GALERKIN SCHEME FOR 2D NON-LINEAR PDE'S COUPLED SYSTEMS SIMULATING HEAT TRANSFER PROBLEM

Mohamed Ghattassi

Mohamed Boutayeb

mohamed.ghattassi@univ-lorraine.fr

mohamed.boutayeb@univ-lorraine.fr

Université de Lorraine, Centre des Recherches en Automatique, CNRS UMR 7039

Campus Sciences, Boulevard des Aiguillettes - BP 70239, 54506, Vandoeuvre lès Nancy, France

Jean R. Roche

jean-rodolphe.roche@univ-lorraine.fr

Université de Lorraine, Institut Elie Cartan de Lorraine, CNRS UMR 7502

Campus Sciences, Boulevard des Aiguillettes - B.P. 70239, 54506, Vandoeuvre lès Nancy, France

Abstract. *The aim of this communication is to present and analyze a numerical method to simulate a combined radiation conduction heat transfer system in a gray, absorbing and non-scattering medium. The model considered is a system of two nonlinear partial differential coupled equations, the first one is a radiation transfer equation (RTE) and the second one a nonlinear parabolic heat conduction equation (CE). Both equations are coupled through a nonlinear source term. The 3D model was reduced to 2D using symmetry. To compute a numerical solution of the RTE we consider the classical SN discrete ordinate method for the angular approximation and discontinuous Galerkin method in an unstructured mesh for spacial discretization. An approximation of the solution of the CE has been obtained using finite element in the same mesh that the RTE equation and using Crank Nicholson scheme in the time variable. The coupled system is solved using a fixed-point method. The poof of existence and uniqueness of a solution of the coupled system is obtained using fixed-point theorem. Performances of the proposed approach, compared to recent results, are analyzed.*

Keywords: *Radiative transfer equation, Energy equation,Discontinuous Galerkin method, Nonlinear System*

1 INTRODUCTION

The objective of this paper is to present and analyze a numerical method to solve a radiative-conductive heat transfer problem in an absorbing and emitting medium. The heat transfer models in participating media is the subject of various fields for engineering and scientific applications such as high temperature processing or thermal insulation. The model considered in this paper involve a transfer radiative equation (RTE) and a conductive equation (CE) with a non-linear coupling term, (Asllanaj et al. 2003). The unknown of the RTE equation is the radiation intensity at any time step, position and direction; the unknown of the conduction equation is the temperature at any time step and position. The 3D model was reduced to 2D using symmetry.

Many simplified models have been used to carry out numerical simulations of radiative transfer phenomena. Generally, two classes of numerical methods have been applied to attempt to solve radiative heat transfer problems namely the Monte Carlo method Howell (1998), Modest (2003) and the deterministic methods (Asllanaj et al. 2007), (Chui et al.1993) and (Furmanski et al, 2004).

In the present work we study the application of high order Galerkin Discontinuous (DG) method based on an upwind stable numerical flux , to solve the RTE. The DG method is known to be conservative stable and high-order accurate. And in the same way that the finite element method it can easily handle complex geometries and unstructured meshes, (Cockburn et al. 2002), (Cui et al. 2005). An approximation of the solution of the CE has been obtained using finite element in the same mesh that the RTE equation and using Crank Nicholson scheme in the time variable, Zlamal (1980). To take account of the non-linearity of the CE we implement a Newton method, we obtain a sequence of semi-linear auxiliary numerical problems. Under some assumptions, convergence of the numerical method. The convergence proof of the coupled system is a consequence of the application of a discrete fixed-point theorem, involving only the temperature fields.

Spatial DG techniques applied to discrete-ordinates radiation transport have been pioneered by Reed and Hill (1973) and (Lesaint et al. 1974). In (Cui et al. 2004), the DG method was extended to the solution of radiative heat transfer problems in absorbing, emitting and scattering media, the authors studies a centered numerical flux and implemented a parallel computing algorithm based around the localized DG formulation. In (Cui et al. 2005) a 3-d model was reduced to 2-d using symmetry and the DG formulation. In (Brezzi et al 2006, 2004), the authors added a stabilization term to penalize the jump of the solution across interior faces of the triangulation and to obtain physical numerical solutions in practical applications. For the transient state, many authors have considered radiation and conduction coupled problem, see (Asllanaj et al, 2007), (Mishra et al., 2005, 2013). (Asllanaj et al. , 2007), simulated transient heat transfer by radiation and conduction in two-dimensional complex shaped domains with structured and unstructured triangular meshes working with an absorbing, emitting and non-scattering grey medium. To solve the RTE, these authors applied a modified FVM based on a cell vertex scheme combined with a modified exponential scheme where temperature was approximated by linear interpolation using nodal values. The PHAML (Parallel Hierarchical Adaptive Multi Level) code was used to solve the heat conduction equation in space, with low or high order finite elements.

The outline of this paper is as follows. In the next section, the continuous model considered in this paper is described and the we recall a solution existence and uniqueness result. The numerical approximation of the RTE and energy equation by the DG method and the finite element method respectively, are discussed in detail in section 3 and we have also established the stability of the DG method with and adequate norm. In section 3 we give also a proof of the existence and uniqueness of the numerical approximation of the coupled system. Finally, in section 4, we give numerical results showing the performances of the proposed numerical method.

2 THE CONTINUOUS PROBLEM

In this section we introduce the coupled RTE-CE system of partial differential equations in a two dimensional gray absorbing and emitting medium considered in this paper. Let $\widetilde{\Omega}$ be a bounded domain in $\mathbb{R}^2$ and $\mathcal{D}$ the unit disk, for the problem considered, $\alpha \in \mathcal{D}$, $\widetilde{x} = (x_1, x_2) \in \widetilde{\Omega}$, $\widetilde{\mathcal{X}} = \widetilde{\Omega} \times \mathcal{D}$ and $\widetilde{t} \in [0, \widetilde{\tau}]$, for $\widetilde{\tau} > 0$. Let ν be the outward unit normal to the boundary $\partial\widetilde{\Omega}$. We denote

$$\partial\widetilde{\Omega}_- = \{(\widetilde{x}, \alpha) \in \partial\widetilde{\Omega} \times \mathcal{D} \text{ such that } \alpha.\nu < 0\}.$$

