ANALYSIS OF PLATE-SOIL INTERACTION BY THE BOUNDARY ELEMENT METHOD

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Abstract. *This work presents an analysis of Reissner's plate with soil-structure interaction by using the Boundary Element Method (BEM). The approach uses the same fundamental solution already employed in previous plate analyses by the BEM. Parcels corresponding to soil reaction are added to the integral equations of the plate. The plate domain in contact with the soil is discretized into constant internal cells and the corresponding integrals are transformed into integrals over the boundary of the cells. Integral equations written for the cell points are added to the original system of plate discretized equations. For the plate boundary discretization, quadratic elements are employed. The soil is modeled as an elastic half-space and the fundamental solution of Boussinesq-Cerruti is used for points at the soil surface. Soil integral equations are written for points at the soil surface in contact with the plate and, in this region, it is used the same cell discretization employed for the plate domain. In order to accomplish the plate-soil interaction, the soil and the plate equations are coupled. The results obtained from the present work were validated by comparing them to numerical and to analytical solutions found in the literature.*

Keywords: *Reissner's theory, Boundary Element Method, Soil–structure interaction, Plate bending.*

1 INTRODUCTION

Theories for plate bending analysis have been developed since the XIX century, with the development of the Kirchhoff's theory, also called classical theory for plate bending, which is applicable to thin plates (Timoshenko and Woinowsky-Krieger, 1970).

As an alternative to Kirchhoff's theory, Reissner has included the effect of transverse shear strains in the solution of the problem. Reissner's theory (Reissner, 1945) allows the analysis of thin or thick plates and admits the consideration of the three physical boundary conditions of the problem. This leads to more accurate results at edges and corners than those obtained from Kirchhoff's theory, which considers only two boundary conditions in each edge and is limited to thin plates.

Several authors have presented works concerning thick plates on elastic supports with the BEM and one can cite Jianguo et al. (1992), Rashed et al. (1998) and Ribeiro and Karam (2009), using Winkler's model; and Jianguo et al. (1992), Rashed et al. (1999) e Altoé and Karam (2009), using Pasternak's model. Syngellakis and Bai (1993) and Paiva and Butterfield (1997) have presented works with the BEM for the analysis of plate-soil interaction considering Kirchhoff's plate theory and the soil as an infinite half-space. Xiao (2001) has presented an analysis of plate-soil interaction considering Reissner's plate theory and the soil as an infinite half-space, by using the thin plate fundamental solution together with the fundamental solution of modified Helmholtz equation.

The boundary element method is used in the present work for the analysis of plate-soil interaction, by considering Reissner's theory for plate bending and the soil as an infinite half-space. The approach adopted in this work uses the fundamental solution already employed in the analysis of plates by the BEM and additional equations to the original system are considered for points of the contact region. This region is discretized into constant triangular or quadrilateral internal cells and the corresponding integrals are transformed into integrals over the boundaries of the cells. The boundary of the plate is discretized into quadratic boundary elements. After the numerical implementation, a system of equation for the plate is obtained. The soil is considered using the three-dimensional theory of elasticity for homogeneous solids. The soil free surface is discretized into constant triangular or quadrilateral boundary elements (surface elements) and the equations for transverse displacements of points of the soil surface are considered in discretized form, in order to obtain an algebraic equation system for the soil. Finally, the equation systems for the plate and for the soil are coupled in order to accomplish the soil-structure interaction.

Cartesian index notation is used along this text, with Greek indices varying from 1 to 2 and Latin ones, from 1 to 3.

2 BASIC FORMULATION

Consider an isotropic, homogeneous and linearly elastic plate, with constant thickness h and subjected to a transversal loading q per unit area, resting on a soil considered as an infinite half-space. It is also considered a transverse loading p per unit area, representing the soil reaction.

The axes x_α are considered at the plate middle plane and x_3 in the transverse direction. The rotations in the planes $x_\alpha - x_3$ are represented by u_α and the transverse displacement is represented by u_3.

