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# A SPECTRAL ELEMENT FORMULATION FOR THE DYNAMIC ANALYSIS OF POLYGONAL THIN PLATES

**Nivaldo B. F. Campos**

**José Maria C. Dos Santos**

campos.nivaldo@gmail.com

zema@fem.unicamp.br

Department of Computational Mechanics, UNICAMP - University of Campinas

Rua Mendeleyev, 200 - Cidade Universitária, Campinas, 13083-970, SP, Brazil

***Abstract.*** *The spectral element method (SEM) is a wave-based method well suited for dynamic analysis of structures at mid and high frequencies. In this method, the exact solution for the differential equation of the problem is used to obtain a dynamic stiffness matrix. These solutions, obtained as an infinite series truncated accordingly to the desired precision, are able to describe an infinite number of modes and are obtained by determining the unknown contribution of the wave factors, what is done by introducing the boundary conditions of the problem. If no discontinuities are present, SEM can model a concave domain with just one spectral element whose shape matches the domain boundary. In this work, a polygonal spectral element formulation for the analysis of rectangular thin plates with arbitrary boundary conditions and domain loading was developed. The plate differential equation was solved using an even Fourier series, which made the resulting matrices smaller than the ones found in previous formulations. For the first time a general solution for the plate differential equation was used in the SEM formulation, making it possible to consider the case of domain loading. Both isotropic and orthotropic cases were considered. The results obtained are compared with those from analytical solutions and from finite element models.*

***Keywords:*** *Wave-based methods, Dynamic stiffness matrix method, Treffz methods, Thin plates, Mid frequencies*

# 1 INTRODUCTION

Finite Element Method (FEM) (Zienkiewicz and Taylor, 2000) and the Boundary Element Method (BEM) (Banerjee and Butterfield, 1981), are most commonly used methods for dynamic simulations of mechanical structures. In these methods, small elements are used to discretize the structure and shape functions are used to represent the field variables. This characteristic is responsible for making the dynamic analyses of structures in mid and high frequencies ranges possible only with an unreasonable computational effort. This happens because in order to capture the structural dynamic behavior the elements have to be smaller than the length wave been modeled. At this frequencies ranges, this will make their size too small and therefore the size of the model will increase.

An alternative to overcome this problem is the use of probabilistic techniques such as the Statistical Energy Analysis (SEA) (Lyon and DeJong, 1995). In SEA results at discrete points of the problem domain cannot be achieved since only averaged energy levels are predicted at subdomains of the model. Good results using SEA can be obtained provided the existence of high modal density and light coupling between subsystems in the frequency range of interest, which is not the case in many situations of interest.

Many attractive methodologies based on an indirect Trefftz approach (Jin et al., 1993, Desmet et al., 2001), which can be classified as wave based methods, have been developed. These methods do not require mesh discretization to model a domain with constant geometric and physical properties, since the pressure or displacement fields are described by wave functions that exactly satisfy the differential Eq. of the problem. The solutions, obtained as an infinite series truncated accordingly to the desired precision, are able to describe an infinite number of modes and are obtained by determining the unknown contribution of the wave factors, what is done by introducing the boundary conditions of the problem. The matrices produced are smaller than the ones from FEM and BEM, and in spite of the fact that they are fully populated and frequency dependent, it has proven that the methods are computationally more efficient for the analysis of steady-state vibroacoustic problems.

A comprehensive overview of the methodologies used in the wave based methods is presented by Desmet (2002). Among them, in the area of structural dynamics, it should be mentioned the superposition method, developed by Gorman (1999), and applied mainly to model free-free plates. The image method, first used in modeling acoustic problems, was extended by Gunda et al. (1995) to treat beams and plates. Kulla (1997) presented a high precision finite element method, which was able to model beams and plates with arbitrary boundary conditions. The same approach was used by Kevorkian and Pascal (2001) and Casemir et al. (2005) on the continuous element method. Lee and Lee (1999) applied the spectral element method to model Levy type plates and Doyle (1997) gave a Fourier approach to it. Arruda et al. (2004) extended the work of Lee and Doyle, developing a spectral element for reinforced panels extended the work of Lee and Doyle, developing a spectral element for reinforced Levy type plates and Campos and Arruda (2008) presented a spectral element capable to model thin plates with and without reinforcing beams and arbitrary boundary conditions. Campos (2011) extended the SEM formulation to model rectangular orthotropic plates. Pereira and Dos Santos (2014) presented a SEM formulation to model the energy flow in plates. Papkov and Banerjee (2015) also presented a formulation to model orthotropic plates but using the Dynamic Spectral Matrix Method

In this work, a polygonal spectral element formulation for the analysis of rectangular thin plates with arbitrary boundary conditions and domain loading was developed. The plate differential equation was solved using an even Fourier series, which made the resulting matrices smaller than the ones found in previous formulations. For the first time a general solution for the plate differential equation was used in the SEM formulation, making it possible to consider the case of domain loading. Both isotropic and orthotropic cases were considered. The results obtained were compared with those from analytical solutions and from finite element models.

# 2 SPECTRAL FORMULATION FOR THIN PLATES

In this section, an even Fourier series solution for the differential equations of isotropic and orthotropic thin plate is used to obtain the matrices of displacements and forces, dependents on x, y and the circular frequency$\omega$, at the boundaries of the plate. From these rectangular matrices it is possible, by a new series expansion, to obtain square matrices dependents only on $\omega$, which combined will result in the dynamic stiffness matrix of the problem.

Using general solutions for the differential equations, transverse loads can be applied to the domain of the problem, what was no possible in previous formulations.