The radiation intensity is the solution of the following transport equation:

$$\alpha.\nabla_{\widetilde{x}}\widetilde{I}(\widetilde{t}, \widetilde{x}, \alpha) + \kappa\widetilde{I}(\widetilde{t}, \widetilde{x}, \alpha) = \frac{\kappa n^2 \sigma_B}{\pi}\widetilde{u}^4(\widetilde{t}, \widetilde{x}) \quad \text{for } (\widetilde{t}, \widetilde{x}, \alpha) \in [0, \widetilde{\tau}] \times \widetilde{\mathcal{X}} \tag{1}$$

$$\widetilde{I}(\widetilde{t}, \widetilde{x}, \alpha) = \widetilde{g}(\widetilde{t}, \widetilde{x}, \alpha), \qquad \text{for } (\widetilde{t}, \widetilde{x}, \alpha) \in [0, \widetilde{\tau}] \times \partial\widetilde{\Omega}_-, \tag{2}$$

$$\widetilde{I}(\widetilde{t}, \widetilde{x}, \alpha) = 0 \qquad \text{for } (\widetilde{t}, \widetilde{x}, \alpha) \in [0, \widetilde{\tau}] \times \partial\widetilde{\Omega}_- \tag{3}$$

where κ is the absorption coefficient of the medium and n is the refractive index. In this work, the refractive index and the absorption coefficient are assumed to be equal to one, $n = 1$, $\kappa = 1m^{-1}$ and $\sigma_B = 5.6698\,10^{-8}\; Wm^{-2}K^{-4}$ is the Stefan-Boltzmann constant. The radiative boundary condition $\widetilde{g}(\widetilde{t}, \widetilde{x}, \alpha)$ takes into account of two quantities, the emitted radiation intensity and the incoming radiation. For an opaque wall with specular reflection, we have

$$\widetilde{g}(\widetilde{t}, \widetilde{x}, \alpha) = \epsilon\frac{\sigma_B}{\pi}\widetilde{u}^4(\widetilde{t}, \widetilde{x}) + (1 - \epsilon)\widetilde{I}(\widetilde{t}, \widetilde{x}, \alpha_x) \quad \text{for } (\widetilde{t}, \widetilde{x}, \alpha) \in [0, \widetilde{\tau}] \times \partial\widetilde{\Omega}_-, \tag{4}$$

where ϵ is the wall emissivity (assumed to be constant), $T^\star(t, s^\star)$ is the temperature at the boundary of the medium and β_s is the specular direction of radiation defined by $\alpha_x = \alpha - 2(\alpha.\nu_\mathrm{i})\nu_\mathrm{i}$ where ν_i is the local outward surface normal. For an opaque wall with diffuse reflection, the boundary condition takes the following from:

$$\widetilde{g}(\widetilde{t}, \widetilde{x}, \alpha) = \epsilon\frac{\sigma_B}{\pi}\widetilde{u}^4(\widetilde{t}, \widetilde{x}) + \frac{(1 - \epsilon)}{\pi}\int_{\nu_\mathrm{i}.\alpha' < 0} \widetilde{I}(\widetilde{t}, \widetilde{x}, \alpha')|\nu_\mathrm{i}.\alpha'|d\alpha' \;\; (\widetilde{t}, \widetilde{x}, \alpha) \in [0, \widetilde{\tau}] \times \partial\widetilde{\Omega}_-. \tag{5}$$

The temperature u is the the unknown of the following nonlinear heat equation (CE):

$$\rho c_p \frac{\partial\widetilde{u}}{\partial\widetilde{t}}(\widetilde{t}, \widetilde{x}) - k_c \triangle\widetilde{u}(\widetilde{t}, \widetilde{x}) = \widetilde{S_{rad}}(\widetilde{t}, \widetilde{x}) \qquad \text{for } (\widetilde{t}, \widetilde{x}) \in [0, \widetilde{\tau}] \times \widetilde{\Omega} \tag{6}$$

$$\widetilde{u}(0, \widetilde{x}) = \widetilde{u}_0(\widetilde{x}) \qquad \text{for all } \widetilde{x} \in \widetilde{\Omega}. \tag{7}$$

with Dirichlet or Robin boundary conditions. The data ρ, c_p, and k_c are the density, the specific heat capacity, and the thermal conductivity of the medium, respectively. In this work, they are assumed to be constant. The energy equation (6) - (7) and the RTE (1) - (2) are strongly coupled by source term $\widetilde{S_{rad}}$. The source term take account of emission and absorption of radiation by the medium and is defined by the following relations:

$$\widetilde{S_{rad}}(t,\widetilde{x}) = \kappa\{\widetilde{G}(\widetilde{t},\widetilde{x}) - 4\sigma_B \widetilde{u}^4(\widetilde{t},\widetilde{x})\} \qquad \text{for } (\widetilde{t},\widetilde{x}) \in [0,\widetilde{\tau}] \times \widetilde{\Omega}, \tag{8}$$

where $\widetilde{G}$ is the incident radiation intensity,

$$\widetilde{G}(\widetilde{t},\widetilde{x}) = \int_{\mathcal{D}} \widetilde{I}(\widetilde{t},\widetilde{x},\alpha)\, d\alpha \qquad \text{for } (\widetilde{t},\widetilde{x}) \in [0,\widetilde{\tau}] \times \widetilde{\Omega}. \tag{9}$$

Both equations can be written in a dimensionless form. Let $t = \dfrac{\gamma\widetilde{t}}{L^2}$ the dimensionless time, where $\gamma = \dfrac{k_c}{\rho c_p}$ is the thermal diffusivity. Let u_{ref} be a given reference temperature, we introduce the conduction radiation number N_s which satisfies the following expression:

$$N_s = \frac{k_c \kappa}{4\sigma_B u_{ref}^3}.$$

Let

$$x = \frac{\widetilde{x}}{L},\ \tau = \frac{\gamma\widetilde{\tau}}{L^2},\ u = \frac{\widetilde{u}}{u_{ref}},$$
$$u_0(x) = \frac{\widetilde{u}_0(\widetilde{x})}{u_{ref}},\ G = \frac{\widetilde{G}}{4\sigma_B u_{ref}^4} \text{ and } S_{rad} = \frac{\widetilde{S_{rad}}}{4\sigma_B u_{ref}^4}.$$