Moments $M_{\alpha\beta}$ and shear forces Q_α, including the contribution of foundation reaction, are given in terms of the generalized displacements by:

$$M_{\alpha\beta} = D\frac{(1-\nu)}{2}\left[u_{\alpha,\beta} + u_{\beta,\alpha} + \frac{2\nu}{1-\nu}\, u_{\gamma,\gamma}\,\delta_{\alpha\beta}\right] + \frac{\nu\left(q+p\right)}{(1-\nu)\,\lambda^2}\delta_{\alpha\beta}$$

$$Q_\alpha = D\frac{1-\nu}{2}\lambda^2\left(u_\alpha + u_{3,\alpha}\right) \tag{1}$$

in which

$$D = \frac{E\ h^3}{12\ \left(1-\nu^2\right)} \text{ = flexural rigidity of the plate;} \tag{2}$$

$$\lambda = \frac{\sqrt{10}}{h} \text{ = constant of Reissner's equations.} \tag{3}$$

and also ν is Poisson's ratio of the plate, $\delta_{\alpha\beta}$ is Kronecker's delta and E is the longitudinal elastic modulus of the plate.

Equilibrium equations for a general domain point, obtained from the equilibrium of an infinitesimal plate element, are written as:

$$M_{\alpha\beta,\beta} - Q_\alpha = 0$$

$$Q_{\alpha,\alpha} + q + p = 0 \tag{4}$$

For a general boundary point, the following boundary tractions are defined:

$$p_\alpha = M_{\alpha\beta} n_\beta$$

$$p_3 = Q_\beta n_\beta \tag{5}$$

with n_β being the direction cosines of the outward normal to the boundary.

In addition, the following boundary conditions are considered:

$$u_k = \bar{u}_k \;\; in \;\; \Gamma_u$$
$$p_k = \bar{p}_k \;\; in \;\; \Gamma_p \tag{6}$$

with Γ_u being the part of the plate boundary where the generalized displacements are prescribed and Γ_p being the part of the plate boundary where the generalized tractions are prescribed. The total boundary is defined as:

$$\Gamma = \Gamma_u \cup \Gamma_p . \tag{7}$$

3 PLATE INTEGRAL EQUATIONS

From a weighted residual method and Reissner's plate bending theory, including the contribution of the elastic foundation reaction, and after performing the limits of the integrals as the source point ends to the boundary, the following integral equation can be obtained, which is valid for points situated in the plate domain Ω or at the plate boundary Γ:

$$\begin{aligned} c_{ij}(\xi)\, u_j(\xi) = & \int_\Gamma u^*_{ij}(\xi, x)\, p_j(x) d\Gamma(x) - \int_\Gamma p^*_{ij}(\xi, x) u_j(x) d\Gamma(x) + \\ & + \int_\Omega q(x) \left[u^*_{i3}(\xi, x) - \frac{\nu}{(1-\nu)\lambda^2} u^*_{i\alpha,\alpha}(\xi, x) \right] d\Omega(x) + \\ & + \int_\Omega p(x) \left[u^*_{i3}(\xi, x) - \frac{\nu}{(1-\nu)\lambda^2} u^*_{i\alpha,\alpha}(\xi, x) \right] d\Omega(x) \end{aligned} \tag{8}$$

where $u^*_{ik}(\xi, x)$ and $p^*_{ik}(\xi, x)$ represent components of the displacement tensor of the fundamental solution for the problem and components of the corresponding traction tensor, respectively, in the direction *k* of the field point x due to the action of a unit concentrated generalized force applied in the direction i of the source point ξ. The coefficient $c_{ij}(\xi)$ depends on boundary geometry at point ξ and one has:

$$c_{ij} = \delta_{ij} \;\; \text{if } \xi \text{ is a domain point;}$$
$$c_{ij} = \delta_{ij}/2 \;\; \text{if } \xi \text{ is a smooth boundary point.} \tag{9}$$

The domain integrals can be transformed into boundary integrals for several kinds of loading. In the present work, the load $q(x)$ is considered as a uniformly distributed load, so that $q(x)=q=$ constant. In addition, the region over the elastic support is considered divided into constant triangular or quadrilateral cells and it is admitted that $p(x)$ is constant in each cell, so that $p(x)=p_c=$ constant in each cell c.

