DIRECT SOLUTION OF 2D HELMHOLTZ EQUATIONS USING THE WAVELET-GALERKIN METHOD

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Abstract. *In the last three decades, the solution of partial differential equations (PDEs) using wavelets has become increasingly popular. Therefore, the use of wavelet scaling functions as a basis in numerical methods holds some promise due to their compact support, orthogonality, localization and multiresolution properties. Daubechies functions have been successfully used as a basis in several schemes like the Wavelet-Galerkin Method (WGM) and the Wavelet Finite Element Method (WFEM). Later, Gilles Deslauriers and Serge Dubuc dyadic subdivision algorithm served as a foundation for the arrival of wavelet scaling functions with interpolating properties, later called Interpolets. In this work, some mathematical foundations and computer implementation aspects regarding wavelet scaling functions, their derivatives and connection coefficients are reviewed. A scheme based on the Galerkin Method is proposed for the direct solution of Helmoltz and Poisson's equations in a meshless formulation using Deslauriers-Dubuc scaling functions (Interpolets). Provided that the analysis domain is rectangular, results can be obtained also for a randomly distributed data set (scattered points). The applicability of the proposed method and some convergence issues are illustrated by means of a few examples.*

Keywords: *Wavelets, Interpolets, Helmholtz equation, Poisson's equation.*

1 INTRODUCTION

The use of wavelet-based numerical schemes has become increasingly popular in the last two decades. Wavelet scaling functions have several properties that are especially useful for representing solutions of differential equations (DE's), such as orthogonality, compact support and exact representation of polynomials of a certain degree. Their capability of representing data at different levels of resolution allow the efficient and stable calculation of functions with high gradients or singularities, which would require a dense mesh or higher order elements in a Finite Element analysis (Qian and Weiss, 1992).

A complete basis of wavelets can be generated through dilation and translation of a mother scaling function. Although many applications use only the wavelet filter coefficients of the multiresolution analysis, there are some which explicitly require the values of the basis functions and their derivatives, such as the Wavelet Finite Element Method (WFEM) (Ma et al., 2003).

Compactly supported wavelets and scaling functions have a finite number of derivatives which can be highly oscillatory. This makes the numerical evaluation of integrals of their inner products difficult and unstable. Those integrals are called connection coefficients and they appear naturally when applying a numerical method for the solution of a DE. Due to some properties of scaling functions, these coefficients can be obtained by solving an eigenvalue problem using filter coefficients.

The most commonly used wavelet family is the one developed by Ingrid Daubechies (1988). All the mathematical foundation for the wavelet theory was formulated for Daubechies wavelets and then extended to other families.

Working with dyadically refined grids, Deslauriers and Dubuc (1989) obtained a new family of functions with interpolating properties, later called Interpolets. Unlike Daubechies wavelets, Interpolets are symmetric, which is especially interesting in numerical analysis. The use of Interpolets instead of Daubechies scaling functions considerably improves the method's accuracy (Burgos et al., 2013).

The use of wavelet scaling functions in numerical schemes such as the Galerkin Method holds some promise due to their multiresolution properties. Accuracy can be improved by increasing either the level of resolution or the order of the wavelet used. In this work, the latter seemed to work better.

Two one-dimensional examples were used as a validation of the proposed method for solving Poisson and Helmoltz equations with known analytical solutions. In order to evaluate the method's capability to solve two-dimensional problems, it was then applied for a homogeneous Helmholtz equation with known analytical solution. In these examples, results were compared to a Finite Differences Method scheme. Finally, scattered data was fed as the non-homogeneous part of a Poisson's equation with excellent results in terms of convergence.

2 WAVELET THEORY

Multiresolution analysis using orthogonal, compactly supported wavelets has been successfully applied in numerical simulation. Wavelet basis are composed of two kinds of functions: scaling functions (φ) and wavelet functions (ψ). The two combined form a

complete Hilbert space of square integrable functions. The spaces generated by scaling and wavelet functions are complementary and both are based on the same mother function.

In the following expression, known as the two-scale relation, a_k are the scaling function filter coefficients and N is its order.

$$\varphi(x) = \sum_{k=0}^{N-1} a_k \varphi(2x-k) = \sum_{k=0}^{N-1} a_k \varphi_k(2x) \,. \tag{1}$$

In general, there are no analytical expressions for wavelet scaling functions, which can be obtained using iterative procedures like Eq. (1). In order to comply with the requirements of orthogonality and compact support, wavelet scaling functions present, in general, an irregular fractal-like shape. Figure 1 shows Daubechies scaling functions of four different orders. It is evident that the higher the order of the function, the greater its smoothness.