## 2.1 Spectral solution for thin plate differential equations

Starting from the governing equation in the frequency domain of an anisotropic thin plate obtained from the work of Lekhnitskii (1968), the lateral mid-surface deflection w satisfies the differential equation

$$D_{11}\frac{\partial^4 w^4}{\partial x^4}+4D_{16}\frac{\partial^4 w}{\partial x^3\partial y}+2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x^2\partial y^2}+4D_{26}\frac{\partial^4 w}{\partial x\partial y^3}+D_{22}\frac{\partial^4 w^4}{\partial y^4}-D_{11}k^4 w=P\left(x,y\right) \quad (1)$$

where

$$k=\sqrt[4]{\omega^2\rho h} \quad (2)$$

with $h$ been the plate thickness and $\rho$ the material density.

Let us now consider an orthotropic rectangular thin plate. The differential Eq. for this kind of structure can be derived from Eq. (1) as

$$D_{11}\frac{\partial^4 w}{\partial x^4}+2\left(D_{12}+2D_{66}\right)\frac{\partial^4 w}{\partial x^2\partial y^2}+D_{22}\frac{\partial^4 w}{\partial y^4}-D_{11}k^4 w=P\left(x,y\right) \quad (3)$$

assuming that $D_{16}=D_{26}=0$

$$\mathbf{D}=\begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66}\end{bmatrix}`=\frac{1}{\det(\mathbf{S})}\frac{h^3}{12}\begin{bmatrix} S_{22}S_{33}-\left(S_{23}\right)^2 & S_{13}S_{23}-S_{12}S_{33} & 0 \\ S_{13}S_{23}-S_{12}S_{33} & S_{11}S_{33}-\left(S_{13}\right)^2 & S_{12}S_{13}-S_{11}S_{23} \\ 0 & 0 & S_{11}S_{22}-\left(S_{12}\right)^2\end{bmatrix} \quad (4)$$

with

$$\mathbf{S} = \begin{bmatrix} \dfrac{1}{E_x} & -\dfrac{\nu}{E_y} & 0 \\ -\dfrac{\nu}{E_x} & \dfrac{1}{E_y} & 0 \\ 0 & 0 & \dfrac{1}{G} \end{bmatrix} \tag{5}$$

and $E_x$ and $E_y$ been the Young Modulus in the $x$ and $y$ direction respectively and $G$ the shear modulus.

In order to solve Eq. (3), in its homogenous form,

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 2\left(D_{12} + 2D_{66}\right)\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w}{\partial y^4} - D_{11}k^4 w = 0 \tag{6}$$

it will be assumed a solution in the form

$$w(x, y; \omega) = C\, e^{px}\, e^{qy} \tag{7}$$

Introducing Eq. (7) into Eq. (6), it will be obtained its characteristic equation as

$$D_{11}p^4 + 2\left(D_{12} + 2D_{66}\right)p^2 q^2 + D_{22}q^4 - k^4 = 0 \tag{8}$$

There are infinite values of $p$ and $q$ that satisfy Eq. (8). Assuming that the solution in the $x$ direction can be expanded as a cosine Fourier series, a general form of $p$ for a rectangular plate with dimensions $L_x$ and $L_y$, and for a given $n \in N$, can be expressed as

$$p_n = \pm\frac{i\, n\, \pi}{L_x} = \pm i\, k_{xn} \quad \text{with} \quad k_{xm} = \frac{n\,\pi}{L_x}, \text{ n = 0, 1, 2, ...} \tag{9}$$

Introducing the expression for $p_n$ into Eq. (8), it will define four values for $q_n$ as

$$q_{1n} = \pm k_{1yn} \quad ; \quad k_{1yn} = \sqrt{\frac{\left(D_{22}k^4 + \left(\left(D_{12} + 2D_{66}\right)^2 - D_{11}D_{22}\right){k_{xn}}^4\right)^{1/2} - \left(D_{12} + 2D_{66}\right){k_{xn}}^2}{D_{22}}} \tag{10}$$

$$q_{2n} = \pm k_{2yn} \quad ; \quad k_{2yn} = \sqrt{\frac{\left(D_{22}k^4 + \left(\left(D_{12} + 2D_{66}\right)^2 - D_{11}D_{22}\right){k_{xn}}^4\right)^{1/2} + \left(D_{12} + 2D_{66}\right){k_{xn}}^2}{D_{22}}} \tag{11}$$

and, therefore, a given $n$ will yield four basis solutions for Eq. (6), which grouped in a set can be expressed as

$$w_{1n} = \left(C_n^1 e^{k_{1yn}y} + C_n^2 e^{-k_{1yn}y} + C_n^3 e^{k_{2yn}y} + C_n^4 e^{k_{2yn}y}\right)\cos\left(k_{xn}\, x\right) \tag{12}$$

where the terms $C_n^1$, $C_n^2$, … , $C_n^8$, are unknown constants to be determined.