If

$$I_{ref} = \frac{\sigma_B}{\pi} u_{ref}^4, \tag{10}$$

we have:

$$I = \frac{\widetilde{I}}{I_{ref}} \text{ and } g = \frac{\widetilde{g}}{I_{ref}} \tag{11}$$

Consequently we obtain the following coupled system of dimensionless equations:

$$\alpha.\nabla_x I(t,x,\alpha) + I(t,x,\alpha) = u^4(t,x), \qquad \text{for } (t,x,\alpha) \in [0,\tau] \times \Omega \times \mathcal{D}, \tag{12}$$
$$I(t,x,\alpha) = g(t,x,\alpha), \qquad \text{for } (t,x,\alpha) \in [0,\tau] \times \partial\Omega_- \times \mathcal{D} \tag{13}$$
$$\frac{\partial u}{\partial t}(t,x) - \triangle u + \theta\, u^4(t,x) = \theta\, G(t,x), \qquad \text{for } (t,x) \in [0,\tau] \times \Omega, \tag{14}$$
$$u(t,x) = f(t,x) \text{ or } -\frac{\partial u}{\partial \nu}(t,x) = u(t,x) - f(t,x),\ (t,x) \in [0,\tau] \times \partial\Omega, \tag{15}$$
$$u(0,x) = u_0(x), \qquad \text{for all } x \in \Omega, \tag{16}$$

where $\theta = \dfrac{\kappa^2 L^2}{N_s}$ is a dimensionless constant.

In order to give an existence and uniqueness of the solution of system (12)-(16) we introduce the following function space:

- $L^p(\Omega)$, $p \in [1, \infty[$ the space of the measurable functions for the measure dx such that

$$\|f\|_{L^p(\Omega)} = \left(\int_{\mathcal{X}} |f(x)|^p dx \right)^{\frac{1}{p}} < +\infty.$$

- $L^p(Q_\tau) = L^p(0, \tau; L^p(\Omega))$ for all $p \in [1, \infty[$, $Q_\tau =]0, \tau[\times \Omega$,
- $L^p(\mathcal{X})$, $p \in [1, \infty[$ the space of the measurable functions for the product measure $dxd\alpha$ such that

$$\|f\|_{L^p(\mathcal{X})} = \left(\int_{\mathcal{X}} |f(x, \alpha)|^p dxd\alpha \right)^{\frac{1}{p}} < +\infty.$$

- $H^1(\Omega)$ is the space of functions belonging to $L^2(\Omega)$ such that it's first order derivatives are in $L^2(\Omega)$. We denote $H^{-1}(\Omega)$ the dual space.

Let E_1 denote the set:

$$E_1 = \{T \in \mathcal{V}; \|T\|_{L^8(Q_\tau)} \leqslant M\},$$

where M is a positive constant and $\mathcal{V}$ is given by

$$\mathcal{V} = L^\infty(0, \tau; L^2(\Omega)) \cap L^{10}(Q_\tau) \cap L^2(0, \tau; H^1_0(\Omega)) \cap H^1(0, \tau; H^{-1}(\Omega)). \tag{17}$$

Let $N_b = \sqrt{\pi} M^4$ and

$$E_2 = \{I \in L^2(0, \tau; L^2(\Omega \times \mathcal{D})); \|I\|_{L^2(0,\tau;L^2(\Omega\times\mathcal{D}))} \leqslant N_b\}.$$

Now we can recall an existence and uniqueness result:
Theorem 2.1. *If the initial data $u_0 \in L^5(\Omega)$ and verifies:*

$$\|u_0\|_{L^5(\Omega)} \leqslant \sqrt[5]{\frac{15\theta\pi M^8}{8}}, \tag{18}$$

then there exists $\tau^\star > 0$, such that for all $\tau \leqslant \tau^\star$, the system of equation (12)-(16) has a unique local solution (u, I) such that $u \in E_1$ and $I \in E_2$.

In fact we have proved that solution of the problem (12)-(16) is a fixed point of a well posed map $\mathbb{H}$ in the set E_1, see Ghattassi (2015). This results means that we can now assume the existence and uniqueness of a solution of system (12)-(16) if the initial data, the initial temperature is small enough.

3 NUMERICAL METHOD

In this section we describe the algorithm presented in this paper to solve the computed system (12) - (16). First we introduce the numerical method to solve the RTE.

To obtain a direction discretization we consider the classical S_N-discrete ordinate method is introduced, see Modest (2003). In the S_N method we approach the radiation intensity

$I(t,x,\alpha)$ by a finite number of radiation intensity values $I^l(t,x) = I(t,x,\alpha_l)$ for all $l \in \{1,...,N_\alpha\}$. The set of $\alpha_l,\ \ l = 1,...,N^\alpha$ was chosen such that:

$$\int_{\mathcal{D}} I(t,x,\alpha)d\alpha \simeq \sum_{l=1}^{N^\alpha} I^l(t,x)w_l \text{ and } \sum_{l=1}^{N^\alpha} w_l = 4\pi. \tag{19}$$

This quadrature formula preserves a symmetry for the opposite $\alpha_{l'} = -\alpha_l$ of discrete ordinate α_l in the quadrature set, the weights are equal $w_{l'} = w_l$, see Modest (2003). The discrete radiation intensity I^l is the solution of a N^α system of partial differential equations over Ω:

$$\begin{cases} \alpha_l.\nabla_x I^l(t,x) + I^l(t,x) = u^4(t,x) & (t,x) \in [0,\tau] \times \Omega \\ \qquad\qquad\qquad I^l(t,x) = g(t,x,\alpha_l) & (t,x) \in [0,\tau] \times \partial\Omega^l_-, \end{cases} \tag{20}$$

where

$$\partial\Omega^l_- = \{x \in \partial\Omega \text{ such that } \alpha_l.\nu < 0\}. \tag{21}$$

We assume that $u(t,.)$ belongs to $L^2(\Omega)$ for all $t \in [0,\tau]$.We extend the boundary datum g to $\partial\Omega$ by setting it to zero outside $\partial\Omega^l_-$ and we assume that

$$g(t,.,\alpha_l) \in L^2(\partial\Omega^l_-) \quad \text{for all } (t,l) \in [0,\tau] \times \{1,...,N_\alpha\}.$$

Theorem 3.1 (Well-posed, see Pietro and al. 2011)**.** *There exist a unique solution of the problem* (20)*,* $I^l(t,.) \in H^1(\Omega)$ *for all* $(t,l) \in [0,\tau] \times \{1,...,N_\alpha\}$.