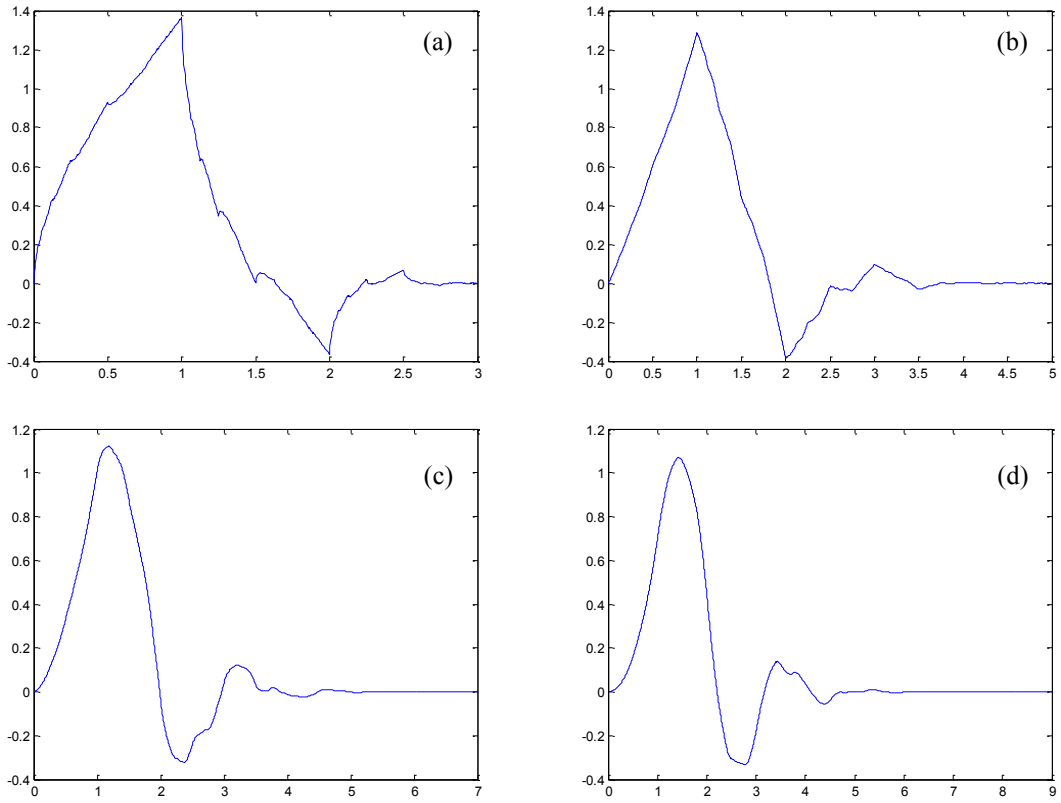

Figure 1. Daubechies scaling functions: (a) $N = 4$, (b) $N = 6$, (c) $N = 8$, (d) $N = 10$

2.1 Wavelet properties

The set of properties summarized in Eq. (2) is valid for Daubechies scaling funtions but can be adapted to other wavelet families, such as Delauriers-Dubuc Interpolets. Some of these properties, like compact support and unit integral, are required for the use of the wavelet family in numerical methods. Others, like orthogonality, are desirable but not necessary.

The last expression in Eq. (2) derives from the vanishing moments property, which states that a set of shifted and scaled Daubechies scaling functions of order N is capable of representing exactly an $N/2 - 1$ degree polynomial.

$$
\begin{aligned}
&\operatorname{supp}(\varphi) = [0 \;\; N-1], \\
&\int_{-\infty}^{+\infty} \varphi(x)dx = 1, \\
&\int_{-\infty}^{+\infty} \varphi(x-i)\varphi(x-j)dx = \delta_{i,j}, \\
&x^m = \sum_{-\infty}^{+\infty} c_k^m \varphi(x-k), \quad m \le N/2-1.
\end{aligned}
\tag{2}
$$

2.2 Wavelet Derivatives

In the process of solving a DE using numerical methods, derivatives of the basis functions tend to appear. As there is no analytical expression for scaling functions, derivatives are obtained in dyadic grid points and the refinement of the solution depends on the level of resolution needed (Lin et. al, 2005). The scale relation in Eq. (1) can be differentiated *d* times, generating the following expression:

$$\varphi^{(d)}(x) = 2^d \sum_{i=0}^{N-1} a_i \varphi^{(d)}(2x-i). \tag{3}$$

Applying Eq. (3) to integer points results in the following system of equations shown in matrix form in Eq. (4), where **A** represents the filter coefficients matrix, **I** is the identity matrix and $\mathbf{\Gamma^{(d)}}$ is the vector containing derivative values at integer points of the grid.

$$
\begin{aligned}
&(2^d \mathbf{A} - \mathbf{I})\mathbf{\Gamma^{(d)}} = 0, \\
&\mathbf{A} = \left[a_{2i-k}\right]_{0 \le i,k \le N-1}.
\end{aligned}
\tag{4}
$$

Equation (4) is an eigenvalue problem which, for unique solution, has to be normalized using the so-called moment equation, derived from the wavelet property of exact polynomial representation. This equation is given by Latto et al. (1992) and provides a relation between derivative values at integer points.

$$
\begin{aligned}
&d! = \sum_{i=0}^{N-1} M_i^d \varphi^{(d)}(x-i), \\
&M_i^j = \frac{1}{2^{j+1}-2} \sum_{k=0}^{j} \binom{j}{k} i^{j-k} \sum_{l=0}^{k-1} \binom{k}{l} M_0^l \left(\sum_{i=0}^{N-1} a_i i^{k-l} \right).
\end{aligned}
\tag{5}
$$

The coefficients *M* in Eq. (5) are the moments of the d^{th} derivative of the scaling function translation *i*. Once the derivative is obtained at integer values, the scale relation can be applied for any $x = k/2^j$ using:

$$\varphi^{(d)}\left(\frac{k}{2^j}\right) = 2^d \sum_{i=0}^{N-1} a_i \varphi^{(d)}\left(\frac{k}{2^{j-1}} - i\right), \quad k = 1,3,5,\ldots \qquad . \tag{6}$$

2.3 Connection Coefficients

Assuming that a function f is approximated by a series of scaling functions, the following may be written:

$$f(\xi) = \sum_k \alpha_k \varphi_k(\xi). \tag{7}$$

In the process of solving a DE of order n by some numerical method, the approximation of the solution using Eq. (7) requires the calculation of inner products of the basis functions and their derivatives up to d_n. These inner products are defined as connection coefficients (Γ) given by:

$$\Gamma_{i,j}^{d_1,d_2} = \int \varphi^{(d_1)}(\xi - i)\varphi^{(d_2)}(\xi - j)d\xi. \tag{8}$$

The values for the limits of the integral in Eq. (8) depend on which method is used to impose boundary conditions. In this work, the limits are given by [0 2^m], where m is the level of resolution. This method allows the use of Lagrange multipliers to deal with boundary conditions, similarly to what is usually done in a meshless scheme (Nguyen et al., 2008). Connection coefficients at level m can be obtained through the calculation at level 0, thus avoiding its recalculation while increasing the level of resolution. Dilation and translation properties allow the calculation of connection coefficients within the interval [0 1] to be summarized by the solution of an eigenvalue problem based only on filter coefficients (Zhou & Zhang, 1998).

