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STABILITY ANALYSIS IN NONLINEAR MECHANICAL SYSTEM

Alcione B. Purcina

Tatiane N. da Costa

alcyone.borges@hotmail.com

tatty_ane_nunes@hotmail.com

Instituto de Matemática e Tecnologia – UFG/Catalão

Av. Dr. Lamartine Pinto de Avelar 1120, 75.704-020, Catalão, Goiás, Brasil

Romes A. Borges

romes@ufg.br

Instituto de Matemática e Tecnologia – UFG/Catalão

Av. Dr. Lamartine Pinto de Avelar 1120, 75.704-020, Catalão, Goiás, Brasil

Abstract. *In this work, the behavior of a nonlinear system with nonlinear absorber will be investigated for the case of primary resonance. The system consists of the main system connected to a mechanical device called dynamic vibration absorber, which are widely used to attenuate undesirable vibrations in a given structure. For the integration of equations of motion will be used Fourth-Order Runge-Kutta Method. Approximate solutions to the first order, are obtained using the multiple scales method, in order to verify how close is achieved to approach of the numerical solution. The method of multiple scales, also, is used to derive a system of first-order nonlinear ordinary-differential equations that describe the modulation of the amplitude and the phase of the response. The steady state solution and stability will be studied. The stability of the system will be investigated using frequency response curve, time domain response, phase plan, Poincare map and Lyapunov exponent. Numerical Simulations show that the system presents periodic and chaotic movements. Such behavior should be demonstrated by applying the Routh-Hurwitz criterion and analysis of Lyapunov exponent. In general, the main characteristics of a dynamic system that experiences the chaotic response will be identified.*

Keywords: *Nonlinear dynamical absorber, Multiple Scales Methods, Numerical results, Bifurcations, Chaotic dynamics.*

1. INTRODUCTION

The Decrease of vibration levels in the response of a system is of great importance to have a reliable and efficient design, since those vibrations are undesirable phenomena that may cause damage, failure, and sometimes destruction of machines and structures. The Vibration Analysis of mechanical systems allows to know, improve and ameliorate in terms of quality, durability and productivity (Sayed, Hamed and Amer, 2011). In this sense, there is growing interest in tools to attenuate unwanted vibrations in various types of systems.

One of the most effective ways to attenuate unwanted vibrations in a given structure is through the Dynamic Vibration Absorbers (Rade et al., 2001). A classic DVA consists of a mass coupled by means of a spring and a damper to a given system, getting a new degree of freedom. The system, then, be tuned to vibrate at higher amplitudes, absorbing thereby partially or fully vibratory energy in the coupling point (Koronev and Reznikov, 1993).

The DVAs may have linear or nonlinear characteristics. However, in recent years, there is growing interest in studying the DVAs with nonlinear characteristics, due to their greater robustness when compared to the linear absorber, based in fact DVA linear operate satisfactorily only in its tuning frequency (Borges, 2008).

In recent years, there is growing interest in the study of nonlinear phenomena due the modernization of structures and employing innovative materials and more flexible, the nonlinearities become more active (Silva, 2006). However, methods for analyzing nonlinear systems, the opposite of linear, are far less known, some are found only partially developed, and the analysis of the equations that model a given problem of difficult to apply (Rodovalho and Costa, 2013).

The nonlinearities offer greater variability in the types of system solutions. Therefore, necessary to formulate means to study and understand the characteristics of nonlinear phenomenon, emphasizing the chaos (Belusso, 2011).

A particularity of the systems that exhibit chaos, which is called sensitivity to initial conditions, i.e. two initial conditions very similar between themselves diverge drastically with time (Amorim, 2013).

The perturbation theory is given by interactive methods that has the purpose of obtain approximate solutions involving the perturbation parameter, this in turn has the condition is a small number (Nayfeh, 2004). Between the perturbation methods stands out the multiple scales method. The basic idea of this method is to achieve the expansion of the solution representing the response as a function of multiple independent variables or multiple scales (Savi, 2004).

The Routh-Hurwitz stability criterion is used to investigate the absolute stability of dynamic systems. Edward John Routh (1831-1907) established a first criterion for a polynomial had roots with negative real part. However, Adolf Hurwitz (1859-1919) classified this as a necessary condition, but not sufficient and developed the called Hurwitz matrix or H matrix, built through of the coefficients of the polynomial. Thus, the criterion went on to establish that, besides the polynomial need to have all your positive coefficients, it was necessary that all the determinants of the matrix were also positive (Belusso, 2011).

One of the most important and used criteria to define the chaos in dynamic systems are the Lyapunov exponents that measuring the exponential average rate of divergence or convergence of trajectories of the phase space from initial points close. Thus, the dynamic behavior of a system can be provided through the signs of the Lyapunov exponents, which indicate the presence of fixed points, periodic movements, almost periodic and chaos (Chierice Júnior, 2007).

This paper has by objective to study the behavior of a nonlinear system with nonlinear absorber in main resonance. For that, initially, we will write the system of equations of motion and applies the Multiple Scales Method to find a general solution to the first order approximation. With the analytical solution, it is verified how close is of the approach calculated numerically by the Fourth-Order Runge-Kutta Method. Next, using the Method Andronov and Vitt, we obtain the frequency response. Routh Hurwitz Criterion will determine the region of stability of the solutions. Subsequently, it presents the time domain response, phase plane, Poincare Map and Lyapunov Exponent obtained through implementations in software Matlab® 8.0. Finally, we analyze the behavior of the system in question, identifying the main characteristics of a dynamic system that experiences the chaotic response.

2. MODELING OF A DISCRETE SYSTEM

Consider the vibratory system of two degrees of freedom (d.o.f), with damping, where will be introduced the concept of Nonlinear Dynamic Vibration Absorber (NDVA), whose model is shown in Fig. 1.