In this case, by using the divergence theorem in order to transform domain the integrals of Eq. (8) into boundary integrals, one obtains:

$$c_{ij}(\xi)\,u_j(\xi)=\int_\Gamma\left[u^*_{ij}(\xi,x)p_j(x)-p^*_{ij}(\xi,x)u_j(x)\right]d\Gamma(x)\;+$$
$$+\,q\int_\Gamma\left[v^*_{i,\alpha}(\xi,x)-\frac{\nu}{(1-\nu)\lambda^2}u^*_{i\alpha}(\xi,x)\right]n_\alpha(x)\,d\Gamma(x)+$$
$$+\sum_{c=1}^{Nc}p_c\int_{\Gamma_c}\left[v^*_{i,\alpha}(\xi,x)-\frac{\nu}{(1-\nu)\lambda^2}u^*_{i\alpha}(\xi,x)\right]n_\alpha(x)\,d\Gamma_c(x) \tag{10}$$

in which N_c is total number of cells and Γ_c is the boundary of cell c.

The generalized displacements at internal points are obtained from Eq. (10), with $C_{ik}=\delta_{ik}$. So, for an internal point ξ, one has:

$$u_j(\xi)=\int_\Gamma u^*_{ij}(\xi,x)p_j(x)d\Gamma(x)\;-\int_\Gamma p^*_{ij}(\xi,x)u_j(x)d\Gamma(x)+$$
$$+\,q\int_\Gamma\left[v^*_{i,\alpha}(\xi,x)-\frac{\nu}{(1-\nu)\lambda^2}u^*_{i\alpha}(\xi,x)\right]n_\alpha(x)\,d\Gamma(x)+$$
$$+\sum_{c=1}^{Nc}p_c\int_{\Gamma_c}\left[v^*_{i,\alpha}(\xi,x)-\frac{\nu}{(1-\nu)\lambda^2}u^*_{i\alpha}(\xi,x)\right]n_\alpha(x)\,d\Gamma_c(x) \tag{11}$$

For calculating moments and shear forces at internal points, the following boundary equations are obtained:

$$M_{\alpha\beta}(\xi)=\int_\Gamma u^*_{\alpha\beta k}(\xi,x)\,p_k(x)d\Gamma(x)-\int_\Gamma p^*_{\alpha\beta k}(\xi,x)\,u_k(x)d\Gamma(x)+q\int_\Gamma w^*_{\alpha\beta}(\xi,x)d\Gamma(x)\;+$$
$$+\sum_{c=1}^{Nc}p_c\int_{\Gamma_c}w^*_{\alpha\beta}(\xi,x)d\Gamma(x)+\frac{\nu}{(1-\nu)\lambda^2}\,(q+p)\delta_{\alpha\beta} \tag{12}$$

and

$$Q_\alpha(\xi) = \int_\Gamma u^*_{3\beta k}(\xi,x)\,p_k(x)\,d\Gamma(x) - \int_\Gamma p^*_{3\beta k}(\xi,x)\,u_k(x)\,d\Gamma(x) + q\int_{\Gamma_c} w^*_{3\beta}(\xi,x)\,d\Gamma(x) +$$

$$+\sum_{c=1}^{Nc} p_c \int_{\Gamma_c} w^*_{3\beta}(\xi,x)\,d\Gamma(x) \qquad (13)$$

Expressions for the tensors corresponding to the fundamental solution are found in Van der Weeen (1982), Karam (1986) and Karam and Telles (1988).

4 NUMERICAL IMPLEMENTATION OF PLATE INTEGRAL EQUATIONS

For the numerical solution of Eq. (10), the boundary Γ is divided into quadratic boundary elements, in which u_j and p_j are calculated by interpolating the nodal values. In addition, each cell boundary Γ_c is divided into cell boundary elements. So, this equation is written in discretized form for each nodal point ξ at Γ, by substituting the boundary integrals and boundary cell integrals in Eq. (10) by summations of boundary element integrals and cell boundary element integrals, respectively.