Appling the same approach to the solution in the $y$ direction, it will be obtained another set of basis solutions as

$$w_{2n} = \left(C_n^5\, e^{k_{1xn}x} + C_n^6\, e^{-k_{1xn}x} + C_n^7\, e^{k_{2xn}x} + C_n^8\, e^{-k_{2xn}x}\right)\cos\left(k_{yn}\, y\right) \tag{13}$$

with

$$k_{1xn} = \sqrt{\frac{\left(D_{11}k^4 + \left(\left(D_{12}+2D_{66}\right)^2 - D_{11}D_{22}\right)k_{yn}{}^4\right)^{1/2} - \left(D_{12}+2D_{66}\right)k_{yn}{}^2}{D_{11}}} \tag{14}$$

$$k_{2xn} = \sqrt{\frac{\left(D_{11}k^4 + \left(\left(D_{12}+2D_{66}\right)^2 - D_{11}D_{22}\right)k_{yn}{}^4\right)^{1/2} + \left(D_{12}+2D_{66}\right)k_{yn}{}^2}{D_{11}}} \tag{15}$$

The solution for the homogenous differential Eq. (6) will be therefore

$$w(x,y;\omega) = \sum_{n=0}^{\infty} w_{1n} + \sum_{n=0}^{\infty} w_{2n} = \sum_{n=0}^{\infty}\left(w_{1n}+w_{2n}\right) \tag{16}$$

whose explicit expression for a given n is

$$\begin{aligned} w(x,y;\omega)_n = & \left(C_n^1 e^{k_{1yn}y} + C_n^2 e^{-k_{1yn}y} + C_n^3 e^{k_{2yn}y} + C_n^4 e^{k_{2yn}y}\right)\cos\left(k_{xn}\,x\right) + \\ & \left(C_n^5 e^{k_{1xn}x} + C_n^6 e^{-k_{1xn}x} + C_n^7 e^{k_{2xn}x} + C_n^8 e^{-k_{2xn}x}\right)\cos\left(k_{yn}\,y\right) \end{aligned} \tag{17}$$

Defining for a given $n$ a vector of basis functions $\Psi_n$ as

$$\boldsymbol{\psi}_n = \left\{\cos\left(k_{xn}\,x\right)e^{k_{1yn}y} \quad \cos\left(k_{xn}\,x\right)e^{-k_{1yn}y} \quad \cdots \quad \cos\left(k_{yn}\,y\right)e^{k_{2yn}y} \quad \cos\left(k_{yn}\,y\right)e^{-k_{2yn}y}\right\} \tag{18}$$

and a vector $c_n$ of constants of integration

$$\mathbf{c}_n = \left\{C_n^1 \quad C_n^2 \quad \cdots \quad C_n^7 \quad C_n^8\right\}^T \tag{19}$$

Eq. (17) becomes

$$w(x,y;\omega)_n = \boldsymbol{\psi}_n \cdot \mathbf{c}_n \tag{20}$$

and a general expression for the displacements, in vector form, will be given by

$$w(x,y;\omega) = \boldsymbol{\psi} \cdot \mathbf{c} \tag{21}$$

For an isotropic plate, the same solution was found (Campos and Arruda, 2008), but since in this case the differential equation is given by

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} - k^4 w = \frac{P\left(x,y\right)}{D} \tag{22}$$

where

$$k = \sqrt[4]{\frac{\omega^2 \rho h}{D}}; \qquad D = \frac{E\,h^3}{12\left(1-\nu^2\right)} \tag{23}$$

the expressions in Eq. 10, 11, 14 and 15 will become

$$k_{1yn} = \sqrt{k^2 - k_{xn}{}^2}; \quad k_{2yn} = \sqrt{k^2 + k_{xn}{}^2}; \quad k_{1xn} = \sqrt{k^2 - k_{yn}{}^2}; \quad k_{2xn} = \sqrt{k^2 + k_{yn}{}^2} \tag{24}$$

## 2.2 Spectral dynamic stiffness matrix

The thin plate spectral dynamic stiffness matrix can be obtained by writing the shear forces and the moments as a function of the displacements and slopes at the boundaries along the x and y directions. These terms are defined by the well-known relations

$$\phi_x\left(x,y;\omega\right) = -\frac{\partial w(x,y;\omega)}{\partial x} \tag{25}$$

$$\phi_y\left(x,y;\omega\right) = -\frac{\partial w(x,y;\omega)}{\partial y} \tag{26}$$

$$\mathrm{M}_x\left(x,y;\omega\right) = -\left(D_{11}\frac{\partial^2 w(x,y;\omega)}{\partial x^2} + D_{12}\frac{\partial^2 w(x,y;\omega)}{\partial y^2}\right) \tag{27}$$

$$\mathrm{M}_y\left(x,y;\omega\right) = -\left(D_{12}\frac{\partial^2 w(x,y;\omega)}{\partial x^2} + D_{22}\frac{\partial^2 w(x,y;\omega)}{\partial y^2}\right) \tag{28}$$

$$\mathrm{V}_x\left(x,y;\omega\right) = -\left(D_{11}\frac{\partial^3 w(x,y;\omega)}{\partial x^3} + \left(D_{12}+4D_{66}\right)\frac{\partial^3 w(x,y;\omega)}{\partial x\,\partial y^2}\right) \tag{29}$$

$$\mathrm{V}_y\left(x,y;\omega\right) = -\left(D_{22}\frac{\partial^3 w(x,y;\omega)}{\partial y^3} + \left(D_{12}+4D_{66}\right)\frac{\partial^3 w(x,y;\omega)}{\partial x\partial y^2}\right) \tag{30}$$