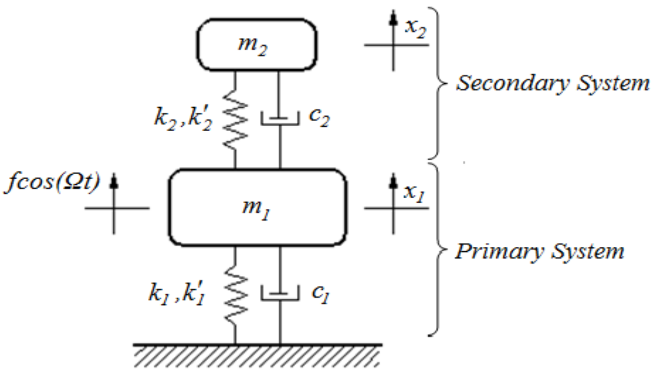

Figure 1. Discrete mechanical system

From the subsystem formed by $(m_2, k_2, k_2^{'}, c_2)$, it is desirable to attenuate the vibration levels the main mass, shown in Fig. 1 by m_1. In this subsystem, gives up the name Dynamic Vibration Absorber (DVA).

Applying Newton's second law in the two masses separately, thus, the system of equations of motion are written, presented in Fig. 1, as being:

$$m_1\ddot{x}_1 + k_1x_1 + k_1^{'}x_1^3 + c_1\dot{x}_1 - k_2(x_2 - x_1) - k_2^{'}(x_2 - x_1)^3 + c_2(\dot{x}_1 - \dot{x}_2) = fcos(\Omega t) \quad (1)$$

$$m_2\ddot{x}_2 + k_2(x_2 - x_1) + k_2^{'}(x_2 - x_1)^3 + c_2(\dot{x}_2 - \dot{x}_1) = 0 \quad (2)$$

This vibrating system is composed of springs $(k_1^{'}, k_2^{'})$ with nonlinear characteristics, which are assumed be sufficiently weak. An external force $f(t) = cos(\Omega t)$ excites the main mass m_1.

From algebraic manipulations, the Eqs. (1) and (2), they can be written as:

$$\ddot{x}_1 + \omega_1^2 x_1 + \varepsilon\alpha_1 x_1^3 + \varepsilon\xi_1\dot{x}_1\varepsilon\alpha_2 x_2 - \varepsilon\alpha_3(x_2 - x_1)^3 + \varepsilon\xi_2(\dot{x}_1 - \dot{x}_2) = \varepsilon f cos(\Omega t) \tag{3}$$

$$\ddot{x}_2 + \omega_2^2(x_2 - x_1) + \varepsilon\beta_1(x_2 - x_1)^3 + \varepsilon\xi_3(\dot{x}_2 - \dot{x}_1) = 0 \tag{4}$$

where

$$\omega_1^2 = \frac{k_1 + k_2}{m_1}, \omega_2^2 = \frac{k_2}{m_2}, \alpha_1 = \frac{k_1^{'}}{m_1}, \alpha_2 = \frac{k_2}{m_1}, \alpha_3 = \frac{k_2^{'}}{m_1}, \tag{5}$$

$$\xi_1 = \frac{c_1}{m_1}, \xi_2 = \frac{c_2}{m_1}, \xi_3 = \frac{c_2}{m_2}, \beta_1 = \frac{k_2^{'}}{m_2}, f = \frac{f_0}{m_1}. \tag{6}$$

and $\varepsilon << 1$ indicates a small greatness assumed.

In Table 1 are listed the parameter values of main system and the absorber.

Table 1 - Parameter Values of the System

Symbol	Variable	Value	Unit
m_1	Mass of nonlinear main system	10	kg
m_2	Absorber mass	0.8	kg
k_1	Linear stiffness of the main system	44	N/m
$k_1^{'}$	Nonlinear stiffness of the main system	8	N/m^3
k_2	Linear absorber stiffness	2	N/m
$k_2^{'}$	Nonlinear absorber stiffness	0.5	N/m^3
c_1	Linear damping constant of the main system	0.1	Ns/m
c_2	Linear absorber damping constant	0.08	Ns/m
f_0	Applied forces	0.2106	N

3. THE MULTIPLE SCALES METHOD APPLIED OF THE NONLINEAR MECHANICAL SYSTEM

Applying the Multiple Scales Method (Nayfeh and Mook, 1979), expected to find a uniform solution of the first order of Eqs. (3) and (4). Considering two terms of the expansion in x, the solution is:

$$x_1(t.;\varepsilon) = x_{10}(T_0, T_1) + \varepsilon x_{11}(T_0, T_1) \tag{7}$$

$$x_2(t.;\varepsilon) = x_{12}(T_0, T_1) + \varepsilon x_{21}(T_0, T_1) \tag{8}$$

where $T_0 = t$ and $T_1 = \varepsilon t$, once, in this work, the expansion is performed until $O(\varepsilon)$, and x_j, $j = 1,2$, are functions to be determined.

The derived of the first and second order of x in relation to time should be expressed in terms of partial derivatives, relatively the T_n such that,

$$\frac{dx}{dt} = \frac{dT_0}{dt}\frac{\partial}{\partial T_0} + \frac{dT_1}{dt}\frac{\partial}{\partial T_1} = D_0 + \varepsilon D_1 \tag{9}$$

$$\frac{d^2x}{dt^2} = \frac{\partial(D_0 + \varepsilon D_1)}{\partial T_0} + \varepsilon\frac{\partial(D_0 + \varepsilon D_1)}{\partial T_1} = D_0^2 + 2\,\varepsilon\, D_0 D_1 \tag{10}$$

where $D_n = \dfrac{\partial}{\partial T_n}(n = 0,1,\ldots)$ are differential operators.