By applying Eq. (10) in discretized form to all nodal points ξ of the boundary, one obtains a system with a number of equations three times the number of nodal points, in the form:

$$(\mathbf{C}+\hat{\mathbf{H}})\,\mathbf{u} = \mathbf{G}\,\mathbf{p} + \mathbf{b} + \mathbf{m} \qquad (14)$$

or

$$\mathbf{H}\,\mathbf{u} = \mathbf{G}\,\mathbf{p} + \mathbf{b} + \mathbf{m} \qquad (15)$$

in which:

$$\mathbf{H} = \mathbf{C} + \hat{\mathbf{H}} \qquad (16)$$

$\mathbf{C}$ = matrix which contains the terms c_{ij};

$\mathbf{u}$ and $\mathbf{p}$ = vectors that contain the nodal values of displacements and tractions, respectively;

$\mathbf{b}$ = vector that contains the applied distributed load contribution;

$\mathbf{m}$ = vector that contains the soil reaction contribution;

$\hat{\mathbf{H}}$ and $\mathbf{G}$ = matrices that contain the integrals over the boundary elements.

The system of equations (15) admits as unknowns the displacements or tractions of the boundary nodes. Since the reactions of the elastic support represent additional unknowns to this system, integral equations for transverse displacements of points situated at the geometric center of the internal cells are also added to this system.

By applying the third of the three integral equations represented in the expression presented in Eq. (11), to all cell nodal points, one has, after discretizing and combining with Eq. (15), the system of equations in the following form:

$$\begin{bmatrix} \mathbf{H} & \mathbf{0} \\ \hat{\mathbf{H}}_\mathbf{i} & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \mathbf{u_i} \end{Bmatrix} = \begin{Bmatrix} \mathbf{G} \\ \mathbf{G_i} \end{Bmatrix} \{\mathbf{p}\} + \begin{Bmatrix} \mathbf{b} \\ \mathbf{b_i} \end{Bmatrix} - \begin{Bmatrix} \mathbf{s_r} \\ \mathbf{s_{ri}} \end{Bmatrix} \{\mathbf{p_r}\} \tag{17}$$

where index **i** indicates that the vectors and matrices are related to internal points.

Displacements at internal points are calculated by using Eq. (11) in discretized form and, for the calculation of moments and shear forces at internal points, Eq. (12) and Eq. (13), respectively, are also considered in discretized form.

5 SOIL INTEGRAL EQUATIONS

Consider a body of arbitrary shape, with domain Ω and bounded by a surface Γ and also that this body is in equilibrium state and has a linear elastic behavior.

From the three-dimensional theory of Elasticity and using a weighted residual method, one can obtain the following equation for the displacements of a general point ξ of the body:

$$\begin{aligned} c_{ij}(\xi)u_i(\xi) = &-\int_\Gamma p^*_{ij}(\xi,x)u_j(x)d\Gamma(x) + \int_\Gamma u^*_{ij}(\xi,x)p_j(x)d\Gamma(x) \\ &+\int_\Omega u^*_{ij}(\xi,x)b_j(x)d\Omega(x) \end{aligned} \tag{18}$$

where the coefficient $c_{ij}(\xi)$ is defined as in Eq. (9) and depends on the geometry of the boundary at point ξ.

By considering a half-space representing the soil, the same Eq. (18) is obtained and, for this case, $c_{ij}(\xi) = \delta_{ij}$ for both domain and boundary points.