To compute a numerical solution (20), for all $\{1,...,N_\alpha\}$ we have implemented a Discontinuous Galerkin (DG) method with a upwind numerical flux. We consider a triangulation $\mathcal{T}_h$ of the domain Ω such that

$$\overline{\Omega} = \bigcup_{T \in \mathcal{T}_h} T,$$

- Each T is a triangle with nonempty internal;
- $\overset{\circ}{T}_1 \bigcap \overset{\circ}{T}_2 = \emptyset$ for each distinct $T_1, T_2 \in \mathcal{T}_h$;
- $F = T_1 \bigcap T_2 \neq \emptyset$ then F is a common face of T_1 and T_2;
- Interfaces are collected in the set $\mathcal{F}_h^{in}$, and boundary faces are collected in the set $\mathcal{F}_h^b$. Henceforth, we set

  $$\mathcal{F}_h = \mathcal{F}_h^{in} \bigcup \mathcal{F}_h^b.$$

  Moreover, for any mesh element $T \in \mathcal{T}_h$, the set

  $$\mathcal{F}_K = \{F \in \mathcal{F}_h | F \in \partial T\},$$

  collects the mesh faces composing the boundary of T.
- $\text{diam}(T) = h$ for each $T \in \mathcal{T}_h$ (structured mesh).

Now, an approximation space $V_h \subset L^2(\Omega)$ is introduced such that

$$V_h = \{v_h \in L^2(\Omega)/\ \forall T \in \mathcal{T}_h,\ v_{h_{|_T}} \in \mathbb{P}^k(T)\}, \tag{22}$$

where $\mathbb{P}^k(T)$ denotes the set of polynomial defined in T of degree less than or equal to k. We define:

$$\mathbb{V}_h = \{V_h\}_{l=1}^{N_\alpha}, \tag{23}$$

and we denote a generic element in $\mathbb{V}_h$ as $I_h(t,.) = \{I_h^l(t,.)\}_{l=1}^{N_\alpha}$. Since $I_h(t,.)$ is discontinuous in the interior face $F \in \mathcal{F}_h^{in}$ separating two mesh cells T_1 and T_2,we define the mean value and jump across $F \in \mathcal{F}_h^{in}$ of the function $I_h^l(t,.) \in V_h$:

$$\{\{I_h^l(t,.)\}\} = \frac{1}{2}(I_1^l(t,.) + I_2^l(t,.)), \quad [\![I_h^l(t,.)]\!] = (I_1^l(t,.) - I_2^l(t,.)),$$

for all $t \in [0,\tau]$, where $I_1^l(t,.) = I_h^l(t,.)_{|_{T_1}}$ and $I_2^l(t,.) = I_h^l(t,.)_{|_{T_2}}$ are the restrictions of $I_h^l(t,.)$ on the mesh cells T_1 and T_2, respectively.

The DG formulation is obtained by multiplying the equation (20) for the direction α_l with the test function $\phi_h^l \in V_h$ and applying the upwind numerical fluxes $\hat{\mathcal{F}}(x)$ to approximate the quantity $(\alpha_l.\nu)I^l(t,.)$ on the elements boundary T:

$$\int_T I_h^l(t,.)\phi_h - (\alpha_l.\nabla_x\phi_h)I_h^l(t,.) + \int_{\partial T} \hat{\mathcal{F}}(x)\phi_h = \int_\Omega u^4(t,.)\phi_h,\ \ \forall t \in [0,\tau]. \tag{24}$$

The upwind numerical flux $\hat{\mathcal{F}}(x)$ at the mesh interface F from K_1 to K_2 is given by:

$$\hat{\mathcal{F}}(x) = \beta_l.\nu_{\mathbf{F}}\{\{I^l(t,x)\}\} + \frac{1}{2}|\alpha_l.\nu_{\mathbf{F}}|[\![I^l(t,x)]\!],\ \ \forall t \in [0,\tau], \tag{25}$$

where $\nu_{\mathbf{F}}$ is the outward unit normal to F at x.

Let denote $\phi_h = \{\phi_h^l\}$, $\phi_h \in \mathbb{V}_h$. Summing over all cells T, integrating by parts a second time, and separating volume and interface terms, we obtain the following global formulation:

$$\begin{aligned}&\text{Find}\ \ I_h(t,.) \in \mathbb{V}_h\ \ \text{such as}\\ &\qquad a_h(I_h(t,.),\phi_h) = l_h(\phi_h)\ \ \text{for all}\ \phi_h \in \mathbb{V}_h\ \ \text{and}\ t \in [0,\tau].\end{aligned} \tag{26}$$

where

$$\begin{aligned} a_h(I_h(t,.),\phi_h) = \sum_{l=1}^{N_\beta} w_l\Big(\sum_{T\in\mathcal{T}_h}&\left[\int_T I_h^l(t,x)\phi_h^l dx - \int_T I_h^l(t,x)(\alpha_l.\nabla_x\phi_h^l)dx\right]\\ &+ \int_{\partial\Omega}(\alpha_l.\nu)^{\oplus} I_h^l(t,x)\phi_h^l d\Gamma(x)\\ &+ \sum_{F\in\mathcal{F}_h^{in}}\int_F(\alpha_l.\nu_{\mathbf{F}})[\![I_h^l(t,x)]\!]\{\{\phi_h^l(x)\}\}d\Gamma(x)\\ &+ \sum_{F\in\mathcal{F}_h^{in}}\int_F\frac{1}{2}|\alpha_l.\nu_{\mathbf{F}}|[\![I_h^l(t,x)]\!][\![\phi_h^l]\!]d\Gamma(x)\Big),\end{aligned} \tag{27}$$

where for a real number x, we define its positive and negative parts respectively as $x^{\oplus} := \frac{1}{2}(|x|+x)$, $x^{\ominus} := \frac{1}{2}(|x|-x)$. The second member l_h is a linear form defined by

$$l(\phi_h) = \sum_{l=1}^{N_\alpha} w_l \left[\int_\Omega u^4(t,x)\phi_h^l dx + \int_{\partial\Omega} (\alpha_l.\nu)^{\ominus} g(t,x,\alpha_l)\phi_h^l d\Gamma(x) \right] \tag{28}$$

for all $\phi_h \in \mathbb{V}_h$ and $t \in [0,\tau]$.