Evaluating Eq. (16), (25) and (26) at the boundaries and assembling the results, a vector $\tilde{\mathbf{d}}$ of displacements on the boundaries, is obtained

$$\tilde{\mathbf{d}} = \tilde{\mathbf{D}}.\mathbf{c} \tag{31}$$

where

$$\tilde{\mathbf{d}}(x,y) = \left\{ w\left(0,y\right) \quad w\left(L_x,y\right) \quad w\left(x,0\right) \quad w\left(x,L_y\right) \quad \phi_x\left(0,y\right) \quad \phi_x\left(L_x,y\right) \quad \phi_y\left(x,0\right) \quad \phi_y\left(x,L_y\right)\right\}^T \tag{32}$$

$$\tilde{\mathbf{D}}(x,y) = \begin{bmatrix} d_{1,0}^1 & \cdots & d_{1,0}^8 & d_{1,1}^1 & \cdots & d_{1,1}^8 & \cdots & d_{1,n}^1 & \cdots & d_{1,n}^1 \\ d_{2,0}^1 & \cdots & d_{2,0}^8 & d_{2,1}^1 & \cdots & d_{2,1}^8 & \cdots & d_{2,n}^1 & \cdots & d_{2,n}^8 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ d_{8,0}^1 & \cdots & d_{8,0}^8 & d_{8,1}^1 & \cdots & d_{8,1}^8 & \cdots & d_{1,n}^8 & \cdots & d_{8,n}^8 \end{bmatrix} \tag{33}$$

$$\mathbf{c} = \left\{ C_0^1 \quad \cdots \quad C_0^8 \quad C_1^1 \quad \cdots \quad C_1^8 \quad \cdots \quad C_n^1 \quad \cdots \quad C_n^8 \right\}^T \tag{34}$$

Proceeding in the same way in relation to the forces at the boundaries, it will be obtained

$$\tilde{\mathbf{f}} = \tilde{\mathbf{F}}\cdot\mathbf{c} \tag{35}$$

where

$$\tilde{\mathbf{f}} = \left[ \text{-}\mathrm{V}_x\left(0,y\right) \quad \mathrm{V}_x\left(L_x,y\right) \quad \text{-}\mathrm{V}_y\left(x,0\right) \quad \mathrm{V}_y\left(x,L_y\right) \quad \text{-}\mathrm{M}_x\left(0,y\right) \quad M_x\left(L_x,y\right) \quad \text{-}\mathrm{M}_y\left(x,0\right) \quad M_y\left(x,L_y\right)\right]^T \tag{36}$$

$$\tilde{\mathbf{F}} = \begin{bmatrix} f_{1,0}^{1} & \cdots & f_{1,0}^{8} & f_{1,1}^{1} & \cdots & f_{1,1}^{8} & \cdots & f_{1,n}^{1} & \cdots & f_{1,n}^{8} \\ f_{2,0}^{1} & \cdots & f_{2,0}^{8} & f_{2,1}^{1} & \cdots & f_{2,1}^{8} & \cdots & f_{2,n}^{1} & \cdots & f_{2,n}^{8} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{8,0}^{1} & \cdots & f_{8,0}^{8} & f_{8,1}^{1} & \cdots & f_{8,1}^{8} & \cdots & f_{8,n}^{1} & \cdots & f_{8,n}^{8} \end{bmatrix} \tag{37}$$

In order to eliminate the dependence on *x* and *y* of $\tilde{\mathbf{d}}, \tilde{\mathbf{D}}, \tilde{\mathbf{f}}$ and $\tilde{\mathbf{F}}$, they will be expanded in a cosine Fourier series. Truncating this series at an adequate number of terms, the resulting matrices containing its coefficients will be square, and expressions for the vector of unknown constants **c** can be found as:

$$\mathbf{f} = \mathbf{F} \cdot \mathbf{c} \quad \Rightarrow \quad \mathbf{c} = \mathbf{F}^{-1} \cdot \mathbf{f} \tag{38}$$

$$\mathbf{d} = \mathbf{D} \cdot \mathbf{c} \quad \Rightarrow \quad \mathbf{c} = \mathbf{D}^{-1} \cdot \mathbf{d} \tag{39}$$

Eliminating **c** from Eq. (38) and (39), yields

$$\mathbf{S} \cdot \mathbf{d} = \mathbf{f} \tag{40}$$

where

$$\mathbf{S} = \mathbf{F} \cdot \mathbf{D}^{-1} \tag{41}$$

with **S** been the dynamic spectral stiffness matrix.

For plates with free (FFFF) or clamped (CCCC) boundary conditions, its natural frequencies and modes can be obtained by setting respectively the vectors **f** or **d** equal to zero on Eq. (40) and solving the resulting eigenproblem. The vector of coefficients **c** can be obtained by solving Eq. (39) or (38) respectively. For mixed boundary conditions (for example FFCC), the non nulls terms of vectors **f** and **d** should be merged in a new vector and the corresponding terms of matrices **F** and **D** in a new matrix, allowing the vector **c** to be determined from an expression similar to Eq. (38) or (39).

## 2.3 Case of inhomogeneous boundary conditions

Forces or displacements can be imposed to the plate as boundary conditions by expanding them in the same Fourier series used in obtaining the dynamic stiffness matrix and inserting the coefficients obtained in the corresponding positions of vectors **d** or **f**. For the particular case of a shear force $V = V_0\ \delta(y-y_0)$ imposed to an edge along the y direction and at an arbitrary position $y=y_0$, the expansion in a Fourier series will produce coefficients given by

$$\mathrm{f}_o = \int_0^{L_y} V_0\, \delta\left(y - y_0\right) dy; \qquad \mathrm{f}_n = \int_0^{L_y} V_0 \delta\left(y - y_0\right) \cos\frac{n\pi y}{L_y} dy, \quad n = 1, 2, 3... \tag{42}$$

which, after integrating results in

$$\mathrm{f}_o = \frac{1}{L_y}; \quad \mathrm{f}_n = \frac{2}{L_y} \cos\frac{n\pi y_0}{L_y}, \quad n = 1, 2, 3.... \tag{43}$$

Introducing these coefficients in **f** and imposing the boundary conditions on Eq. (38), in the same way as it is done in the finite element method, the value of **c** can be obtained which, introduced in Eq. (21), will completely determine the displacement field for the homogeneous case as

$$w_H(x,y;\omega) = \boldsymbol{\psi}\cdot\mathbf{F}^{-1}\cdot\mathbf{f} \tag{44}$$

## 2.4 Domain loads

In order to introduce domain loads to the spectral element method formulation it is necessary to find a general solution for the thin plate differential equation.