In the present work, only the third of the three equations represented by Eq. (18) is used, written for the vertical displacement of a point situated at the region of the soil surface in contact with the plate. In addition, by considering that the parcel which corresponds to tractions related to the fundamental solution in Eq. (18) vanishes at the soil surface and by neglecting the body forces, the integral equation for the vertical displacement of such point is given by:

$$u_3(\xi) = \int_\Gamma u^*_{33}(\xi,x)p_3(x)d\Gamma(x) \tag{19}$$

Considering the fundamental solution of Boussinesq-Cerruti and the soil surface in contact with the plate divided into constant triangular or quadrilateral surface elements, so that in each element the vertical traction is constant, Eq. (19) is written as:

$$u_3\left(\xi\right)=\sum_{c=1}^{Ncel} p_{3c}\,\frac{1-\nu_s}{2\pi G_s}\int_{\Gamma_c}\frac{1}{r}\,d\Gamma_c \tag{20}$$

in which N_{cel} is the number of surface elements in the soil surface in contact with the plate, p_{3c} the is constant vertical traction on boundary surface element c, ν_s is Poisson's ratio of the soil and G_s is the transversal elasticity modulus of the soil.

In order to eliminate the singularity of Eq. (20), a polar coordinate system is used and the integrals over the domains of the surface elements are transformed into integrals over the boundary of the surface elements and this equation becomes as:

$$u_3\left(\xi\right)=\sum_{c=1}^{Ncel} p_{3c}\,\frac{1-\nu_s}{2\pi G_s}\int_{\Gamma_{bc}} r,_n\,d\Gamma_{bc} \tag{21}$$

where $r,_n$ is the derivative of r with reference to the normal direction.

6 NUMERICAL IMPLEMENTATION OF SOIL INTEGRAL EQUATIONS

For the numerical solution of the soil equations, let the boundary Γ_{bc} of each surface element be divided into boundary elements and Eq. (21) be written, in discretized form, for a soil surface element node ξ of Γ_c, situated at the geometric center of the surface element, by substituting the integrals over the boundaries of the surface elements by summations of integrals over the elements of the boundaries of the surface elements.

By considering Eq. (21) in discretized form for all surface element nodes, one has the system of equations for the soil given by:

$$\{\mathbf{u_i}\}=[\mathbf{A}]\{\mathbf{p_r}\} \tag{22}$$

in which

$\{\mathbf{u_i}\}$= vector formed with the vertical displacements of all surface element nodes;

$\{\mathbf{p_r}\}$= vector formed with the tractions of all surface element nodes;

$[\mathbf{A}]$= matrix related to the fundamental solution.

By inverting Eq. (22), one has the system of equations for the soil given by:

$$\{\mathbf{p_r}\}=[\mathbf{B}]\{\mathbf{u_i}\}. \tag{23}$$

7 PLATE-SOIL INTERACTION

In order to perform the plate-soil interaction, the system of equations for the plate and for the soil, given by Eq. (17) and Eq. (23), respectively, can be coupled. To this purpose, one can substitute the vector expressed by Eq. (25) into the system given by Eq. (17) and this results in a system of the following form:

$$\begin{bmatrix} \mathbf{H} & \mathbf{0} \\ \hat{\mathbf{H}}_\mathbf{i} & \mathbf{I} \end{bmatrix}\begin{Bmatrix} \mathbf{u} \\ \mathbf{u_i} \end{Bmatrix} = \begin{Bmatrix} \mathbf{G} \\ \mathbf{G_i} \end{Bmatrix}\{\mathbf{p}\} + \begin{Bmatrix} \mathbf{b} \\ \mathbf{b_i} \end{Bmatrix} - \begin{Bmatrix} \mathbf{s_r} \\ \mathbf{s_{ri}} \end{Bmatrix}[\mathbf{B}]\{\mathbf{u_i}\} \tag{24}$$

The system given by Eq. (24) can also be rewritten, by locating all unknowns in a vector $\mathbf{x}$ and all known values in a vector $\mathbf{f}$ and so, one obtains the system in the form

$$\mathbf{A}\ \mathbf{x} = \mathbf{f} \tag{25}$$

The unknowns of this system are displacements or tractions in each generalized direction of boundary nodes and vertical displacements at each cell nodal point.

It is important to mention that the discretizations of the plate domain and the soil surface must be the same.

8 NUMERICAL EXAMPLES

In order to validate the developed formulation, some examples are presented and the corresponding results are compared with results obtained from literature.