Substituting the Eqs. (7) and (8) in Eqs. (3) and (4) and ordering each term in relation the potencies of ε^0 and ε^1, has, respectively, the following set of equations:

- To order ε^0:

$$(D_0^2 + \omega_1^2)x_{10} = 0 \tag{11}$$

$$(D_0^2 + \omega_2^2)x_{20} = \omega_2^2 x_{10} \tag{12}$$

- To order ε^1:

$$\begin{aligned}(D_0^2 + \omega_1^2)x_{11} = {} & f\cos(\Omega c_0) - 2D_0D_1x_{10} - \xi_1 D_0 x_{10} - \alpha_1 x_{10}^3 + \alpha_2 x_{20} + \alpha_3(x_{20} - x_{10})^3 \\ & - \xi_2 D_0(x_{10} - x_{20})\end{aligned} \tag{13}$$

$$(D_0^2 + \omega_2^2)x_{21} = -2D_0D_1x_{20} + \omega_2^2 x_{11} - \beta_1(x_{20} - x_{10})^3 - \xi_3 D_0(x_{20} - x_{10}) \tag{14}$$

The general solutions of the Eqs. (11) and (12) can be expressed in the following ways:

$$x_{10} = Ae^{i\omega_1 T_0} + cc \tag{15}$$

$$x_{20} = Be^{i\omega_2 T_0} + \Gamma_1 Ae^{i\omega_1 T_0} + cc \tag{16}$$

where $\Gamma_1 = \dfrac{\omega_2^2}{\omega_2^2 - \omega_1^2}$, A and B are unknown functions in T_1, can be determined from the elimination of secular terms on the side of approach, and cc represents the conjugate of previous terms. The particular solutions of Eqs. (13) and (14), after eliminating the secular terms, are given by:

$$x_{11} = Ce^{i\omega_1 T_0} + U_1 e^{i\,\Omega T_0} + H_1 e^{3i\omega_1 T_0} + H_2 e^{i\omega_2 T_0} + H_3 e^{3i\omega_2 T_0} + H_4 e^{i(2\omega_1 + \omega_2)T_0}$$
$$+ H_5 e^{i(2\omega_1 \quad \omega_2)T_0} + H_6 e^{i(\omega_1 + 2\omega_2)T_0} + H_7 e^{i(\omega_1 \quad 2\omega_2)T_0} + cc \tag{17}$$

$$x_{21} = De^{i\omega_2 T_0} + U_2 e^{i\,\Omega T_0} + H_8 e^{i\omega_1 T_0} + H_9 e^{3i\omega_1 T_0} + H_{10} e^{3i\omega_2 T_0} + H_{11} e^{i(2\omega_1 + \omega_2)T_0}$$
$$+ H_{12} e^{i(2\omega_1 - \omega_2)T_0} + H_{13} e^{i(\omega_1 + 2\omega_2)T_0} + H_{14} e^{i(\omega_1 - 2\omega_2)T_0} + cc \tag{18}$$

where C, $D, U_j, j = 1,2$ and $H_i, i = 1,...,14$ are complex functions in T_1.

The general solution of x_1 and x_2 until the first order approximation is given by:

$$x_1 = x_{10} + \varepsilon x_{11} \tag{19}$$

$$x_2 = x_{20} + \varepsilon x_{21} \tag{20}$$

3.1 Evaluation of the response obtained by Multiple Scale Method

The search for precisely approximate solutions of an equation can be performed in two ways: Numerical Methods and Analytical Methods. An analytical approach is obtained when a parameter of the problem is small, hence the name Perturbation Methods (Borges, 2000).

With the solution calculated analytically can be verified how close is achieved approach the numerical solution of the system in question. In this sense, follow, is shown the numerical solution of Eqs. (3) and (4), and evaluates with the solution obtained by the perturbation theory, Eqs. (19) and (20).

The Figure 2 presents the analytical solutions, obtained from the Multiple Scale Method, and numerical, obtained by the Fourth-Order Runge-Kutta Method to evaluate the efficiency of the Perturbation Method applied to the nonlinear system, in question. The values are listed in Tab. 1, presented in the previous Section.

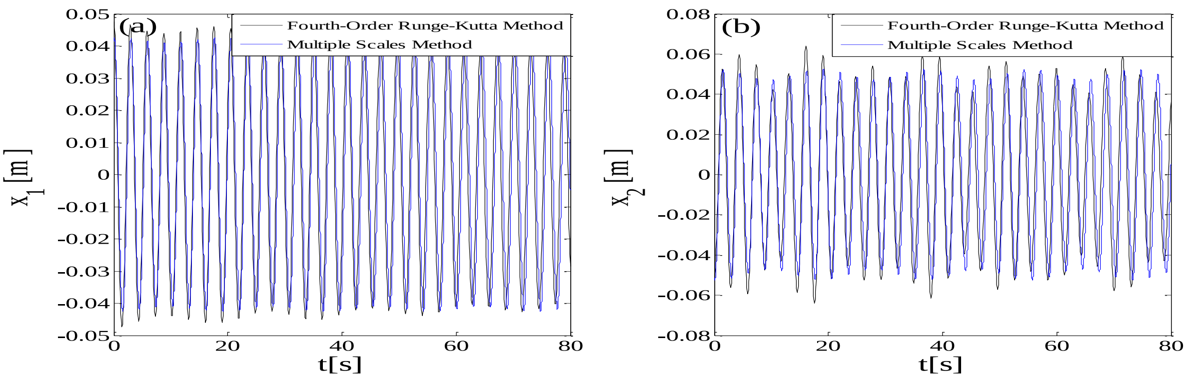

Figure 2. Solutions of Eqs. (3) and (4) obtained by Fourth-Order Runge-Kutta Method and General Solutions, Eqs. (19) and (20), obtained by the Multiple Scales Method for (a) main system and (b) absorber.

Observes that for the main system, Fig. 2 (a) and for the absorber, Fig. 2 (b), the general solution of x_1 and x_2, Eqs. (19) and (20) respectively, found by applying the Multiple Scales Method, differs little from numerical solutions obtained by the Fourth-Order Runge-Kutta Method for Eq. (3) and (4).

In this way, verifies that the general solution obtained by the perturbation method, to the first order approximation is sufficiently close to the numerical solution for the nonlinear system, although it present some discrepancy points of the oscillation amplitude. For better accuracy of the analytical solution, generally, terms of higher orders have that be considered in the expansion (Marques, 2013).