$$\begin{aligned} &\left(\mathbf{P} - \frac{1}{2^{d_1+d_2-1}}\mathbf{I}\right)\mathbf{\Gamma}^{\mathbf{d_1},\mathbf{d_2}} = 0, \\ &P_{i,j:k,l} = a_{k-2i}a_{l-2j} + a_{k-2i+1}a_{l-2j+1}. \end{aligned} \tag{9}$$

Since Eq. (9) leads to an infinite number of solutions, there is the need for a normalization rule that provides a unique eigenvector. This unique solution comes with the inclusion of an adapted version of the moment equation mentioned before (Latto et al., 1992).

$$\sum_i \sum_j M_i^k M_j^k \, \Gamma_{i,j}^{d_1,d_2} = \frac{(k!)^2}{(k-d_1)!(k-d_2)!(2k-d_1-d_2+1)}. \tag{10}$$

2.4 Delauriers-Dubuc Interpolets

The term Interpolet was first used by Donoho (1992) to designate wavelets with interpolating characteristics. The basic characteristics of interpolating scaling functions require that the mother scaling function satisfies the following condition (Shi et al., 1999):

$$\begin{aligned} &\varphi(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}, \quad n \in \mathbb{Z} \\ &\varphi(x) = \sum_{k=1-N}^{N-1} c_k \varphi(2x-k), \quad c_k = \sum_{m=0}^{N-1} a_m a_{m-k}. \end{aligned} \tag{11}$$

The filter coefficients c_k for Deslauriers-Dubuc scaling function of order N can be obtained by an autocorrelation of the same order Daubechies' filter coefficients (a_m), as shown in Eq. (11). Interpolets satisfy the same requirements as other scaling functions, specially the two-scale relation, which is fundamental for their use in numerical methods. Figure 2 shows the Interpolet IN6 (autocorrelation of DB6, Daubechies wavelet of order 6). Its symmetry and interpolating properties are evident. There is only one integer abscissa which evaluates to a non-zero value.

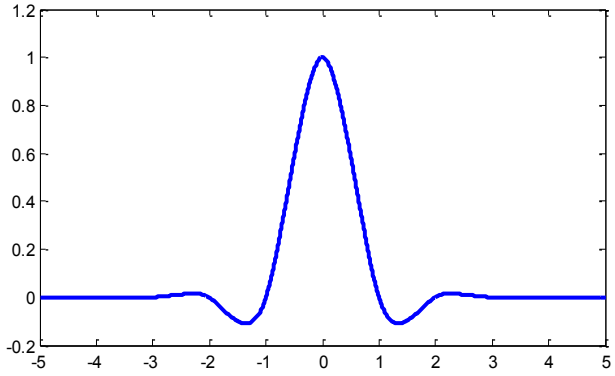

Figure 2. Interpolet IN6 scaling function with its full support.

All expressions used for the calculation of derivatives, connection coefficients and moments of Daubechies scaling functions can be applied to Interpolets. Of course, due to the correlation, the support [0 N−1] in the expressions for Daubechies becomes [1−N N−1] for Interpolets.

3 WAVELET-GALERKIN METHOD

The numerical solution of differential equations is one of the possible applications of the wavelet theory. The Wavelet-Galerkin Method (WGM) results from the use of wavelet scaling functions as the interpolating basis in a traditional Galerkin scheme. In the following sections, the WGM will be applied to solve one and two-dimensional differential equations.

3.1 Poisson's equation in 1-D (Truss equation)

The one-dimensional Poisson's equation is equivalent to the differential equation of an axially loaded bar (truss element):

$$\frac{d^2u}{d\xi^2} = f(\xi)\,. \tag{12}$$

Using interpolating scaling functions as a basis, multiplying by $\varphi(\xi - i)$ and integrating:

$$\begin{aligned} &u(\xi) = \sum_j d_j \varphi(\xi - j) \rightarrow \frac{d^2u}{d\xi^2} = \sum_j d_j \varphi''(\xi - j), \\ &\sum_j d_j \int_0^1 \varphi(\xi - i)\varphi''(\xi - j)d\xi = \int_0^1 f(\xi)\varphi(\xi - i)d\xi. \end{aligned} \tag{13}$$

The system in Eq. (13) is in wavelet space and can be written in matrix form as:

$$\mathbf{kd} = \mathbf{f}, \quad \mathbf{k} = \mathbf{\Gamma}^{02}. \tag{14}$$

Vector **f** can have an analytical solution depending on the function $f(\xi)$. The matrix **k** is composed by connection coefficients, which are obtained analytically for interpolating scaling functions by:

$$\Gamma_{ij}^{02} = \int_0^1 \varphi_i(\xi)\varphi_j''(\xi)d\xi. \tag{15}$$

Boundary conditions. Some algebraic transformation needs to be performed in order to impose essential boundary conditions. Usually, Lagrange multipliers can be employed and they may or may not have physical meaning.

First, the matrix of connection coefficients is rewritten using integration by parts. The integration is performed in [0 1] resulting in extreme values of the scaling functions and their derivatives:

$$\begin{aligned} &\mathbf{\Gamma}^{02} = -\mathbf{\Gamma}^{11} + \mathbf{\Phi(1)}^{\mathrm{T}}\mathbf{\Phi}'\mathbf{(1)} - \mathbf{\Phi}(0)^{\mathrm{T}}\mathbf{\Phi}'(0), \\ &\mathbf{\Gamma}^{02}\mathbf{d} = -\mathbf{\Gamma}^{11}\mathbf{d} + \mathbf{\Phi(1)}^{\mathrm{T}} \underbrace{\qquad}_{u'(1)} \qquad \underbrace{\qquad}_{u'(0)} \end{aligned} \tag{16}$$