8.1 Example 1: square plate with free edges

This example consists of a square plate with free edges over a soil which is considered as a semi-infinite continuum medium, subjected to a uniformly distributed transverse loading q.

The plate is considered with side $b = 12m$, thickness $h = 0.1m$ and Poisson's ratio $\nu_p = 0.3$. The soil in which the plate is supported has Young's modulus $E_s = 0.26\text{x}10^6 kN/m^2$ and Poisson's ratio $\nu_s = 0.3$.

The elasticity modulus of the plate E_p is obtained from a relative rigidity factor K between the plate and the soil, presented in Shen et al (1999) and expressed by:

$$K = \frac{4}{3}\frac{E_p(1-\nu_p^2)h^3}{E_s(1-\nu_s^2)b^3} \tag{26}$$

which is valid for the general case of a rectangular plate of sides a and b, with b being the smaller side of the plate.

For the obtained transverse displacement, the following non-dimensional factor is used:

$$I_w = \frac{E_s u_3}{qb(1-\nu_s^2)} \tag{27}$$

In Fig. 1, it is shown the discretization used, consisting of 36 quadratic boundary elements and 81 constant internal cells for the plate and 81 constant surface elements for the soil surface.

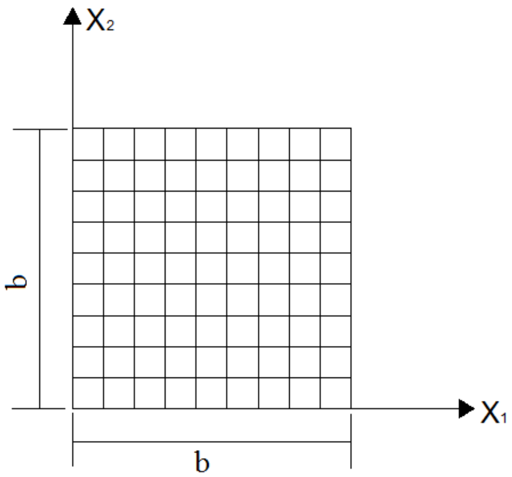

Figure 1. Square plate: discretization in 36 elements and 81 internal cells

The obtained results were compared with results of Shen et al. (1999), who utilized Kirchhoff's theory for plate bending and an energy method to solve the problem. In Fig. 2, one can observe the values of factor I_w for a variation of the relative rigidity from 0.001 to 100, in comparison to the values obtained by Shen et al. (1999), at the points indicated in Fig. 3. It can be observed that the values of I_w for the point A, located at the plate center, are coincident, even for a mesh not so refined. It can also be observed that the values of I_w for points B and C, located at the boundary of the plate, are close to those obtained by Shen et al. (1999). For points B and C, results coincident to results obtained by Shen et al. (1999) were obtained by using a more refined mesh, with 76 boundary elements and 361 cells and soil surface elements.

