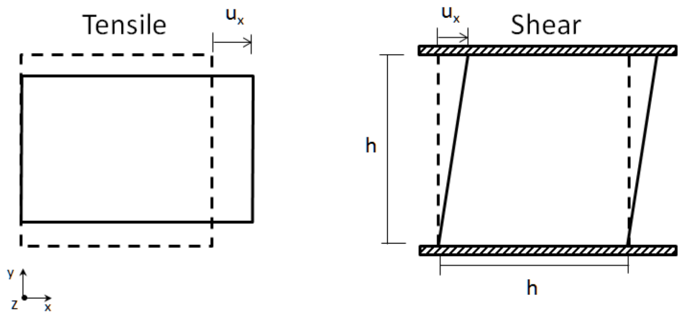

Figure 1: Schematic representation of tensile and shear tests. Dimension h= 1.0 **mm.**

damage). Second, only plastic damage is taken into account (Plastic damage). And finally, total damage effects are presented (Total damage). Hydrolytic degradation effects can be clearly noticed by comparing the results for short- and long-term tests. It can also be pointed out that the hydrolytic degradation increases with deformation and time, followed by a reduction in its evolution rates once the material degradation achieves higher levels (see Fig. 3). This is likely to be a typical characteristic of materials undergoing hydrolytic degradation. As depicted in Fig. 4 and 5, the material resulting behavior under shear conditions is quite similar to those presented for uniaxial tests.

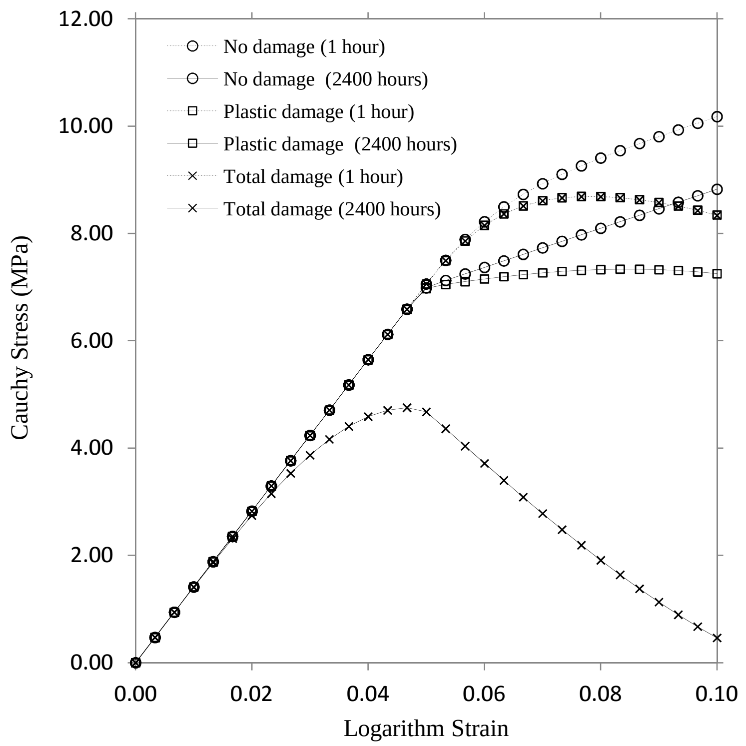

Figure 2: Cauchy stress vs. logarithm strain for short-and long-term tests with constant strain rates.

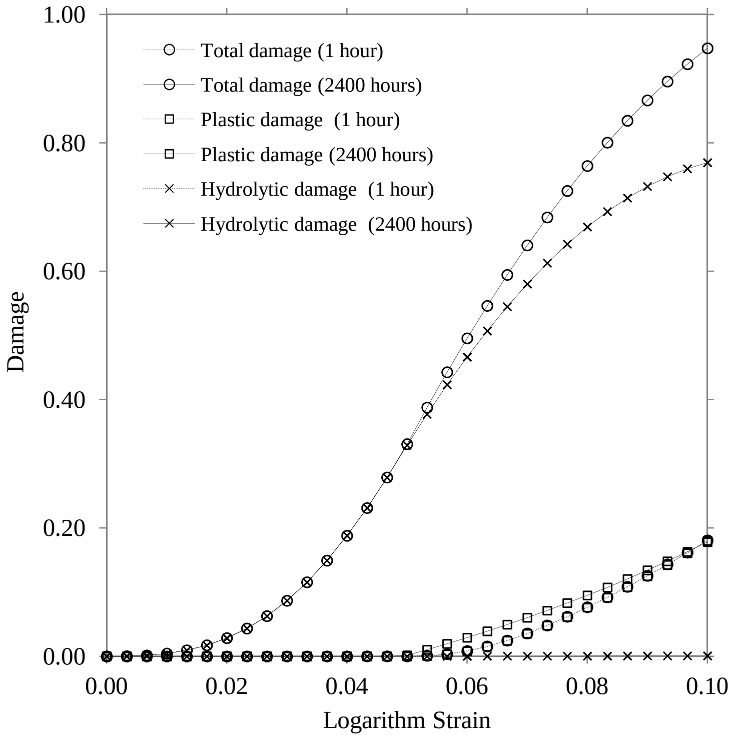

Figure 3: Damage vs. logarithm strain for short-and long-term tests with constant displacement rates.