4. STUDY OF THE STABILITY IN MECHANICAL SYSTEM

The stability of the vibratory system of two degrees of freedom (d.o.f), with damping, is investigated for the case of primary resonance, i.e., in which the frequency of the external excitation Ω is very close to the natural frequency ω_1, instead of use the excitation frequency Ω as a parameter, introduces a tuning parameter σ, such that:

$$\Omega = \omega_1 + \varepsilon\sigma \tag{21}$$

that quantitatively describes the proximity of Ω to ω_1 and $\sigma = O(1)$.

Substituting the Eq. (21) in Eqs. (13) and (14) and eliminating the secular terms leads to the following conditions of solvency:

$$\begin{aligned}2i\omega_1 D_1 A = {} & [-\xi_1 i\omega_1 + \alpha_2\Gamma_1 + \xi_2 i\omega_1(\Gamma_1 - 1)]A - 3[\alpha_1 + \alpha_3(\Gamma_1 - 1)^3]A^2\bar{A} + 6\alpha_3(\Gamma_1 - 1)AB\bar{B} \\ & + \frac{f}{2}e^{i\sigma\sigma_1}\end{aligned} \tag{22}$$

$$\begin{aligned}2iD_1\omega_2 B = {} & \frac{\omega_2^2}{(\omega_1^2 - \omega_2^2)}[6\alpha_3(\Gamma_1 - 1)^2 A\bar{A}B + (\xi_2 i\omega_2 + \alpha_2)B + 3\alpha_3 B^2\bar{B}] - 6\beta_1(\Gamma_1 - 1)^2 A\bar{A}B \\ & - \xi_3 i\omega_2 B - 3\beta_1 B^2\bar{B}\end{aligned} \tag{23}$$

Expressing the complex functions A and B in polar form as:

$$A = \frac{1}{2}ae^{i\theta} \tag{24}$$

$$B = \frac{1}{2}be^{i\gamma} \tag{25}$$

where, a, b, θ and γ are real. Substituting the Eqs. (24) and (25) in Eqs. (22) and (23) and separating the real and imaginary parts, there is obtained the following set of equations:

$$a' = c_1 a - dsen(\eta) \tag{26}$$

$$a\eta^{'} = c_2 a + c_3 a^3 + c_4 ab^2 - dcos(\eta) \tag{27}$$

$$b^{'} = c_5 b \tag{28}$$

$$b\gamma^{'} = c_6 b + c_7 b^3 + c_8 a^2 b \tag{29}$$

where

$$d = \frac{f}{2\omega_1}, c_1 = \frac{1}{2}[-\xi_1 + \xi_2(\Gamma_1 - 1)], c_2 = -\frac{1}{2\omega_1}\alpha_2\Gamma_1, c_3 = \frac{3}{8\omega_1}[\alpha_1 - \alpha_3(\Gamma_1 - 1)^3] \tag{30}$$

$$c_4 = -\frac{3}{4\omega_1}\alpha_3(\Gamma_1 - 1), c_5 = \frac{1}{2}(\xi_2\omega_2^2\Gamma_2 - \xi_3), c_6 = -\frac{1}{2}\alpha_2\Gamma_2\omega_2 \tag{31}$$

$$c_7 = -\frac{3}{8}\alpha_3\omega_2\Gamma_2 + \frac{3}{8\omega_2}\beta_1, \mathbf{c}_8 = -\frac{3}{4}\alpha_3(\Gamma_1 - 1)^2\omega_2\Gamma_2 + \frac{3}{4\omega_2}\beta_1\Gamma_1 - 1)^2 \tag{32}$$

where $\Gamma_2 = \dfrac{1}{\omega_1^2 - \omega_2^2}$, $\eta = \theta - \sigma T_1$, a and θ are amplitude and phase of the main system, respectively , b and γ are amplitudes and phase the absorber.

In the search for stationary solutions, has, $a^{'} = b^{'} = \eta^{'} = \gamma^{'} = 0$ in (26) - (29). Thus, the stationary values become solutions of algebraic system of equations:

$$c_1 a - dsen(\eta) = 0 \tag{33}$$

$$(c_2 - \sigma)a + c_3 a^3 - c_4 ab^2 - dcos(\eta) = 0 \tag{34}$$

$$c_5 b = 0 \tag{35}$$

$$c_6 b + c_7 b^3 + c_8 a^2 b = 0 \tag{36}$$

Resolving the resulting algebraic equations (33) - (36) produces three possibilities for the fixed points, namely:

Case (1): $a \neq 0, b = 0$;
Case (2): $a = 0, b \neq 0$ and
Case (3): $a \neq 0, b \neq 0$.

which only the first can happen (Saberi and Nahvi, 2014).

In this way, considering $b = 0$ and squaring the two sides of each of the Eqs. (33) and (34) and summing them, we obtain:

$$[(c_2 - \sigma + c_3 a^2)^2]a^2 + c_1^2 a^2 = d^2 \tag{37}$$

which is the frequency response for the system given in Fig.1.

It is worth mentioning that, the values of a and η are solutions of the Eqs. (33) and (34), since, it is considered that the case in which $b = 0$.

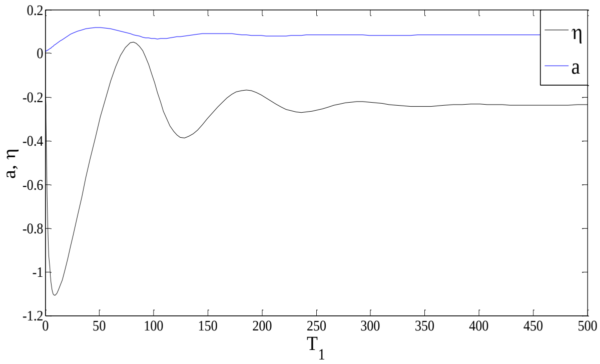

Figure 3 - Variation in a and η with T_1 numerically calculated (25) and (26) to $\sigma = 0$, $b = 0$, $a(0) = 0.01$ and $\eta(0) = 0.01$

The Figure 3 shows the variation in a and η in relation T_1 calculated by numerical integration of the equations (33) and (34). It is noted that, in principle a and η exhibit oscillations, but in the steady state becomes constant.