For the case of a transverse harmonic point force applied at a point ($x_o$, $y_o$), a force $P(x,y,t)$ can be expressed as:

$$P(x,y;t) = P_0\,\delta\left(x-x_0\right)\left(y-y_0\right)e^{i\omega t} \tag{45}$$

In the frequency domain, this force can be assumed in the form of a double cosine Fourier series as

$$P(x,y;\omega) = \frac{P_0}{Lx\,Ly}\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}C_{m,n}\cos\left(\frac{\pi\mathrm{m}\,\mathrm{x}}{Lx}\right)\cos\left(\frac{\pi\mathrm{n}\,y}{Ly}\right)\right) \tag{46}$$

where

$$C_{m,n} = \alpha_{m,n}\cos\left(\frac{\pi\mathrm{m}\,\mathrm{x}_0}{Lx}\right)\cos\left(\frac{\pi\mathrm{n}\,y_0}{Ly}\right) \tag{47}$$

and

$$\begin{aligned}
&\alpha_{m,n} = 1 \;\; for\;\; m=0 \;and\; n=0\\
&\alpha_{m,n} = 2 \;\; for\;\; m=0 \;or\; n=0\\
&\alpha_{m,n} = 4 \;\; for\;\; m\neq 0 \;and\; n\neq 0
\end{aligned} \tag{48}$$

Assuming that a particular solution $w_p$ for the plate differential equation has the same form as the force applied, results in

$$w_P = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{m,n}\cos\left(\frac{\pi\mathrm{m}\,\mathrm{x}}{Lx}\right)\cos\left(\frac{\pi\mathrm{n}\,y}{Ly}\right) \tag{49}$$

were for the isotropic case

$$A_{m,n} = \frac{\alpha_{m,n}\,P_0}{D\,Lx\,Ly\left(\pi^4\left(\dfrac{n^2}{Lx^2}+\dfrac{m^2}{Ly^2}\right)^2-k^4\right)}\cos\left(\frac{\pi\mathrm{m}\,\mathrm{x}_0}{Lx}\right)\cos\left(\frac{\pi\mathrm{n}\,y_0}{Ly}\right) \tag{50}$$

and for the orthotropic case

$$A_{m,n} = \frac{\alpha_{m,n}\,P_0}{D_{11}Lx\,Ly\left(\dfrac{D_{22}Lx^4m^4+D_{11}Ly^4n^4+\left(2D_{11}+4D_{66}\right)Lx^2Ly^2m^2n^2}{D_{11}Lx^4Ly^4}\pi^4-k^4\right)}\cos\left(\frac{\pi\;\mathrm{m}\,\mathrm{x}_0}{Lx}\right)\cos\left(\frac{\pi\;\mathrm{n}\,y_0}{Ly}\right) \tag{51}$$

Therefore a general solution $w_G$ will be

$$w_G = w_H + w_P \tag{52}$$

Introducing the expression for the displacement from Eq. (52) into Eq. (22) to (27), expressions for rotations, bending and shear forces for the general case are obtained. Equations (38) and (39) can then be rewritten as

$$\tilde{\mathbf{f}}_H - \tilde{\mathbf{f}}_P = \tilde{\mathbf{F}} \cdot \mathbf{c} \;\; \Rightarrow \;\; \mathbf{c} = \mathbf{F}^{-1} \cdot \left( \tilde{\mathbf{f}}_H - \tilde{\mathbf{f}}_P \right) \tag{53}$$

$$\tilde{\mathbf{d}}_H - \tilde{\mathbf{d}}_P = \tilde{\mathbf{D}}.\mathbf{c} \;\; \Rightarrow \;\; \mathbf{c} = \mathbf{D}^{-1} \cdot \left( \tilde{\mathbf{d}}_H - \tilde{\mathbf{d}}_P \right) \tag{54}$$

and the expression for the plate transverse displacement will be given by

$$w_G(x, y; \omega) \;\; = \boldsymbol{\psi} \cdot \mathbf{c} + w_p(x, y; \omega) \tag{55}$$

# 3 NUMERICAL VALIDATION

## 3.1 Isotropic simple supported rectangular plate

An isotropic simple supported rectangular plate subjected to a point load has a modal solution for the transverse displacements in the frequency domain given by the series

$$w(x, y; \omega) = \frac{4 P_0}{D\, Lx\, Ly \left( \pi^4 \left( \frac{n^2}{Lx^2} + \frac{m^2}{Ly^2} \right)^2 - k^4 \right)} \sin\left( \frac{\pi \mathrm{m}\, \mathrm{x}_0}{Lx} \right) \sin\left( \frac{\pi \mathrm{n}\, y_0}{Ly} \right) \sin\left( \frac{\pi \mathrm{m}\, \mathrm{x}}{Lx} \right) \sin\left( \frac{\pi \mathrm{n}\, y}{Ly} \right) \tag{56}$$

This solution was applied to a rectangular thin plate with the following properties: E = 70 MPa, ρ = 2700 Kg/m$^3$, ν = 0.3, h = 0.001 m, $L_x$ = 0.20 m, $L_y$= 0.30 m. An unitary harmonic shear force is imposed at a point ($x_o$,$y_o$)=(0.13, 0.16), and the resulting displacements are measured at the same point.