Now we introduce the following norm in $\mathbb{V}_h$

$$\|I_h\|_h^2 = \sum_{l=1}^{N_\alpha} w_l \left[\|I_h^l\|_{L^2(\Omega)}^2 + \int_{\partial\Omega} \frac{1}{2}|\alpha_l.\nu| {I_h^l}^2 d\Gamma(x) + \sum_{F\in\mathcal{F}_h^{in}} \int_F \frac{1}{2}|\alpha_l.\nu_{\mathbf{F}}| [\![I^l]\!]^2 d\Gamma(x) \right], \tag{29}$$

for all $I_h \in \mathbb{V}_h$. Then we verifies that the bilinear form (27) is in fact the square of a norm.

Lemma 3.2.

$$a_h(I_h(t,.), I_h(t,.)) = \|I_h(t,.)\|_h^2 \quad \textit{for all} \quad I_h(t,.) \in \mathbb{V}_h \quad \textit{and} \quad t \in [0,\tau] \tag{30}$$

The proof of these lemma can be find in, Ghattassi (2015). This stability result have the following corollary:

Theorem 3.3. *The problem (26) has a unique solution in* $\mathbb{V}_h$.

Proof. The second member l_h is a continuous map from $\mathbb{V}_h$ to $\mathbf{R}$. The lemma 3.2 imply the continuity and the ellipticity of the bilinear form (27), consequently thanks to the Lax Milgram lemma, the weak problem (26) has a unique solution. □

Moreover, if we assume $I(t,.) \in (H^r(\Omega))^{N_\alpha}$ for $r > 0$, applying classical results for DG methods, (Han and al, 2010), we have the following a priori error estimate:

$$\|I(t,.) - I_h(t,.)\|_h \leqslant Ch^{\min\{r,k+1\}-\frac{1}{2}} \left[\sum_{l=1}^{N_\alpha} w_l \|I^l(t,.)\|_r^2 \right]^{\frac{1}{2}} \quad \text{for all } t \in [0,\tau]. \tag{31}$$

To give the weak formulation of the non-linear CE problem we introduce the following space of function:

$$\mathcal{V} = \{v \in L^2(0,\tau,H^1(\Omega)) \text{ such that } v(t,x) = f(t,x) \text{ for } (t,x) \in [0,\tau]\times\partial\Omega\}. \tag{32}$$

In the case of the Dirichlet boundary conditions, thanks to the Green formula we obtain the following weak formulation of problem (14)-(16):

Find $u \in \mathcal{V}$ such that

$$(\frac{du}{dt},\varphi)_\Omega + (\nabla_x u, \nabla\varphi)_\Omega + \theta(u^4,\varphi)_\Omega = \theta(G,\varphi)_\Omega, \tag{33}$$

for all $\varphi \in H_0^1(\Omega)$ and satisfying the initial condition:

$$(u(0,.),\varphi)_\Omega = (u_0(.),\varphi)_\Omega \quad \forall\varphi \in L^2(\Omega). \tag{34}$$

The existence and uniqueness of a solution of (51)-(34) was proved in, (Casenave et al., 1998).

To obtain a numerical approximation of the solution of (51)-(34) we apply finite element method using the same mesh that was used for the RTE numerical approach.

Let

$$W_h = \{p_h \in \mathcal{C}^0(\Omega)/ \ \forall T \in \mathcal{T}_h, \ \ p_{h_{|_T}} \in \mathbb{P}^k(T)\}, \tag{35}$$

This space is finite dimensional subspace of $H^1(\Omega)$, we denote N_w the dimension. Let $\{\phi_1, \phi_2, ..., \phi_{N_w}\}$ be a base. Then we consider the following approximation of the temperature u:

$$u_h(t,x) = \sum_{i=1}^{N_w} u_{hi}(t)\phi_i(x). \tag{36}$$

Consequently we obtain the following semi-discrete system:

Find $u_h : [0, \tau] \to u_h(t, .) \in W_h$ such that $u_h(t, x) = f(t, x)$ in $\partial\Omega$ and

$$(\frac{d\,u_h}{d\,t}, \phi_i)_\Omega + (\nabla u_h, \nabla\phi_i)_\Omega + \theta(u_h^4, \phi_i)_\Omega = \theta(G_h, \phi_i)_\Omega \quad \forall\phi_i \in W_h, \tag{37}$$

satisfying the initial condition:

$$(u_h, \phi_i)_\Omega = (u_0, \phi_i)_\Omega \quad \forall\phi_i \in W_h. \tag{38}$$

Let $\triangle t$ denote the time step, $\triangle t = \dfrac{\tau}{N}$ where N is given. If we consider Crank-Nicolson scheme for time discretization the numerical approximation of equation (37) can be written in the following form:

for all $n \in \{0, 1, ..., N-1\}$.

$$\begin{aligned}(\frac{u_h^{n+1} - u_h^n}{\triangle t}, \phi_i)_\Omega + (\nabla_x(\frac{u_h^{n+1} + u_h^n)}{2}, \nabla_x\phi_i)_\Omega + \theta(\frac{u_h^{n+1^4} + u_h^{n4}}{2}, \phi_i)_\Omega \\ = \theta(G^n, \phi_i)_\Omega \quad \forall\phi_i \in W_h.\end{aligned} \tag{39}$$

where $u_h^n(x) = u_h(t^n, x)$ when $t^n = n\,\triangle t$.

A proof of the existence and uniqueness of the solution of discrete system (39) was given in Zlamal (1980). A priori estimates of numerical error can be found in Chrysafinos (2002). In the case of Robin boundary condition the same approach can be used to compute a numerical approximation of the solution.