Substituting in the expression for the differential equation leads to:

$$\mathbf{\Gamma}^{02}\mathbf{d} = \mathbf{f} \rightarrow \begin{bmatrix} \mathbf{\Gamma}^{11} & \mathbf{\Phi(0)}^{\mathrm{T}} & \mathbf{\Phi(1)}^{\mathrm{T}} \end{bmatrix} \begin{Bmatrix} \mathbf{d} \\ u'(0) \\ -u'(1) \end{Bmatrix} = -\mathbf{f}, \quad \begin{matrix} \Phi(0)_j = \varphi(-j) \\ \Phi(1)_j = \varphi(1-j) \end{matrix}. \tag{17}$$

The derivatives $u'(0)$ and $u'(1)$ appear naturally (see Eq. (7)) and their values can be both unknown. If one of them is known then it can be passed to the right side of the equation. For instance, if one knows the value for $u'(1)$ the system becomes:

$$\mathbf{\Gamma}^{02}\mathbf{d} = \mathbf{f} \rightarrow \begin{bmatrix} \mathbf{\Gamma}^{11} & \mathbf{\Phi(0)}^{\mathrm{T}} \end{bmatrix} \begin{Bmatrix} \mathbf{d} \\ u'(0) \end{Bmatrix} = -\mathbf{f} + \mathbf{\Phi(1)}^{\mathrm{T}} u'(1). \tag{18}$$

If this form is used, boundary conditions can be imposed in a similar form as in meshless methods, adding a transpose of the vector **g** in order to obtain a square matrix, as shown in Eq. (19). In this case, λ is a vector of Lagrange multipliers. Another advantage is that the basis only needs to have a first derivative. This allows the use of lower order functions. For most scaling functions (including Daubechies and Interpolets), the rank of the stiffness matrix is one unit less than its size, i. e., only one boundary condition needs to be imposed for the system to have a solution. Nevertheless, one needs to have at least two conditions for the system (for example, the displacement at one side and the derivative at the other).

$$\begin{bmatrix} \mathbf{\Gamma}^{11} & \mathbf{g}^{\mathrm{T}} \\ \mathbf{g} & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{d} \\ \lambda \end{Bmatrix} = -\begin{Bmatrix} \mathbf{f} \\ 0 \end{Bmatrix}. \tag{19}$$

In the case of a truss equation, $u'(0)$ and $u'(1)$ have physical meaning: they are the strains at $x = 0$ and $x = 1$, and therefore they are proportional to the stress at those points. In a truss, generally if a displacement is known, the applied force in its direction is unknown and vice-versa. For instance, in a truss with axial stiffness EA, with the left-end fixed ($u(0) = 0$) and an applied compressive force F at the right-end ($F = EA\,u'(1)$), the reaction force R at $x = 0$ and the displacement at $x = 1$ are unknown. The system would become:

$$\begin{bmatrix} \boldsymbol{\Gamma}^{11} & \boldsymbol{\Phi}(\mathbf{0})^{\mathrm{T}} \\ \boldsymbol{\Phi}(\mathbf{0}) & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{d} \\ \dfrac{R}{EA} \end{Bmatrix} = \begin{Bmatrix} -\mathbf{f} + \boldsymbol{\Phi}(\mathbf{1})^{\mathrm{T}} \dfrac{F}{EA} \\ 0 \end{Bmatrix}. \tag{20}$$

In the more general case the system is given by:

$$\begin{bmatrix} \boldsymbol{\Gamma}^{11} & \boldsymbol{\Phi}(\mathbf{0})^{\mathrm{T}} & \boldsymbol{\Phi}(\mathbf{1})^{\mathrm{T}} \\ \boldsymbol{\Phi}(\mathbf{0}) & 0 & 0 \\ \boldsymbol{\Phi}(\mathbf{1}) & 0 & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{d} \\ u'(0) \\ u'(1) \end{Bmatrix} = \begin{Bmatrix} -\mathbf{f} \\ u(0) \\ u(1) \end{Bmatrix}. \tag{21}$$

The last equations of the system are prescribed displacements equations, i. e. the matrix **g** in Eq. (19). If interpolets are used, then the row and column of the correspondent degree of freedom can be removed, since the equations will become $d_j = u(0)$ and $d_{j+1} = u(1)$. This approach prevents numerical issues that can arise from the inversion of an ill-conditioned matrix. It's important to emphasize that the system described above is not the weak form of the DE: it is equivalent to the conventional strong form system and was obtained through integration by parts and not by using a variational approach.

3.2 Helmholtz equation in 2-D

For the solution of the two-dimensional Helmholtz equation in wavelet space, a similar procedure is adopted. Some extra algebraic transformations have to be performed, leading to some interesting findings. The equation is given by:

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + k^2 u(\xi,\eta) = f(\xi,\eta). \tag{22}$$

Provided that function f can be written in separate variables and applying the same form of solution used in 1-D form, the following can be written:

$$\left[\boldsymbol{\Gamma}^{\mathbf{02}} \otimes \boldsymbol{\Gamma}^{\mathbf{00}} + \boldsymbol{\Gamma}^{\mathbf{00}} \otimes \boldsymbol{\Gamma}^{\mathbf{02}} + k^2 \left(\boldsymbol{\Gamma}^{\mathbf{00}} \otimes \boldsymbol{\Gamma}^{\mathbf{00}}\right)\right] \boldsymbol{\alpha} = \int_0^1 \boldsymbol{\Phi}^{\mathrm{T}} f(\xi)\, d\xi \otimes \int_0^1 \boldsymbol{\Phi}^{\mathrm{T}} f(\eta)\, d\eta. \tag{23}$$

In Eq. (23), the symbol $\otimes$ indicates Kronecker product. Using the same integration by parts and rearranging the terms, the solution can be achieved by the following system:

$$\begin{bmatrix} \boldsymbol{\Gamma}^{\mathbf{11}} \otimes \boldsymbol{\Gamma}^{\mathbf{00}} + \boldsymbol{\Gamma}^{\mathbf{00}} \otimes \boldsymbol{\Gamma}^{\mathbf{11}} + k^2 \left(\boldsymbol{\Gamma}^{\mathbf{00}} \otimes \boldsymbol{\Gamma}^{\mathbf{00}}\right) & \mathbf{g}^{\mathrm{T}} \\ \mathbf{g} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{d} \\ \boldsymbol{\lambda} \end{Bmatrix} = -\begin{Bmatrix} \mathbf{f} \\ \mathbf{0} \end{Bmatrix}. \tag{24}$$