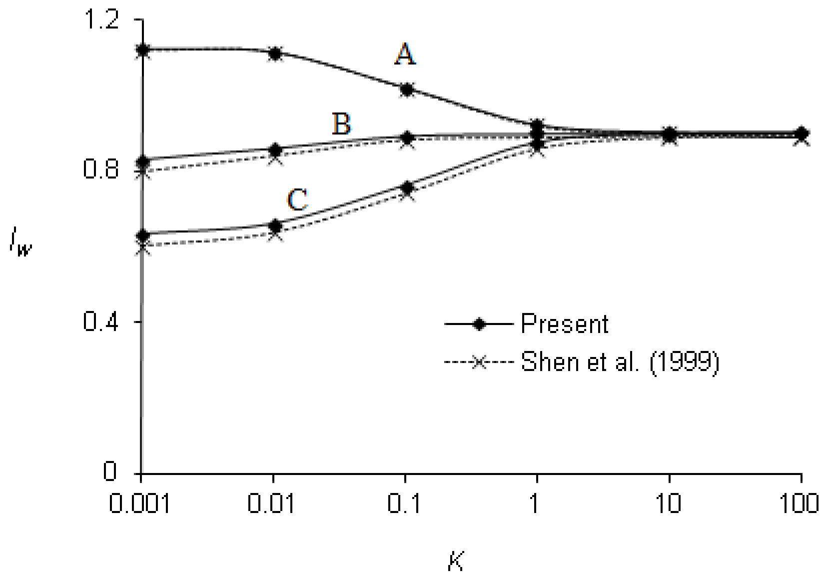

Figure 2. Square plate: factor I_w versus relative rigidity K

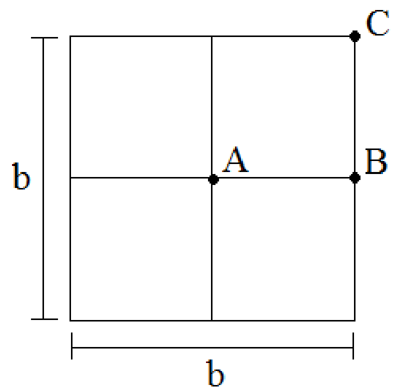

Figure 3. Square plate with indicated points

8.2 Example 2: circular plate with free edge

In this example, a circular plate with free edge subjected to a uniformly distributed load q, over a semi-infinite continuum medium is analyzed. The plate is considered with radius $a = 8m$, thickness $h = 0.1m$ and Poisson's ratio $\nu_p = 0.3$. The soil in which the plate is supported has elasticity modulus $E_s = 0.26 \text{x} 10^6 kN/m^2$ and Poisson's ratio $\nu_s = 0.3$.

The elasticity modulus of the plate E_p is obtained from a relative rigidity factor X between the plate and the soil, defined in Zaman et al. (1988), as:

$$X = \left(1 - \nu_s^2\right)\left(\frac{E_p}{E_s}\right)\left(\frac{t}{a}\right)^3 \tag{28}$$

The plate was analyzed for values of X varying from 0.01 to 10.

The plate was discretized into 40 quadratic boundary elements and 200 constant internal cells, as it can be seen in Fig. 4 and the soil surface was discretized with 200 surface elements.

In Fig. 5, the variation of the transverse displacement as the plate becomes more rigid, for the point situated at the plate center. The transverse displacements are written in non-dimensional form, as:

$$u_0 = u_3 \frac{E_s}{qa(1-\nu_s^2)} \tag{29}$$

with u_3 being the calculated displacements

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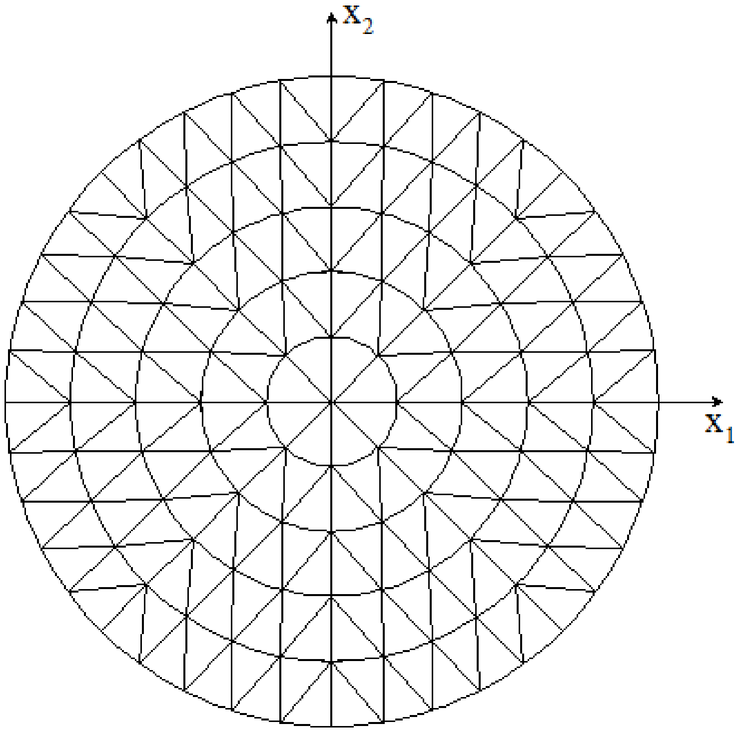