In conclusion, we obtain the following algorithm to compute a numerical approximation to the solution of the coupled system (12)-(16) :

Given $u_h^0(x) = u_0(x)$, N^α, N, $\mathcal{T}_h$,

for $n = 0, ..., N-1$

•

$$\begin{aligned}&\text{Find } \ I_h(t^{n+1}, .) \in \mathbb{V}_h \ \text{ such as} \\ &a_h(I_h(t^{n+1}, .), \phi_h) = l_h(\phi_h) \ \text{ for all } \phi_h \in \mathbb{V}_h\end{aligned} \tag{40}$$

- $$G_h^n(.) = \sum_{l=1}^{N_\alpha} w_l I_h^l(t^{n+1}, .) \tag{41}$$

- $$\begin{aligned}
&\text{Find } \; u_h^{n+1} \in W_h \;\; \text{ such as, } \; u_h^{n+1}(x) = f(t^{n+1}, x) \;\text{ in }\; \partial\Omega \\
&\quad (\frac{u_h^{n+1} - u_h^n}{\triangle t}, \phi_i)_\Omega + (\nabla(\frac{u_h^{n+1} + u_h^n)}{2}), \nabla\phi_i)_\Omega + \\
&\quad + \theta(\frac{u_h^{n+1^4} + u_h^{n^4}}{2}, \phi_i)_\Omega \\
&\quad = \theta(G_h^n, \phi_i)_\Omega \quad \forall\phi_i \in W_h.
\end{aligned} \tag{42}$$

3.1 Existence and uniqueness solution of the semi-discrete problem

In this section will be probed that there exist a unique solution of the coupled system (26),(37) and (38) if we assume that the initial data are small enough. To this end we introduce a fixed point formulation of the algorithm. First we define the followings function set:

$$\begin{aligned}
F_1 :=& \{v_h \in \mathcal{C}([0,\tau]; W_h); \|v_h\|_{L^8(Q_\tau)} \leqslant M\} \\
F_2 :=& \{I_h \in \mathcal{C}([0,\tau]; \mathbb{V}_h); \|I_h\|_{L^2(Q_\tau)} \leqslant N_b\}, \\
F_3 :=& \{G_h \in \mathcal{C}([0,\tau]; W_h) \;\; ; \|G_h\|_{L^2(Q_\tau)} \leqslant \sqrt{\pi} N_b\}.
\end{aligned}$$

and the following three map:

- $\mathbb{A}_1 : F_1 \longrightarrow F_2$ where for all $u_h \in F_1$, $\mathbb{A}_1(u_h) \in F_2$ is the solution of (26).
- $\mathbb{A}_2 : F_2 \longrightarrow F_3$ such as for all $I_h \in F_2$ we have $\mathbb{A}_2(I_h) = G_h \in F_3$
- $\mathbb{A}_3 : F_3 \longrightarrow F_1$ such that for all $G_h \in F_3$, $\mathbb{A}_3(G_h) \in F_1$ is the solution of (37) and (38).

Now we define the composition map $\mathbb{A} = \mathbb{A}_3 \circ \mathbb{A}_2 \circ \mathbb{A}_1$ from $F_1 \longrightarrow F_1$.

Let

$$\|I_h(t,.)\|_d^2 = \sum_{l=1}^{N_\alpha} w_l \|I_h^l(t,.)\|_{L^2(\Omega)}^2 \tag{43}$$

and

$$\|I_h\|_{\star,2} = \Big(\int_0^\tau \|I_h(t,.)\|_d^2 \, dt\Big)^{\frac{1}{2}} \tag{44}$$

We assume that the initial data $u_h^0 \in W_h$ is fixed and satisfies

$$\|u_h^0\|_{L^5(\Omega)} \leqslant \sqrt[5]{\frac{15\theta\pi M^8}{8}}. \tag{45}$$

Since u_h^0 verify (45) thanks to the theorem 3.3 the equation (26) have a unique solution in $\mathcal{C}([0,\tau]; \mathbb{V}_h)$.

Moreover:

$$\|I_h\|_{\star,2} \leqslant \sqrt{\pi}\|u_h\|^4_{L^8(Q_\tau)} \leqslant \sqrt{\pi}M^4 = N_b.$$

Thus, $I_h \in F_2$ consequently $\mathbb{A}_1$ is a well posed map.

Now, we prove that the map $\mathbb{A}_1$ is continuous. We denote by $I_{1,h}$ and $I_{2,h}$ two solutions of (26) associated to two data, $u_{1,h}$ and $u_{2,h}$ respectively . Let $t \in [0,\tau]$, we have

$$\|I_{1,h}(t) - I_{2,h}(t)\|^2_d \leqslant \sum_{l=1}^{N_\alpha} w_l \|u^4_{1,h}(t) - u^4_{2,h}(t)\|^2_{L^2(\Omega)}.$$

Using generalized Holder's inequality and integrating in $[0,\tau]$ we obtain:

$$\begin{aligned}\|I_{1,h} - I_{2,h}\|^2_{\star,2} &\leqslant 16\sum_{l=1}^{N_\alpha} w_l M^3 \|u_{1,h} - u_{2,h}\|^2_{L^8(Q_\tau)}\\ &\leqslant 16\sqrt{\pi}M^6\|u_{1,h} - u_{2,h}\|^2_{L^8(Q_\tau)}\end{aligned}$$

hence,

$$\|I_{1,h} - I_{2,h}\|_{\star,2} \leqslant 4\sqrt{\pi}M^3\|u_{1,h} - u_{2,h}\|_{L^8(Q_\tau)}. \tag{46}$$

Therefore $\mathbb{A}_1$ is a continuous map.

Thanks to the Cauchy-Schwarz inequality it follows that:

$$\|G_h\|^2_{L^2(Q_\tau)} \leqslant \sqrt{\pi}\int_0^\tau \sum_{l=0}^{N_\alpha} w_l \|I^l_h(t)\|^2_{L^2(\Omega)},$$

then,

$$\|G_h\|_{L^2(Q_\tau)} \leqslant \sqrt{\pi}\|I_h\|_{\star,2}, \tag{47}$$

thus, $\mathbb{A}_2(I_h) = G_h \in F_3$ is well posed. Moreover since $\mathbb{A}_2$ is linear and bounded, $\mathbb{A}_2$ is a continuous map.