3.3 Interpolation of scattered data

If a set of scattered data **y**(x) is provided as a domain action instead of the independent function on the right side of the equation, this information should be written in terms of interpolating coefficients by solving the following system:

$$\begin{aligned} &y_i = \sum_j c_j \varphi_j(x_i), \\ &\mathbf{Ac} = \mathbf{y}, \quad A_{i,j} = \varphi_j(x_i). \end{aligned} \tag{25}$$

After obtaining the interpolating coefficients c_j, the function in the right side of the equation can be written in terms of scaling function translations and the force vector becomes:

$$f(x) = \sum_j c_j \varphi_j(x) \rightarrow \mathbf{f} = \int_0^1 \varphi_i(x) f(x) dx = \sum_j c_j \int_0^1 \varphi_i(x) \varphi_j(x) dx \,. \tag{26}$$

In 2-D, this is done in an analogous manner, for **z**(x,y):

$$z_i = \sum_j c_j \Phi_j(x_i, y_i), \quad A_{i,j} = \Phi_j(x_i, y_i) = \varphi_p(x_i) \varphi_q(y_i). \tag{27}$$

4 EXAMPLES

4.1 1-D Poisson's equation with known analytical solution

As a first example, the one-dimensional form of Poisson's equation is solved. To test the capability of dealing with high order functions, a non-homogeneous cubic function is used. The DE and its particular solution are given:

$$\frac{d^2u}{dx^2} = 1 + x + x^2 + x^3, \quad u_p(x) = \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{12} + \frac{x^5}{20}. \tag{28}$$

To simulate the behavior of a truss element subjected to a force given by Eq. (28), boundary conditions were set as $u(0) = 0$ and $u'(1) = 0$. The solution is shown in Fig. 3.

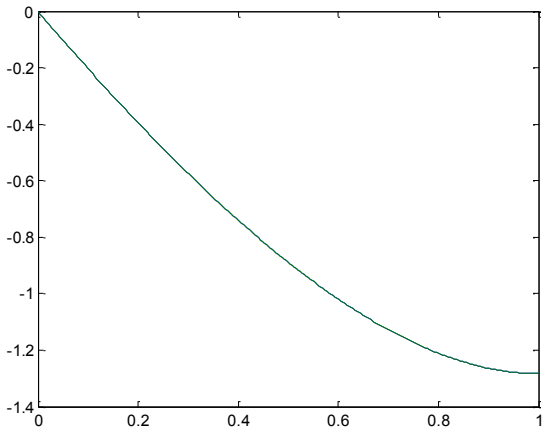

Figure 3. Solution for Example 1

The convergence of the method was studied and compared to a Finite Differences (FD) solution, as shown in Fig. 4. The Deslauriers-Dubuc scaling function used in this example was IN4 (i. e., DD function based on a Daubechies scaling function with 3 vanishing moments).

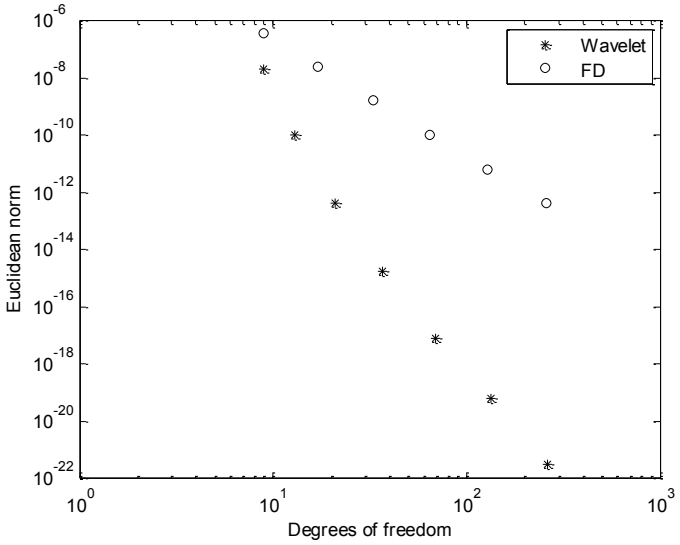

Figure 4. Convergence study for Example 1

4.2 1-D Helmholtz equation (Euler column)

Figure 5 shows a simply supported beam-column subjected to transversal (q) and axial (P) loading. Equilibrium analysis of the bending moment at any point x leads to a second order differential equation as a function of the transversal displacement $v(x)$:

$$EI\frac{d^2v}{dx^2} + Pv = \frac{q}{2}\left(Lx - x^2\right), \tag{29}$$

where EI is the flexural rigidity and L is the beam's length.

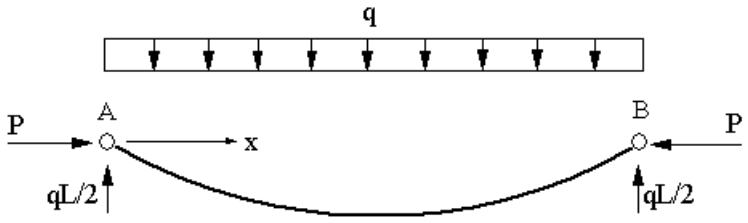

Figure 5. Simply supported beam under transversal and axial loading

Rearranging the terms, Eq. (29) can be written as a 1-D Helmholtz equation with known analytical solution:

$$\frac{d^2v}{dx^2} + k^2 v = f(x), \quad k^2 = \frac{P}{EI}, \quad f(x) = \frac{q}{2EI}\left(Lx - x^2\right). \tag{30}$$

The solution of Eq. (30) is given below for the boundary conditions shown in Fig. 5:

$$v(x)=\frac{q}{k^2P}\left[1-\cos(kx)-\tan\left(\frac{kL}{2}\right)\sin(kx)\right]+\frac{q}{2P}\left(Lx-x^2\right). \tag{31}$$

Results using $q = 1$, $P = 1$, $EI = 1$ and $L = 1$ are shown in Figs. 6 and 7 for displacement and bending moment, respectively. The Finite Element Method (FEM) was used as a standard for comparison, with the same number of degrees of freedom. It's evident that both methods give good results in terms of displacements, but for values that depend on higher derivatives (the bending moment depends on the second) the wavelet-based scheme performs better.