The system stability under primary resonance was studied using the method of Andronov and Vitt (Saberi and Nahvi, 2014, apud Nayfeh and Mook, 1979). This way, is considering the following relations:

$$\begin{cases} a = a_0 + a_1 \\ \eta = \eta_0 + \eta_1 \\ b = b_0 + b_1 \end{cases} \tag{38}$$

where a_0, η_0 and b_0 are solutions of steady state of the Eqs. (33) – (36) and a_1, η_1 and b_1 are small perturbations.

Substituting the Eq. (38) in Eqs. (33) – (36) is obtained the following matrix:

$$\begin{bmatrix} a_1' \\ \eta_1' \\ b_1' \end{bmatrix} = J \begin{bmatrix} a_1 \\ \eta_1 \\ b_1 \end{bmatrix} \tag{39}$$

where J is the Jacobian matrix given by:

$$J = \begin{bmatrix} c_1 & -(c_2 - \sigma)a_0 - c_3 a_0^3 & 0 \\ \dfrac{c_2 - \sigma}{a_0} + 3c_3 a_0 & c_1 & 0 \\ 0 & 0 & c_5 \end{bmatrix} \tag{40}$$

The eigenvalues are

$$\lambda^3 + c_9\lambda^2 + c_{10}\lambda + c_{11} = 0 \tag{41}$$

where c_9, c_{10} and c_{11} are constant given by

$$c_9 = -2c_1 - c_5 \tag{42}$$

$$c_{10} = 3a_0{}^4 c_3{}^2 + 4a_0{}^2 c_2 c_3 - 4a_0{}^2 c_3\sigma + c_2{}^2 - 2c_2\sigma + c_1{}^2 + 2c_5 c_1 + \sigma^2 \tag{43}$$

$$c_{11} = -3c_5 a_0^4 c_3{}^2 - 4c_5 a_0^2 c_2 c_3 + 4c_5 a_0^2 c_3\sigma - c_5 c_2{}^2 + 2c_5 c_2\sigma - c_5 c_1^2 - c_5\sigma^2 \tag{44}$$

The stability region of the frequency response curve, Eq. (37), it will be determined by Routh-Rurwitz criterion. A system is considered stable if all eigenvalues of the Jacobian matrix associated with a particular point of balance, have negative real part (Savi, 2004). According to the Routh-Hurwitz criterion, for this to occur it is necessary that all coefficients and the determinants of Hurwitz matrix (H) are positive. If one of these values is negative, the Jacobian matrix have at least one eigenvalue with positive real part, making the system unstable (Belusso, 2011).

In this sense, considering the Eq. (41), builds the matrix H :

$$H = \begin{bmatrix} c_9 & c_{11} & 0 \\ 1 & c_{10} & 0 \\ 0 & c_9 & c_{11} \end{bmatrix} \tag{47}$$

The necessary and sufficient conditions for the coefficients of Eq. (41) and the determinants of Hurwitz Matrix, Eq. (47), real parts are negatives:

$$c_9 > 0, c_{10} > 0, c_{11} > 0, c_9 c_{10} - c_{11} > 0 \text{ and } c_9 c_{10} c_{11} - c_{11}{}^2 > 0 \tag{48}$$

When these conditions are satisfied, the fixed points are stable, otherwise, are unstable.

4.1 Saddle-node bifurcation

The real solution of Eq. (37) determines the amplitude of the main system response in primary resonance. In it, there are three real solutions between two points of vertical tangent, which, they are called of limit point bifurcation, known in the literature as saddle-node bifurcation (Elnaggar and Khalil, 2014).

The saddle-node bifurcation is a nonlinear phenomenon and is related a nonlinear model as a quadratic equation (Medeiros, 2010). At the saddle-nodes, the tangency of the frequency response curve is vertical. The locations of the jumping points will are obtained through differentiation of the frequency response of the Eq. (37) with relation a^2 and considering $\frac{d\sigma}{da^2} = 0$. This way, the resulting expression is,

$$(c_3a^2 + c_2 - \sigma)^2 + c_1^2 + 2a^2c_3(c_3a^2 + c_2 - \sigma) = 0 \tag{45}$$

with solutions:

$$\sigma_{\pm} = c_2 + 2c_3a^2 \pm \sqrt{c_3^2a^4 - c_1^2} \tag{46}$$

For there is an interval $\sigma_- < \sigma < \sigma_+$ in which three real and positive solutions of Eq. (37) exist.

4.2 Frequency response for the system

The Equation (37) is a nonlinear algebraic equation solved numerically using implementations in software MATLAB® and the result is show in Fig. 4. In this figure, the frequency response curve is composed of continuous and dashed lines, which represent the stable and unstable solutions, respectively, determined in accordance with the Routh-Hurwitz criterion as described in the previous subsection.