The same problem was solved using SEM with 25 terms of the Fourier series expansion. The FRFs obtained with both methods are compared in Fig. 1 and they are in excellent agreement.

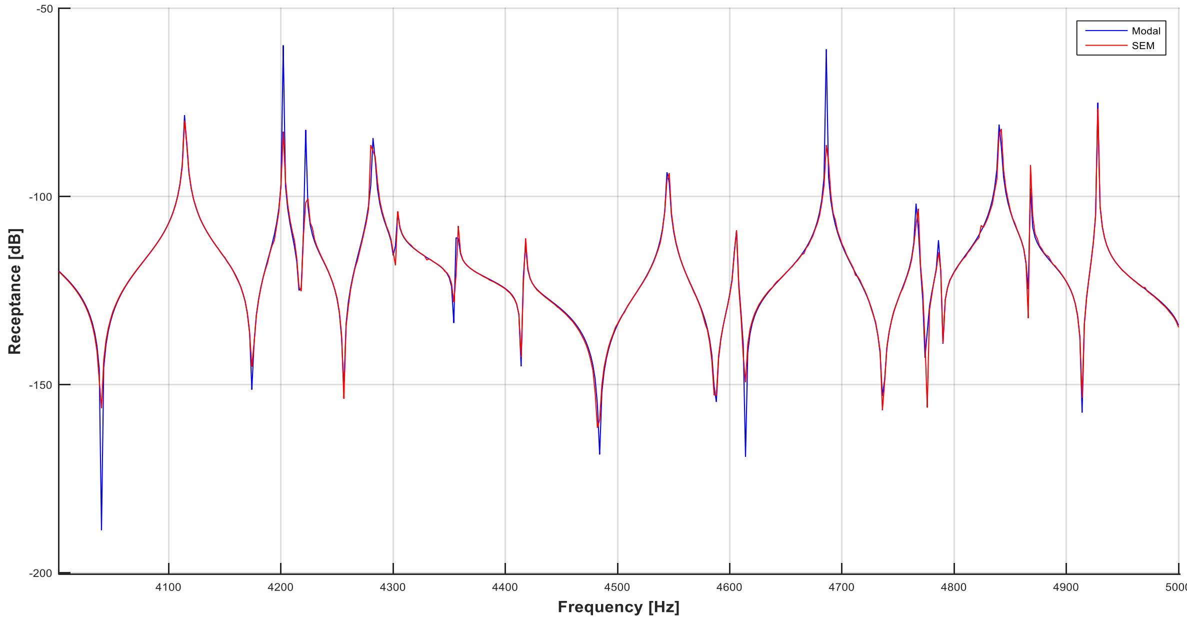


**Figure 1: Simple support plate with a punctual unitary shear force imposed at an edge.**

## 3.2 Isotropic free (FFFF) rectangular plate

A rectangular thin plate with the following properties: E = 70 MPa, $\rho$ = 2700 Kg/m$^3$, $\nu$ = 0.3, h = 0.001 m, $L_x$ = 0.20m, $L_y$= 0.20 m and a unitary harmonic shear force imposed at a point $(x_o,y_o)$=(0.04, 0.08), and the resulting displacements measured at the same point was obtained through an ANSYS model with 250 shell63 elements.

Solving the problem again using SEM with 12 terms of the Fourier series expansion and plotting in Fig. 2 both FRFs obtained, it can be observed again an excellent agreement also between these two methods.