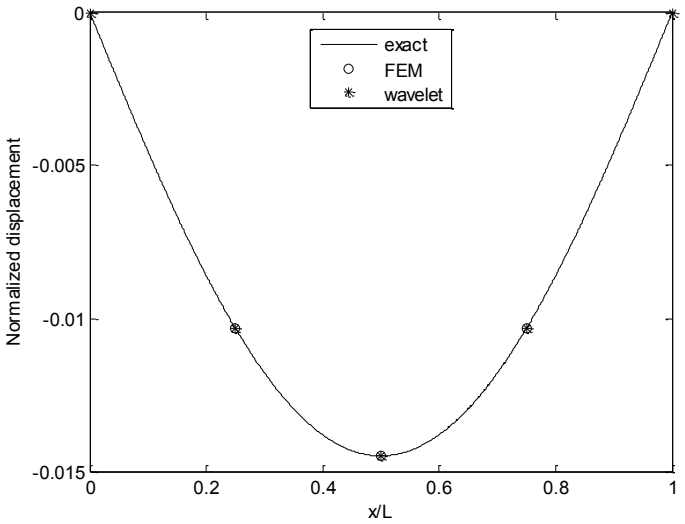

Figure 6. Transversal displacements of the beam in Example 2

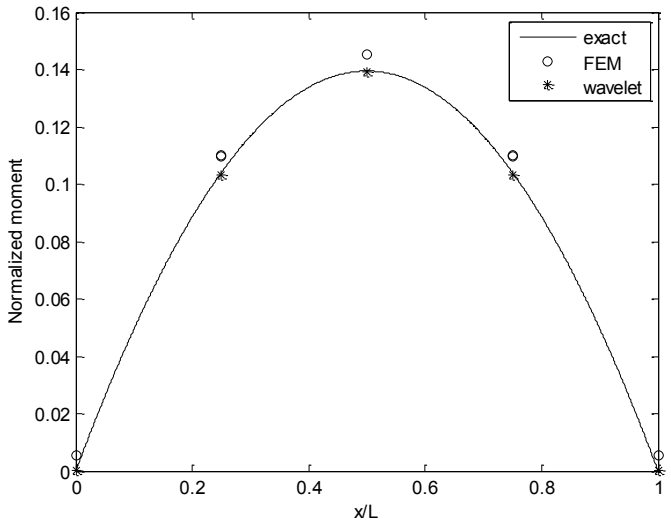

Figure 7. Bending moments of the beam in Example 2

The Deslauriers-Dubuc scaling function used in this example was IN6 (i. e., DD function based on a Daubechies scaling function with 3 vanishing moments). Figures 8 and 9 show the error obtained in terms of Euclidean norm for displacements and bending moments, respectively. Results are once more compared to the FEM ones, showing the method's robustness.

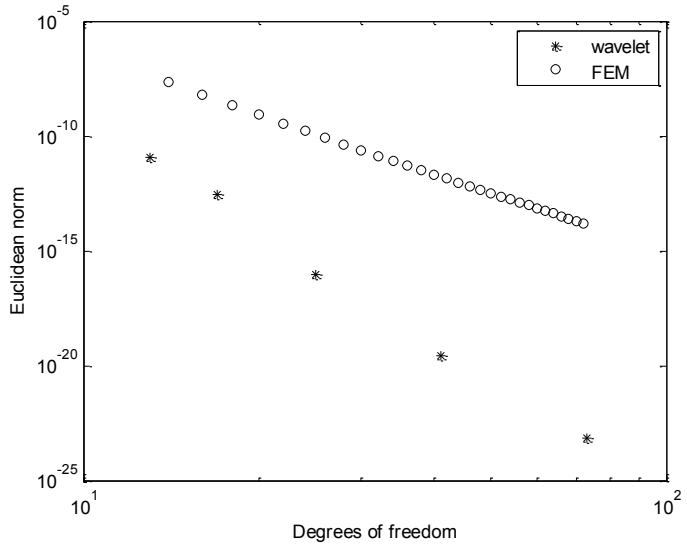

Figure 8. Convergence study in terms of displacements for Example 2

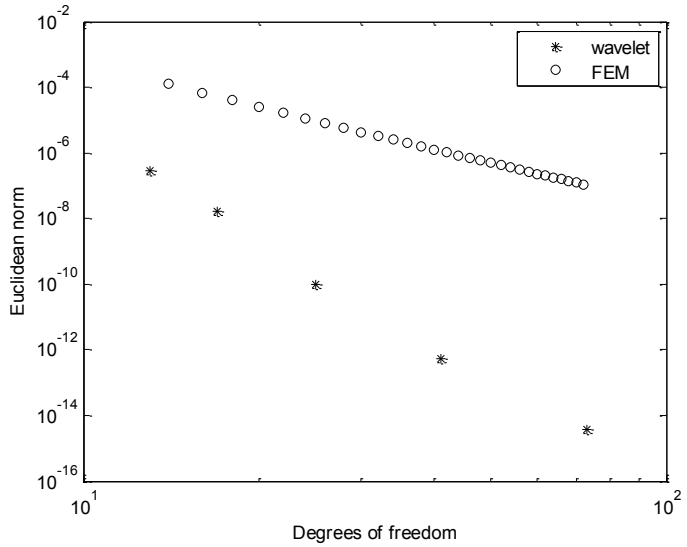

Figure 9. Convergence study in terms of bending moments for Example 2

The homogeneous version of Eq. (30), i. e. $q = 0$, describes what is known as the Euler column. In this case, the only non-trivial solutions for $v(x)$ are given for specific values of the axial load P, known as the critical loads P_{cr}, given by:

$$P_{cr} = \frac{n^2\pi^2 EI}{L^2}, \quad v_n(x) = A_n \sin\left(\frac{n\pi x}{L}\right), \quad n = 1,2,\ldots. \tag{32}$$

In Eq. (32), the function $v_n(x)$ is known as the buckling mode of the beam and A_n is an arbitrary amplitude. Generally, only the three first modes are of interest for this kind of beam. Table 1 shows their analytical values in terms of EI/L^2.