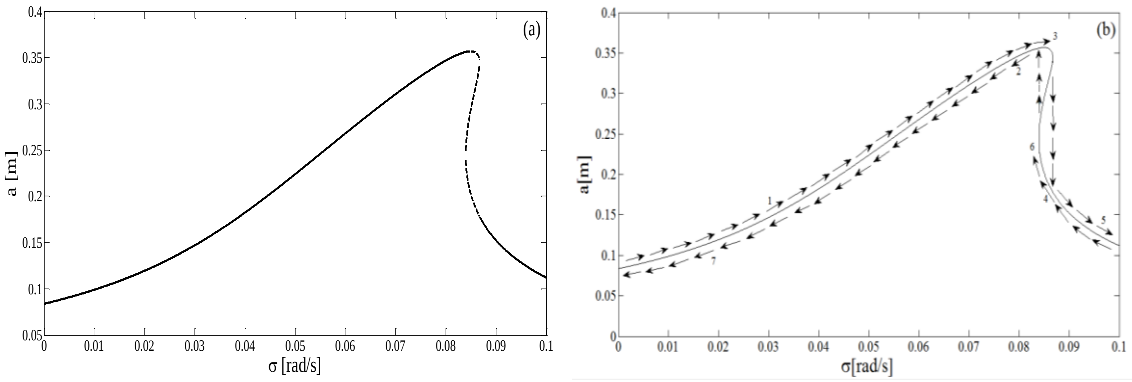

Figure 4 – (a) Amplitude of the main system in case of a primary resonance with $f_0 = 0.2106\ N$, (b) Jump Phenomenon

In Figure 4 (a), it was observe that, the curve shows the jump phenomenon that is one of the outstanding characteristics of nonlinear systems. The jumping phenomenon occurs where the steady state behavior changes dramatically due to a transition from one stable solution to another stable solution when the tuning parameter σ is varied (Balachandran and Nagy, 2011). In Figure 4 (b) can accompany the jump phenomenon. If the experiment begin in the point 1 and the frequency is gradually increased, a jump occurs from point 3 to 4. On the other hand, decreasing the frequency value, the response follows the curve along the passages 4 - 6 - 2 - 7. The stretch of response between 3 - 4 is unstable.

In this Figure, also, it was observed that, the frequency response for the main system, has peak amplitude in $a_{max} = 0.3568\ m$ that occurs in $\sigma = 0.0850\, rad/s$ and presents an unstable region in the interval of $0.08399 < \sigma < 0.08669$, as shown eigenvalues listed in Tab. 2.

Table 2 – Eigenvalues of the system with nonlinear absorber in force $f_0 = 0.2106\ N$

σ	λ_1	λ_2	λ_3
0	-0.013762 + 0.05872i	-0.013762 - 0.05872i	-0.045238
0.08	-0.013762 + 0.014051i	-0.013762 - 0.014051i	-0.045238
0.08398	-0.013762 - 0.0073204i	-0.013762 + 0.0073204i	-0.045238
0.08399	-0.013762 + 0.007265i -0.028148 -0.026818	-0.013762 - 0.007265i 0.00062373 -0.00070583	-0.045238
0.08669	-0.013762 +0.01465i -0.026881 -0.028113	-0.013762 - 0.01465i -0.00064263 0.00058928	-0.045238
0.08671	-0.013762 + 0.01472i	-0.013762 - 0.01472i	-0.045238
0.09	-0.013762 + 0.023129i	-0.013762 - 0.023129i	-0.045238
0.1	-0.013762 + 0.03856i	-0.013762 - 0.03856i	-0.045238

Is worth mentioning that is determined the curve of stability region by applying the Routh Hurwitz criterion as described above.

4.3 Time domain response analysis

The stability of the nonlinear system with nonlinear absorber it is evaluated according to two important criterion of nonlinear dynamics, namely, the Routh-Hurwitz criterion and analysis of Lyapunov exponent. Thus, it is intended confirm the graphic analysis in the previous subsection.

The numerical solution of the nonlinear system will be obtained by integration of Eqs. (3) and (4), carried out through the implementation of the Fourth-Order Runge-Kutta Method, the software MATLAB® (Yang, Cao, Chung and Morris, 2005). In this way, follow, presents the responses in the time domain, phase planes and Poincare Sections of the main system and the absorber, Fig. 5, considering the case of the primary resonance $\Omega \cong \omega_1$. The values of the parameters are listed in Tab. 1, presented in Section 2.

It can be seen that, both time domain response of the main system , Fig. 5 (a), as the absorber, Fig. 5 (b) show low amplitude oscillations around the zero point which are originally are irregular and after a period of time, become regular.