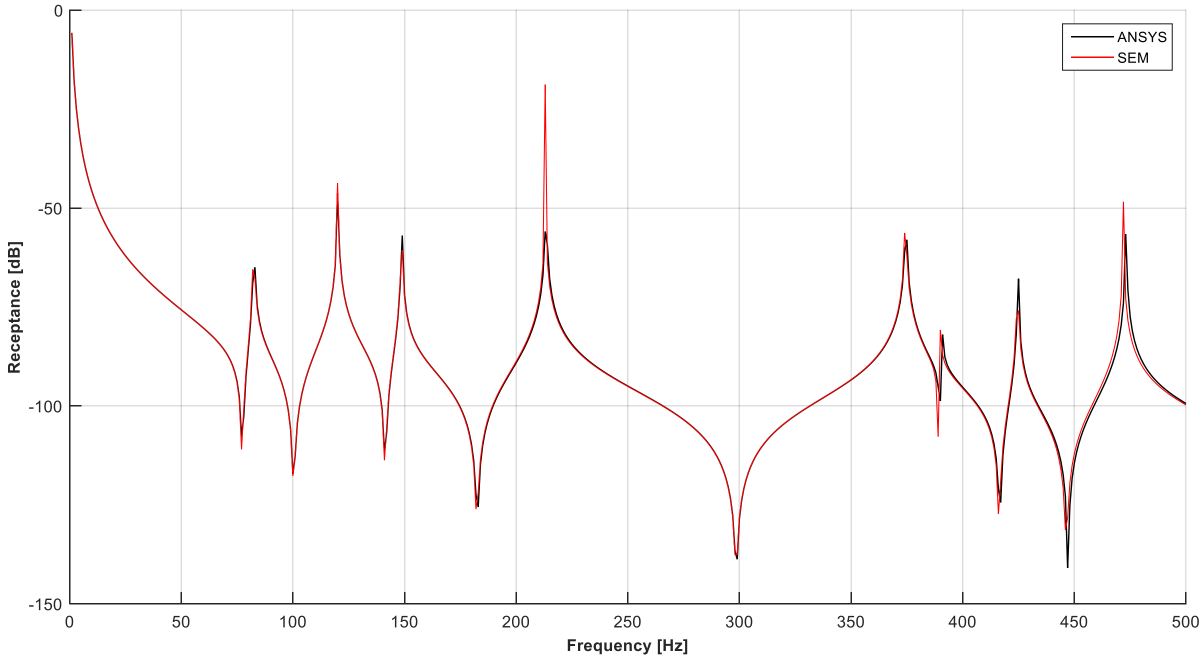


**Figure 2: Isotropic FFFF plate with a punctual unitary shear force imposed at an edge.**

## 3.3 Orthotropic free (FFFF) rectangular plate

In order to verify the SEM formulation using only an even Fourier series in place of a series having cosine and sine terms, it was performed a dynamic analysis of a free-free plate used by Grédiac and Paris (1996) in obtaining the elastic constants of composite materials. This analysis has also been done by Campos (2011) using the usual SEM formulation. The plate has the following physical and geometric properties: $E_x$ = 120 MPa, $E_y$ = 10 MPa, G = 4.9 MPa, $\rho$ = 1510 Kg/m$^3$, $\nu$ = 0.3, h = 0.001 m, $L_x$ = $L_y$= 0.20 m and a unitary harmonic shear force is imposed at a point in its boundary, as showed in Fig. 3.

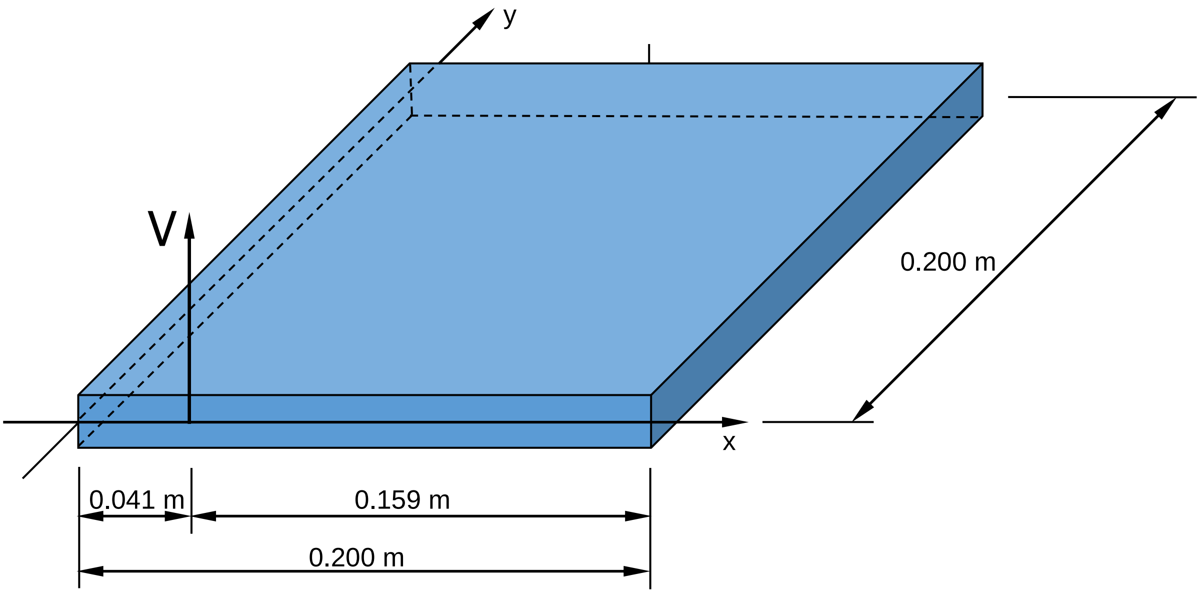


**Figure 3: Free-Free plate with a punctual unitary shear force imposed at an edge.**

The frequency response function (FRF) of the plate at the loading point was obtained using just one SEM element and an increasing number of Fourier series terms. The results showed that for this frequency range convergence is achieved with just 5 terms, and the FRF for this case is showed in Fig. 4.

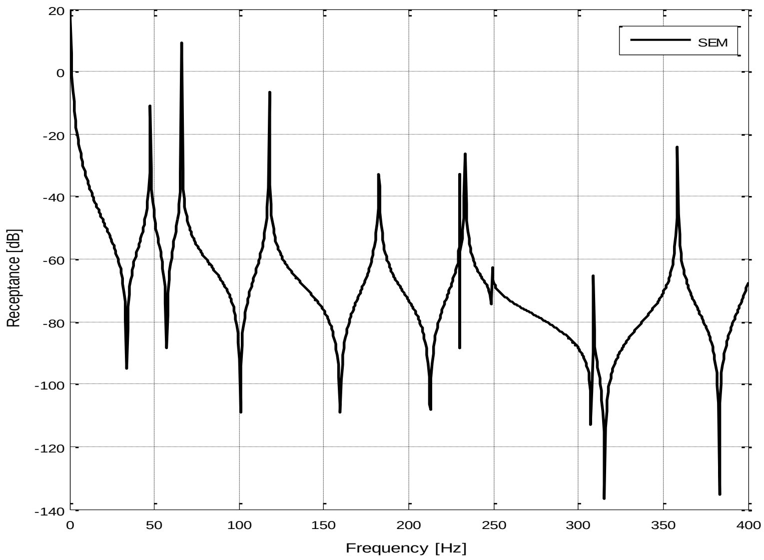


**Figure 4: Frequency response function for a low frequency range**

The plate first nine natural frequencies were extracted from the FRF and compared with that obtained by Gréniac. Both results are in good agreement as shown in Table 1, and there is no difference in the results obtained in this work and in the work of Campos (2011) with the usual SEM formulation