Let $\bar{u}_h(t,x) = \mathbb{A}_3(G_h(t,x)$. Using the Young's inequality, we obtain:

$$\|\bar{u}_h\|^8_{L^8(Q_\tau)} \leqslant \frac{1}{12\pi^2}\|G_h\|^2_{L^2(Q_\tau)} + \frac{1}{15\theta\pi}\|\bar{u}^0_h\|^5_{L^5(\Omega)}, \tag{48}$$

As $G_h \in F_2$, we get

$$\|\bar{u}_h\|^8_{L^8(Q_\tau)} \leqslant \frac{M^8}{4} + \frac{1}{15\theta\pi}\|\bar{u}^0_h\|^5_{L^5(\Omega)}. \tag{49}$$

Since the initial data satisfies (45), we have

$$\|\bar{u}_h\|^8_{L^8(Q_\tau)} \leqslant M^8. \tag{50}$$

Then $\bar{u}_h \in F_1$.

To establish the continuity of $\mathbb{A}_3$ we denote $\bar{u}_{1,h}, \bar{u}_{2,h}$ the solutions of (37)-(38) associated to $G_{1,h}, G_{2,h} \in F_3$, respectively. The difference $\bar{u}_{1,h} - \bar{u}_{2,h}$ verifies:

$$(\frac{d\,(\bar{u}_{1,h} - \bar{u}_{2,h})}{d\,t}, \varphi)_\Omega + (\nabla_x(\bar{u}_{1,h} - \bar{u}_{2,h}), \nabla\varphi)_\Omega + \theta(\bar{u}^4_{1,h} - \bar{u}^4_{2,h}, \varphi)_\Omega = \theta(G_{1,h} - G_{2,h}, \varphi)_\Omega, \quad (51)$$

for all $\varphi \in H^1_0(\Omega)$.

Let $\mathbb{H}$ the semigroups of contraction generated by the Laplacian $\triangle$ in $L^5(\Omega) \cap H^1_0(\Omega)$. It follows, (Cazenave et al., 1998), that:

$$(\bar{u}_{1,h} - \bar{u}_{2,h})(t,.) = -4\pi\theta \int_0^t \mathbb{H}(t-s)(\bar{u}^4_{1,h} - \bar{u}^4_{2,h})(s,.)ds + \theta \int_0^t \mathbb{H}(t-s)(G_{1,h} - G_{2,h})(s,.)ds. \quad (52)$$

Since, Cazenave (1998), for $1 \leqslant q \leqslant p \leqslant \infty$

$$\|\mathbb{H}(t)\phi\|_{L^p(\Omega)} \leqslant \frac{1}{(4\pi t)^{(\frac{1}{q} - \frac{1}{p})}} \|\phi\|_{L^q(\Omega)}, \quad \forall t > 0, \forall \phi \in H^1_0(\Omega). \quad (53)$$

taking $p = 8$ and $q = 2$ we obtain:

$$\begin{aligned} \|(\bar{u}_{1,h} - \bar{u}_{2,h})(t,.)\|_{L^8(\Omega)} \leqslant & 4\pi\theta \int_0^t \frac{1}{(4\pi(t-s))^{\frac{1}{2} - \frac{1}{8}}} \|\bar{u}^4_{h,1}(s,.) - \bar{u}^4_{h,2}(s,.)\|_{L^2(\Omega)} ds \\ & + \theta \int_0^t \frac{1}{(4\pi(t-s))^{\frac{1}{2} - \frac{1}{8}}} \|G_{h,1}(s,.) - G_{h,2}(s,.)\|_{L^2(\Omega)} ds. \end{aligned} \quad (54)$$

Applying Holder's inequality and Cauchy-Schwarz inequality we obtain:

$$\|(\bar{u}_{1,h} - \bar{u}_{2,h})(t,.)\|_{L^8(\Omega)} \leqslant 4\pi\theta \frac{2\sqrt[8]{\tau}}{(4\pi)^{\frac{3}{8}}} 4M^3 \Big(\int_0^t \|(\bar{u}_{1,h} - \bar{u}_{2,h})(s,.)\|^8_{L^8(\Omega)} ds\Big)^{\frac{1}{8}} \quad (55)$$

$$+ \theta \frac{2\sqrt[8]{\tau}}{(4\pi)^{\frac{3}{8}}} \|G_{h,1} - G_{h,2}\|_{L^2(Q_\tau)}. \quad (56)$$

By the inequality $(a+b)^8 \leqslant 128(a^8 + b^8)$ for all $(a,b) \in \mathbb{R}^2_+$, it holds that

$$\begin{aligned} \|(\bar{u}_{1,h} - \bar{u}_{2,h})(t,.)\|^8_{L^8(\Omega)} \leqslant & \frac{\theta^8}{\pi^3} 2^{25} M^{24} \tau \int_0^t \|(\bar{u}_{1,h} - \bar{u}_{2,h})(s,.)\|^8_{L^8(\Omega)} ds \\ & + \frac{\theta^8}{\pi^3} 2^9 \tau \|G_{h,1} - G_{h,2}\|^8_{L^2(Q_\tau)}. \end{aligned}$$

Using The Gronwall's inequality, (Cazenave et al., 1998), we deduce that

$$\|(\bar{u}_{1,h} - \bar{u}_{2,h})\|^8_{L^8(Q_\tau)} \leqslant \frac{\theta^8}{\pi^3} 2^9 \tau^2 e^{\frac{\theta^8}{\pi^3} 2^{25} M^{24} \tau^2} \|G_{h,1} - G_{h,2}\|^8_{L^2(Q_\tau)}. \quad (57)$$

Hence, the map $\tilde{\mathbb{H}}_3$ is continuous.

Consequently, thanks to the estimates (46), (47) and (57) we have:

$$\|(\bar{u}_{1,h} - \bar{u}_{2,h})\|^8_{L^8(Q_\tau)} \leqslant \gamma(\tau, M)\|(u_{1,h} - u_{2,h})\|^8_{L^8(Q_\tau)} \tag{58}$$

where $\gamma(\tau, M) = \pi \sqrt[8]{\frac{\theta^8}{\pi^3} 2^{25} M^{24} \tau^2}\, exp(\sqrt[8]{\frac{\theta^8}{\pi^3} 2^{25} M^{24} \tau^2})$. There exists $\tau^\star(M) > 0$ such that for all $\tau \leqslant \tau^\star(M)$, we have

$$\gamma(\tau, M) < 1. \tag{59}$$

Since F_1 is a closed set in $L^8(Q_\tau)$ applying the fixed point Banach theorem, Martin (1976), the coupled system (26),(37) and (38) have a unique solution for $\tau \leqslant \tau^\star(M)$.