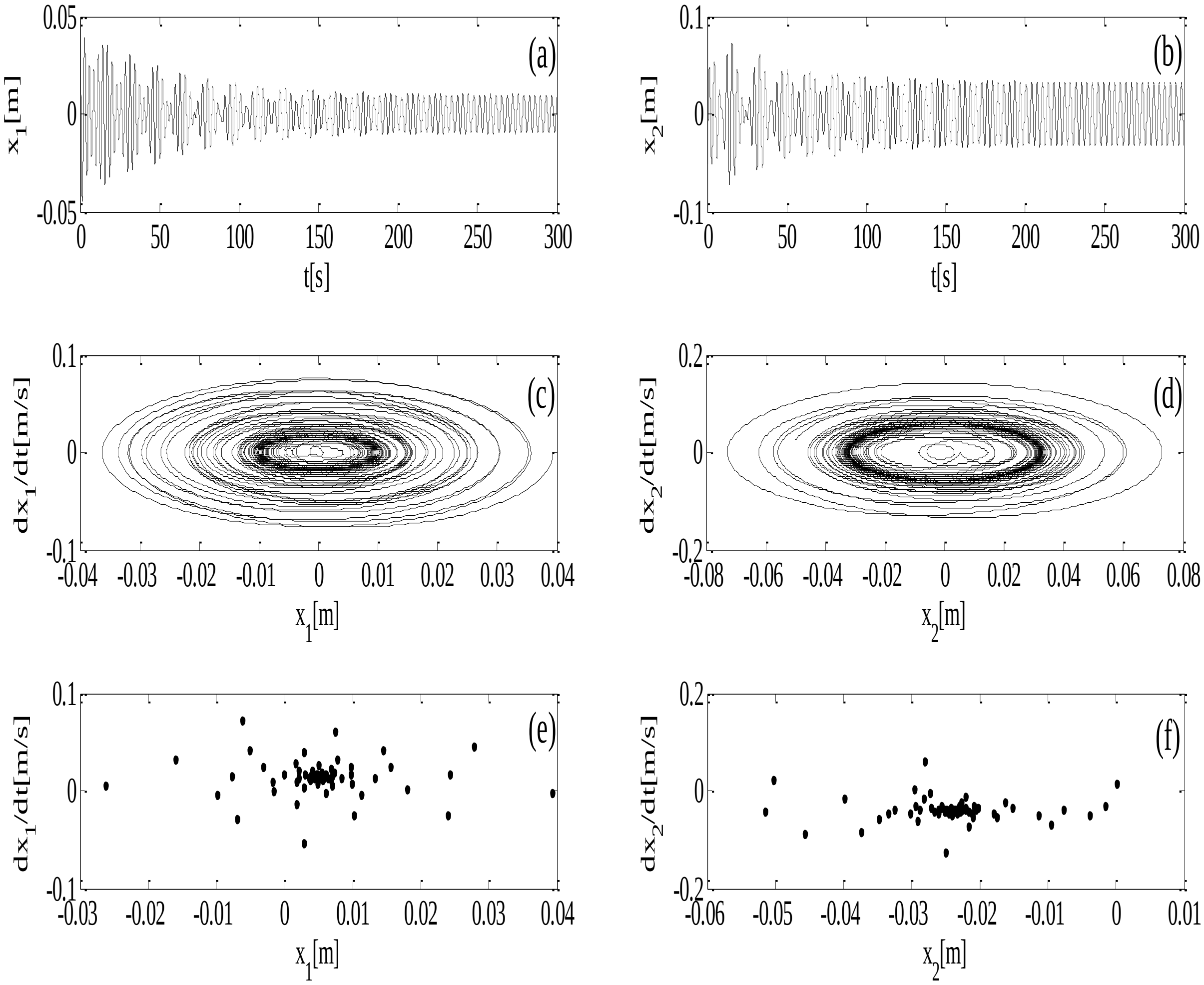

Figure 5 – Responses in the time domain (a) and (b), phase plans (c) and (d) and sections of Poincare (e) and (f) the main system and the absorber, respectively, in primary resonance $\Omega \cong \omega_1$ **.**

In Figure 5 (c) and (d) has the visualization of the phase plane of the main system and the absorber, respectively. Observed that such systems have multi-limit cycle and dynamic chaos.

The chaotic behavior is evidenced by the analysis of Poincare Sections of the main system and the absorber, Fig. 5 (d) and (e), respectively, since they have a great number of points around an irregularly in plan.

Classification of the stability According the Routh-Hurwitz Criterion. Considering the nonlinear system with nonlinear absorber, Eqs. (3) and (4), it is first necessary to find your equilibrium points. However, it is know that, is a non-autonomous system, ie, a system that explicitly depends on the time variable. In order to eliminate this variable, it turns this system on a standalone system:

$$\dot{x}_1 = y_1 \tag{49}$$

$$\dot{y}_1 = \omega_1^2 x_1 - \varepsilon\alpha_1(x_1^3) - \varepsilon\xi_1 y_1 + \varepsilon\alpha_2 x_2 + \varepsilon\alpha_3(x_2 - x_1)^3 - \varepsilon\xi_2(y_1 - y_2) + \varepsilon f cos(\omega z) \tag{50}$$

$$\dot{x}_2 = y_2 \tag{51}$$

$$\dot{y}_2 = -\omega_2^2(x_2 - x_1) - \varepsilon\beta_1(x_2 - x_1)^3 - \varepsilon\xi_3(y_2 - y_1) \tag{52}$$

$$\dot{z} = 1 \tag{53}$$

where is composed of five state variables x_1, y_1, x_2, y_2 and z .

A remarkable feature of nonlinear systems is the infinity of equilibrium points. However, through the transformation of a non-autonomous system in autonomous, besides eliminating the time variable, you can get a single break-even point for each predetermined initial condition (Belusso, 2011). Thus, solving the Eqs. (49) – (53) through of the implementations in the software Matlab® given the initial conditions $y_1(0) = 0$, $y_2(0) = 0$, $y_3(0) = 0$, $x_1(0) = 0$ and $x_2(0) = 0$, is obtained:

$$y_1 = 0, y_2 = 0, y_3 = 0, x_1 = 0.1826, x_2 = 0.3035 \tag{54}$$

The Jacobian matrix of the autonomic system, Eqs. (49) – (53), for the corresponding equilibrium point, is given by:

$$J = \begin{bmatrix} 0 & 0 & g_5 & 0 & 0 \\ g_6 & g_7 & g_8 & g_5 & 0 \\ 0 & 0 & 0 & g_5 & 0 \\ g_1 & g_2 & g_3 & g_4 & 0 \\ 0 & 0 & 0 & 0 & g_5 \end{bmatrix} \tag{55}$$

where

$$g_1 = \omega_2^2 + 3\beta_1\varepsilon(x_1 - x_2)^2, g_2 = -\omega_2^2 - 3\beta_1\varepsilon(x_1 - x_2)^2, g_3 = \varepsilon\xi_3, g_4 = -\varepsilon\xi_3, g_5 = \varepsilon\xi_2 \tag{56}$$

$$g_6 = -\omega_1^2 - 3\varepsilon\alpha_3(x_1 - x_2)^2 - 3\varepsilon\alpha_1 x_1^2, g_7 = \varepsilon\alpha_2 + 3\varepsilon\alpha_3(x_1 - x_2)^2, g_8 = -\varepsilon\xi_1 - \varepsilon\xi_2 \tag{57}$$

The eigenvalues were obtains by:

$$det(J - \lambda I) = 0 \tag{58}$$

where I is the identity matrix of the same order of the Jacobian.

Substituting the matrix (55) in (58), it is obtaining the following characteristic equation:

$$\lambda^2 + 3.5882\lambda - 4.5882 = 0 \tag{59}$$

Knowing the coefficients of the Eq. (59) is possible the construction of Hurwitz matrix, given by:

$$H = \begin{bmatrix} 3.5882 & 0 \\ 1 & -4.5882 \end{bmatrix} \tag{60}$$

The values of the coefficients of Eq. (57) and the determinants of the Hurwitz matrix (60) are listed in Table 3.