Figure 4. Circular plate: discretization in 40 elements and 200 internal cells

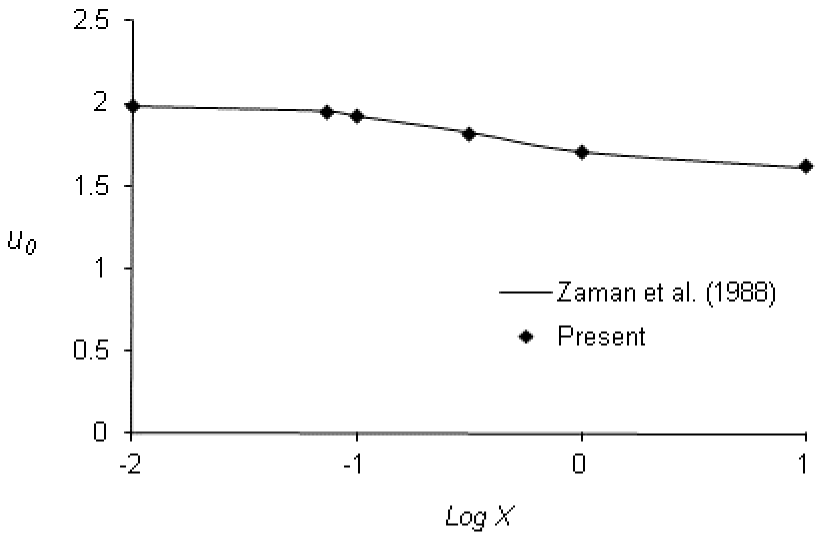

Figure 5. Circular plate: displacement u_o versus relative rigidity factor

In Fig. 6, it is presented the variation of transverse displacements along a radius of the plate, also in non-dimensional form, for a plate of intermediate rigidity, $X = 0.1$.

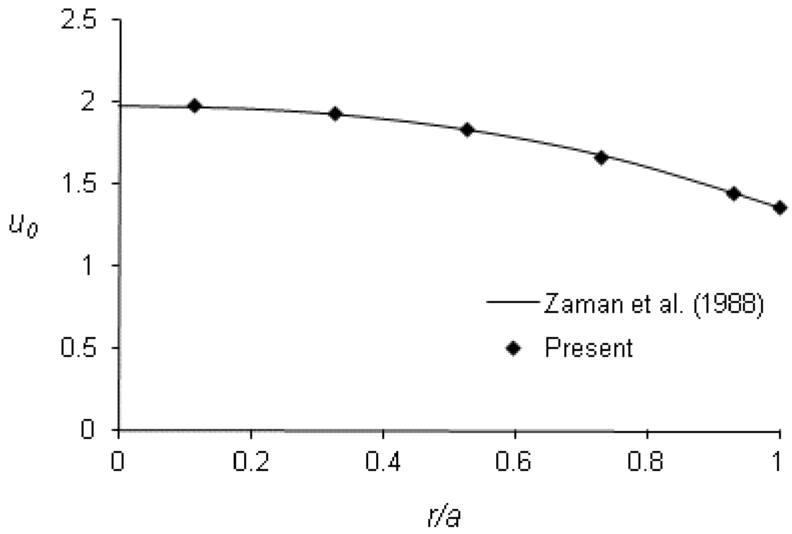

Figure 6. Circular plate: transverse displacement u_0 for $X = 0.1$

As it can be observed, the results obtained in the present work for displacements and soil reactions are in excellent agreement with the results obtained by Zaman et al. (1988), who have developed an analytical solution based on an energy method.

9 CONCLUSIONS

A formulation for analysis of plate-soil interaction using the BEM was presented in this work, considering Reissner's theory for plate bending and with the soil modeled as an elastic half-space. In order to validate the presented development, the obtained results were compared with results obtained from the literature using energy and variational methods. It was observed that these results are in a very good accordance.

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