**Table 1. Comparison of natural frequencies obtained by Gréniac e by SEM**

| Natural Frequencies | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Gréniac | 47,5 | 65,7 | 117,4 | 180,4 | 227,9 | 232,6 | 247,1 | 306,3 | 354,9 |
| SEM | 47,6 | 66,1 | 117,7 | 182,5 | 229,7 | 233,3 | 249,1 | 309,0 | 358,5 |
| Variation (%) | 0,21 | 0,61 | 0,26 | 1,16 | 0,79 | 0,30 | 0,81 | 0,88 | 1,01 |

# 4 FINAL REMARKS

It was developed a spectral element that can be used to model isotropic and orthotropic thin plates with arbitrary boundary conditions with point loads that can be applied to the domain of the problem and a detailed description on how to obtain all the terms needed to implement it was presented.

The use in the formulation of an even Fourier series have been proved to be as precise as the usual SEM formulation the advantage of the resulting matrices been much smaller. Also for the first time it was presented a SEM formulation that make it possible to apply loads to problem domain, overcoming one of the main limitations of this method.

The results obtained proved to be appealing and its accuracy and computational cost when compared with FEM make it a potential tool for structural analysis in mid and high frequency ranges. Further development to allow the application of any kind of loads, domains with polygonal shapes other than rectangular and structures made of anisotropic materials are been carrying out in order to make it possible to apply the method to a larger number of structures.

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# REFERENCES

Arruda, J.R.F., Donadon, L.V., Nunes, R.F. and Albuquerque, E.L., 2004, “On the Modelling of Reinforced Plates in the mid-frequency range”, INTERNOISE 2004 - The 13th International Congress and Exposition on Noise Control Engineering, Prague, Czech Republic, August 22-25.

Brebbia, C, Dominguez, J., 1989,“Boundary Element an Introductory Course”, Computation Mechanics Publications, 2nd Edition, Southampton.

Campos, N. B. F., ARRUDA, J.R.F., 2008, “On the Modeling of Beam Reinforced Thin Plates Using the Spectral Element Method”, In: Special Issue of the journal Shock and Vibration.

Campos, N.B.F. 2011, “Structural Modeling of Orthotropic Thin Plates by the Spectral Element Method”, DINAME 2011 - XIV International Symposium on Dynamic Problems of Mechanics, São Sebastião, SP , Brazil, March 13th - March 18th, 2011.

Casimir, J.B., Kevorkian, S. and Vinh, T., 2005, “The dynamic stiffness matrix of two-dimensional elements: application to Kirchhoff’s plate continuous elements”, Journal of Sound and Vibration 287, pp. 571–589.

Desmet, W., 2002, "Mid-frequency vibro-acoustic modelling: challenges and potential solutions", in Proceedings of ISMA2002 - Volume II, Leuven, Belgium, pp. 835-862.

Desmet, W., Sas, P. and Vandepitte, D., 2001, "An indirect Trefftz-method for the steady-state dynamic analysis of coupled vibro-acoustic systems", Computer Assisted Mechanics and Engineering Sciences, 8, 271-288.

Doyle, J.F., 1997, "Wave propagation in structures: spectral analysis using fast discrete Fourier transforms", 2nd ed., Springer-Verlag.

Gorman, D. J., 1999, "Vibration Analysis of Plates by the Superposition Method" (Series on Stability, Vibration and Control of Systems Series a, Volume 3), World Scientific Publishing Company, 270 p.

Grédiac, M., Paris, P.A., 1996, "Direct Identification of Elastic Constants of Anisotropic Plates By Modal Analysis: Theoretical And Numerical Aspects", Journal of Sound and Vibration, 195(3), 401-415.

Gunda, R., Vijayakar, S.M. and Singh, R., 1995, "Method of images for the harmonic response of beams and rectangular plates", Journal of Sound and Vibration 185(5), pp. 791-808.

Jin, W.G., Cheung, Y.K. and Zienkiewicz, O.C., 1993, "Trefftz method for Kirchhoff plate bending problems", International Journal for Numerical Methods in Engineering 36, 765-781.

Kevorkian, S. and Pascal, M., 2001, "An accurate method for free vibration analysis of structures with application to plates", Journal of Sound and Vibration 246 (5), pp. 795–814.

Kulla, P.H., 1997, "High precision finite elements", Finite Elements in Analysis and Design 26, pp. 97-114.

Lee, U. and Lee, J., 1999, "Spectral-element method of Levy-type plates subjected to dynamic loads", Journal of Engineering Mechanics, February, pp. 243-247.

Lyon, R.H. and DeJong, R.G., 1995, "Theory and Application of Statistical Energy Analysis", Butterworth-Heinemann, Boston.

Papkov, S.O. and Banerjee, J.R., 2015, "A new method for free vibration and buckling analysis of rectangular orthotropic plates", Journal of Sound and Vibration, 339, 342-358.

Pereira, V. S., Dos Santos J. M. C., 2014, "Coupled plate energy models at mid-and high-frequency vibrations", Computers & Structures, vol. 134, pp. 48-61.

Zienkiewicz, OC., 1977, "The Finite Element Method", 3rd Ed., McGraw-Hill.