Table 3 – Coefficients and determinants of Hurwitz matrix for the system

Coefficients	Determinants
$a_0 = 1$ $a_1 = 3.5882$ $a_2 = -4.5882$	$\Delta_1 = 3.5882$ $\Delta_2 = -16.4631$

Analyzing the values presented, note the presence of negative values for the coefficient a_2 and for the determinants Δ_2. Thus, it is know that an eigenvalue of the Jacobian matrix has positive real part, so that equilibrium point of the system unstable.

Lyapunov Exponent. The system behavior, in question, will be study again, now with the help of the analysis of Lyapunov exponent, which classifies the stability according to the signs of the exponents. Briefly, if your system has, the highest negative Lyapunov exponent has the indication the presence of a fixed point; if null presents a periodic or almost periodic system; and if it is positive, it is a system with chaotic response (Belusso, 2011).

For the analysis of Lyapunov exponents it will be necessary reduce the Eq. (3) and (4) in non-autonomous in a standalone system. Therefore, it is necessary to add another dimension to the phase plane by introducing one more variable, associated with time. Then, the Lyapunov exponent were obtained by implementation performed in software MATLAB®, considering the values of the parameters described in Tab. 1, the initial conditions $x_1(0) = 0.04761, x_2(0) = -0.05129, \dot{x}_1(0) = 0, \dot{x}_2(0) = 0$ and $\dot{z}(0) = 1$ the integrative function, ode45. The dynamics of Lyapunov exponent is show in Fig. 6.

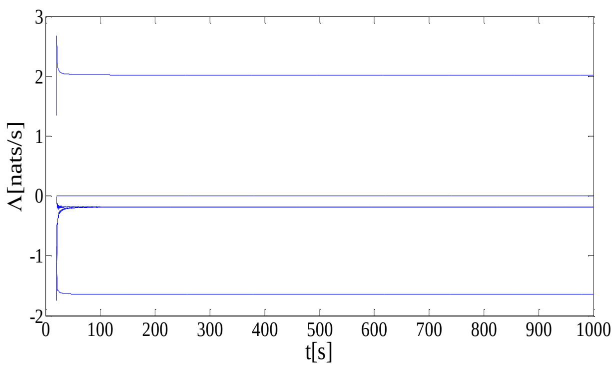

Figure 6 - Evolution of the Lyapunov exponents of the system with nonlinear absorber

The Figure 6 shows the evolution of the Lyapunov exponent of the system with nonlinear absorber. Here, was seen the presence of a positive Lyapunov exponent, leading to chaotic behavior indications of the system in question. They are listed in Tab. 3 the values of the Lyapunov exponent.

Table 4. Lyapunov exponent of the system with nonlinear absorber

Lyapunov Exponent Λ	Values	Unit
Λ_1	-1.6441	$nats/s$
Λ_2	-0.18301	$nats/s$
Λ_3	-0.18245	$nats/s$
Λ_4	0	$nats/s$
Λ_5	2.0196	$nats/s$

Analyzing the values of the Lyapunov exponent, Tab. 4, said that, the system displays a chaotic behavior, as one of the Lyapunov exponent was positive. The reaffirm that the observations made in the previous subsections, by means of other criteria for stability analysis. Moreover, it is observed that one of the Lyapunov exponent is equal to zero, corresponding to the point of bifurcation, stable periodic system (limit cycle).

5. CONCLUSION

In this paper, we investigated the behavior of a nonlinear system with nonlinear absorber in primary resonance, thus, discussed both characteristics of the theory of mechanical vibration as the nonlinear systems. It was noted that, the application of the Multiple Scales Method describes well the numerical solution. For the case of primary resonance, the amplitude of the frequency response of main system, with force $f_0 = 0.2106\ N$, is composed of a continuous curve with stable and unstable solutions and the presence of the jumping phenomenon. The region of stability of the solutions are determined by Routh-Rurwitz criterion, which proved efficient in analyzing for both situations stability, how much, instability.

In the interval of $0.08399 < \sigma < 0.08669$ comprises a region in which the criterion is not satisfied and therefore, they are called, unstable region. These unstable regions are due to the emergence of an eigenvalue with positive real part.

The numerical results are exhibited in different ways: through graphs showing time domain responses, in the form of phase planes, Poincare sections and Lyapunov exponents. The simulations were performed with the help of software MATLAB® through computer programs using the integrative function ode45. The time domain responses show trajectories, initially irregular that, with the passage of time, become regular. In the visualization of phase plane, the chaotic behavior is evident, because the system responds with a regular oscillation represented by multi-limit cycles that are intermittently interrupted by irregular intervals. The Poincare sections have irregular oscillations with number of periods not defined.

The stability of the system in question, it verified with the assistance of two important criteria of nonlinear dynamics, namely, of Routh-Hurwitz criterion and analysis of Lyapunov exponents. The Routh-Hurwitz criterion showed excellent results, confirming the occurrence of chaos in the system in question, since it was obtained negative value, sometimes to a coefficient of the characteristic polynomial or a determinant of Hurwitz matrix. The theory of the greater Lyapunov exponent, also, proved to be efficient in the analysis of stability, whose simulation showed a positive exponent, proving chaos, and a zero exponent, demonstrating the appearance of periodic response.

Therefore, that both the Routh-Hurwitz criterion as the analysis of Lyapunov exponents, showed acceptable results in regard the stability of dynamic systems. It is worth mentioning that, the results obtained were not influenced the transformation of the non-autonomous system in autonomous, proving that the whole process was developed correctly and effectively.

However, it is noted that the methods applied to the nonlinear system with nonlinear absorber, provided accurate results. Furthermore, it was found that when these are used with the assistance of software provide a better agility for the study of differential equations, with tables and graphs, enabling an analysis of the results obtained by the numerical method and perturbation method.

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