Table 1. Critical loads for the first three buckling modes of an Euler beam

MODE Nº.	1	2	3
$P_{cr}L^2/EI$	9.8696	39.478	88.826

Usually, critical loads are obtained through the solution of an eigenvalue problem, therefore numerical errors increase along with the mode number. For this reason, the third buckling mode was used for the convergence study shown in Fig. 10. Interpolets IN4 and IN6 were used in this case for comparison purposes.

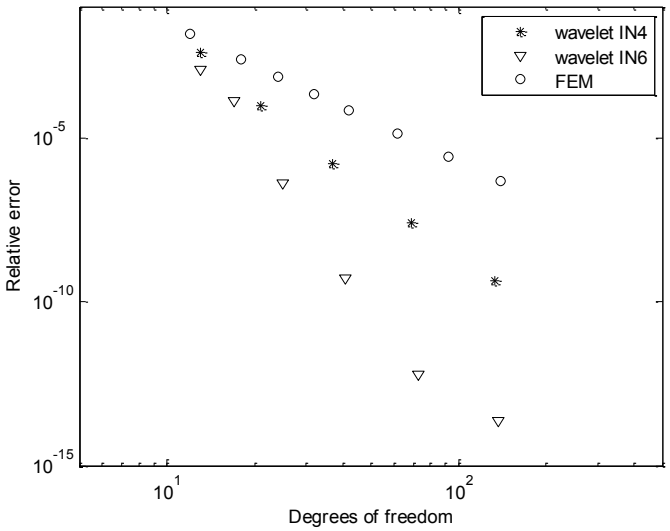

Figure 10. Relative error in the third critical load for the Euler beam

4.3 2-D Helmholtz homogeneous equation

In this example, the formulation was extended to a two-dimensional homogeneous problem in a unitary domain given by:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+k^2 u=0. \tag{33}$$

Using $k = 2\pi$ and imposing null Dirichlet boundary conditions at edges $x = 0$, $y = 0$ and $y = 1$, the solution can be obtained analytically and is given by:

$$u(x,y)=\frac{\sin\left(x\pi\sqrt{3}\right)\sin\left(\pi y\right)}{\sin\left(\pi\sqrt{3}\right)},\quad x,y\in\left[0\,1\right]. \tag{34}$$

The solution given in Eq. (34) is shown in Fig. 11. Figure 12 shows a convergence study in which the error is measured using Euclidean norm and compared to the errors given by the Finite Difference Method (FDM).

4.4 2-D Poisson's equation with scattered data

Finally, scattered data was used as the non-homogeneous term in a 2-D Poisson's equation. The following equation was used as a standard for comparison:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\sin\left(2\pi x\right)\sin\left(2\pi y\right). \tag{35}$$

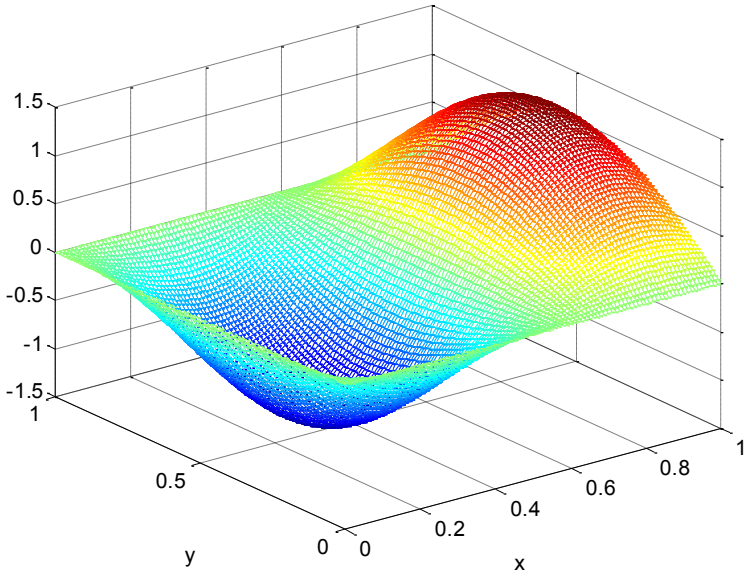

Figure 11. Homogeneous solution given in Eq. (34)

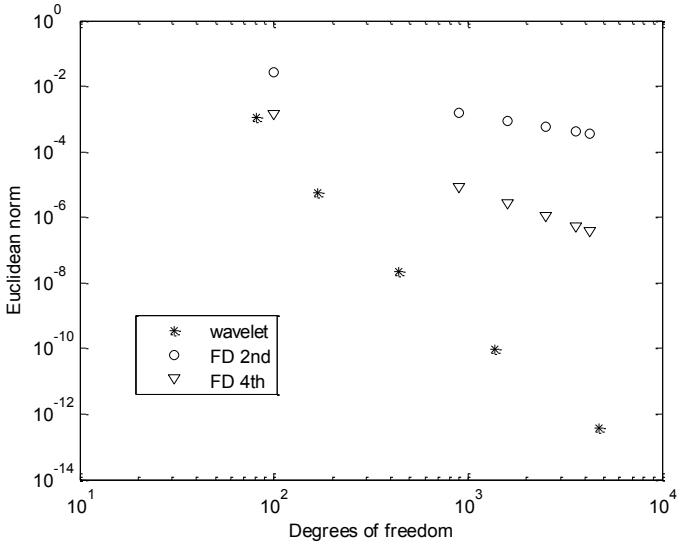

Figure 12. Convergence study for Example 3

An arbitrary particular solution is given below in Eq. (36) and shown in Fig. 13 for a domain given by a square in [0 1]×[0 1]:

$$u_p = -\frac{\sin(2\pi x)\sin(2\pi y)}{8\pi^2} \tag{36}$$

Scattered data was fed using the non-homogeneous function in Eq. (35) measured at randomly distributed points, whose distribution for a particular resolution (64×64) is shown in Fig. 14. Dirichlet boundary conditions were used again and set to zero in all borders, since this is the case for the particular solution. Figure 15 shows the error in terms of Euclidean norm using increasing wavelet resolutions and 4096 scattered points. Since the FD model would require interpolation of the scattered data to the grid, it was not used in this example for comparison.