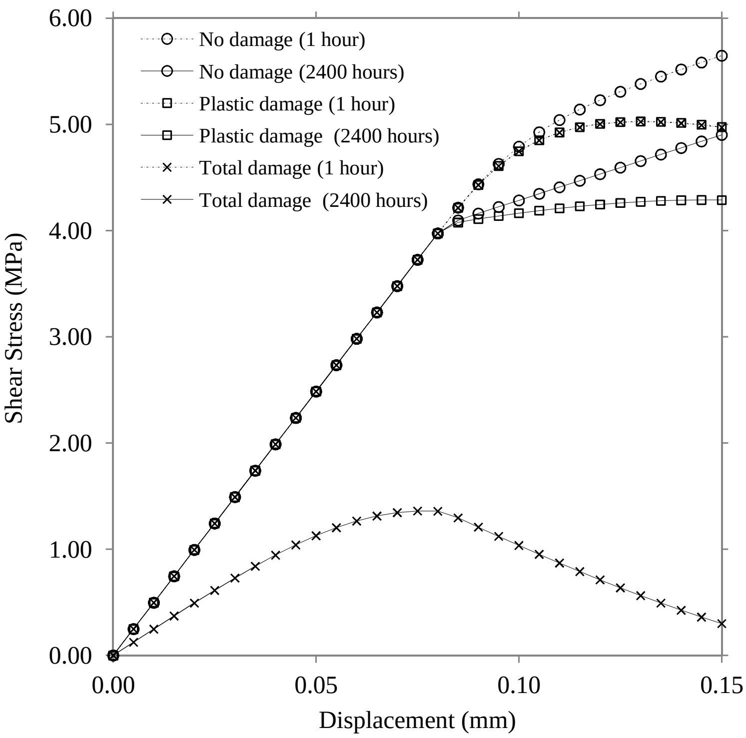

Figure 4: Shear stress vs. displacement for short-and long-term tests with constant strain rates.

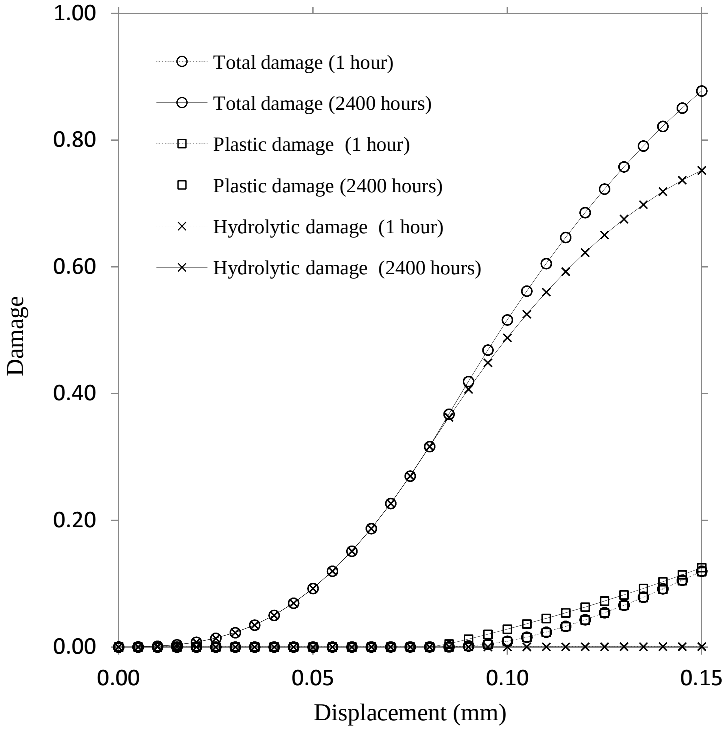

Figure 5: Damage vs. displacement for short-and long-term tests with constant displacement rates.

5 CONCLUSIONS

In this paper, a constitutive model for viscoplastic materials subjected to plastic-hydrolytic damage was presented. By means of preliminary results, and by comparison with those results obtained by Soares et al. (2008) and Fancello et al. (2014), we conclude the current model has sound predictive capabilities to represent the mechanical behavior of bioabsorbable materials. Thus, the grounds for variants of new constitutive models has been established. Future models may include, for instance, alternative forms of hardening formulation, viscoelastic effects, suitable finite element formulation for nearly incompressible and incompressible materials, as well as an analytical tangent modulus for the global Newton-Raphson scheme. Methods for the identification of material parameters can also be proposed and validation procedures may be carried out.

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