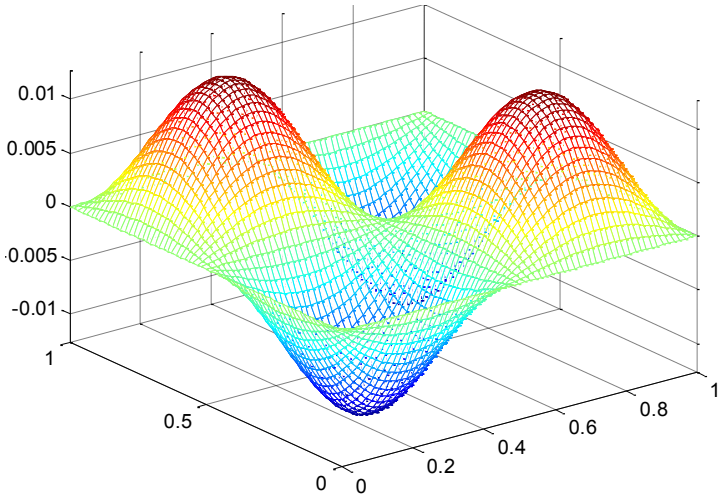

Figure 13. Particular solution given in Eq. (34)

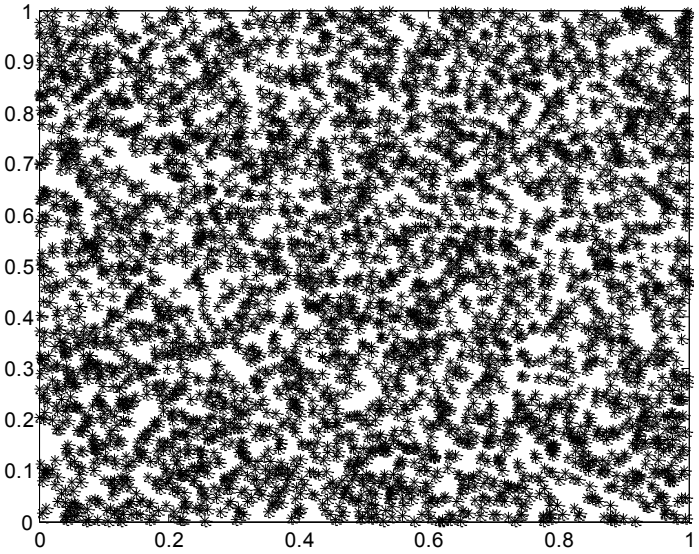

Figure 14. Randomly distributed points in [0 1]x[0 1]

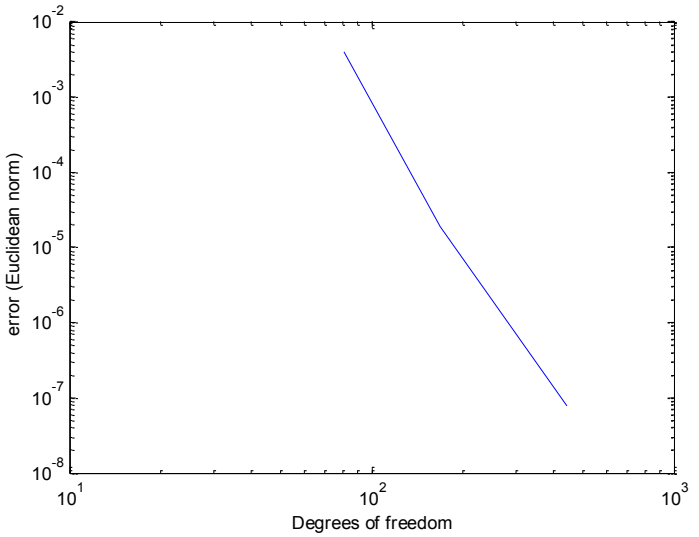

Figure 15. Convergence study for Example 4

5 CONCLUSIONS

This work presented the formulation and validation of the Wavelet-Galerkin Method using Deslauriers-Dubuc Interpolets for the solution of problems based on Poisson and Helmholtz equations.

As in the traditional FEM and other numerical methods, the accuracy of the solution can be improved either by increasing the level of resolution or the scaling function order. In one of the examples, two different orders of scaling functions were tested and the higher order one performed better regardless of the level of resolution.

For two-dimensional problems, results were extremely good, although only regular geometry problems were studied. The extension of the method to irregular geometries remains a challenge.

When scattered data is provided, the method showed a very impressive ability to represent functions given at randomly distributed points. This is almost impossible to obtain with methods that depend on regularly distributed data, as the FDM. To use FDM, some interpolation would have to be performed beforehand. For that reason, in this particular case (Example 4), the convergence study did not use FDM results. Nevertheless, it is shown that accuracy increases along with the level of discretization.

Since the unknowns of the method are interpolation coefficients instead of nodal values, it is possible to obtain a smooth representation of solutions even with a reduced number of degrees of freedom.

All matrices involved can be stored and operated in a sparse form, since most of their components are null, thus saving computer resources. Due to the compact support of scaling functions, the sparseness of matrices increases along with the level of resolution. Nevertheless, it must be taken into account that higher order scaling functions at high levels of resolution can lead to ill-conditioned matrices.

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