4 NUMERICAL RESULTS

First in this section, we shall present the numerical results concerning test problems found on the references Mishra et al. (2005) and Asllanaj et al. (2007). In a second time we shall present the numerical result about a new case, with opaques surfaces with combined specular and diffuse reflection.

The reference temperature considered is the hot temperature $u_{ref} = 1000K$ and in all examples the absorption coefficient was $\kappa = 1m^{-1}$. Initially, the medium was assumed to be at a constant temperature equal to $\widetilde{u}_0 = 500K$.

The angular S_4 discretization was used to maintain a reasonable level of accuracy. The dimensionless time step was fixed at $\triangle t = 5.10^{-4}$, and iterations were terminated when the steady state was achieved with a tolerance set to 10^{-5}.

All calculations were performed on a Mac OS X version 10.6.6 2.22GHz Intel Core 2 Duo using the MATLAB software.

4.1 Non-homogeneous thermal Dirichlet boundary conditions.

In this section, we consider the case with non-homogeneous Dirichlet conditions (15). The conduction radiation number was set to $N_s = 1$ and $N_s = 0.1$. The simulations were performed for a non-structured mesh refined in high-temperature region, composed of 1534 nodes, Figure 1. We have considered DG method of order 2 to the approximation of the RTE solution and to solve the CE we have implemented finite element method with an order of approximation equal to 2.

In this first test case, the medium boundaries were assumed to be black surfaces with imposed temperatures. This problem was investigated by Mishra et al. (2005). These authors have implemented a Lattice Boltzmann method (LBM) to solve the energy equation and the collapsed-dimension method (CDM) was used to compute the radiative source term in the energy equation. This method was developed for 2-d simple geometries based on structured grids. Let us note that the same transient problem was also studied by Asllanaj et al. (2007). In this latter work, the RTE was solved by using a FVM based on a cell vertex scheme and associated to a modified exponential scheme. The PHAML code was used to solve the energy equation using finite elements method. The numerical solutions by Mishra et al. (2005) were used as reference to check those obtained with our numerical method.

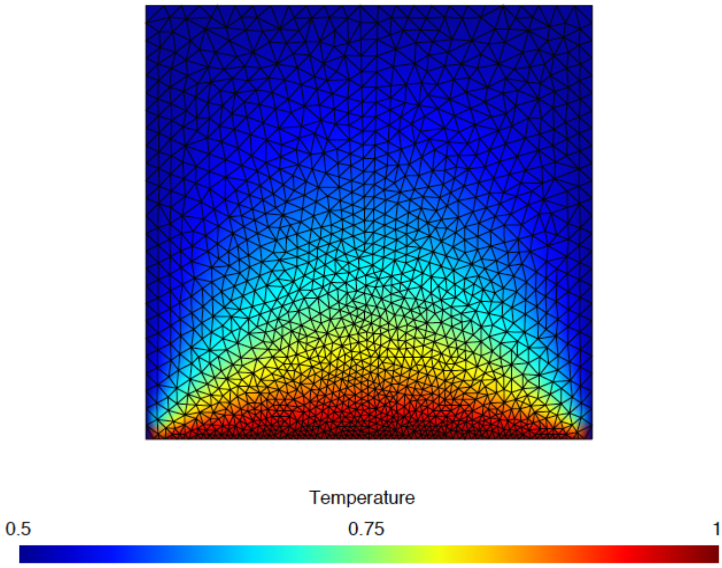
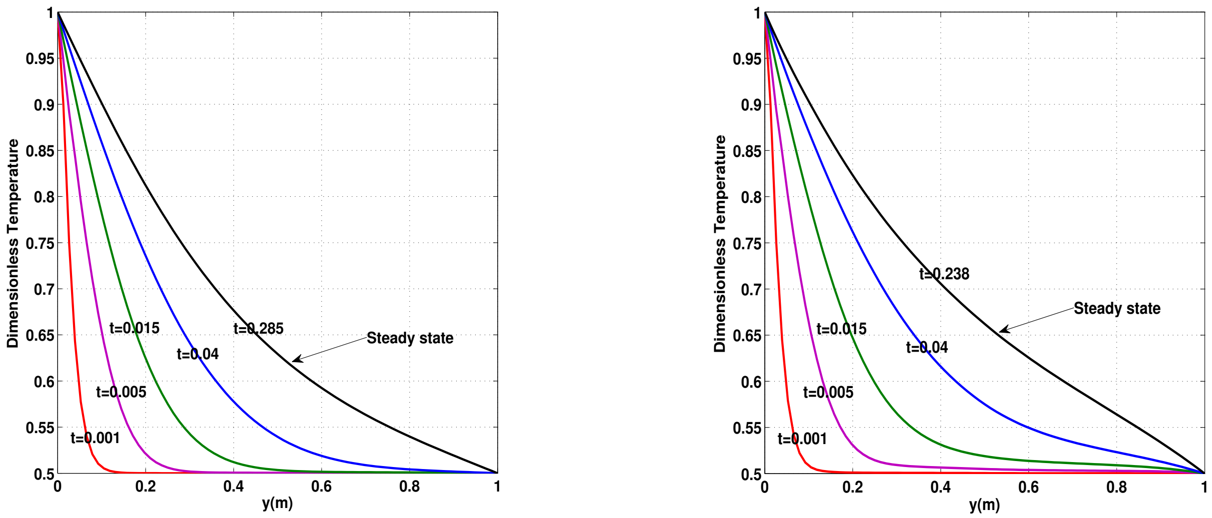


Figure 1: The dimensionless temperature distribution for $N_s = 1$ for a non-structured mesh refined in high-temperature region composed of 1534 nodes.

Figure 2 show the evolution of the dimensionless temperature along the centerline position $x_1 = 0.5$ at different times and for different number $N_s = 0.1$ and $N_s = 1$. It should be noted that the computational time needed to achieve convergence of our algorithm increases with N_s. Obviously, more important the radiative part is, more quickly steady state was reached. Our results were in agreement with those of Asllanaj et al. (2007) and Mishra et al. (2003), confirming the validity of our numerical method for black surfaces.



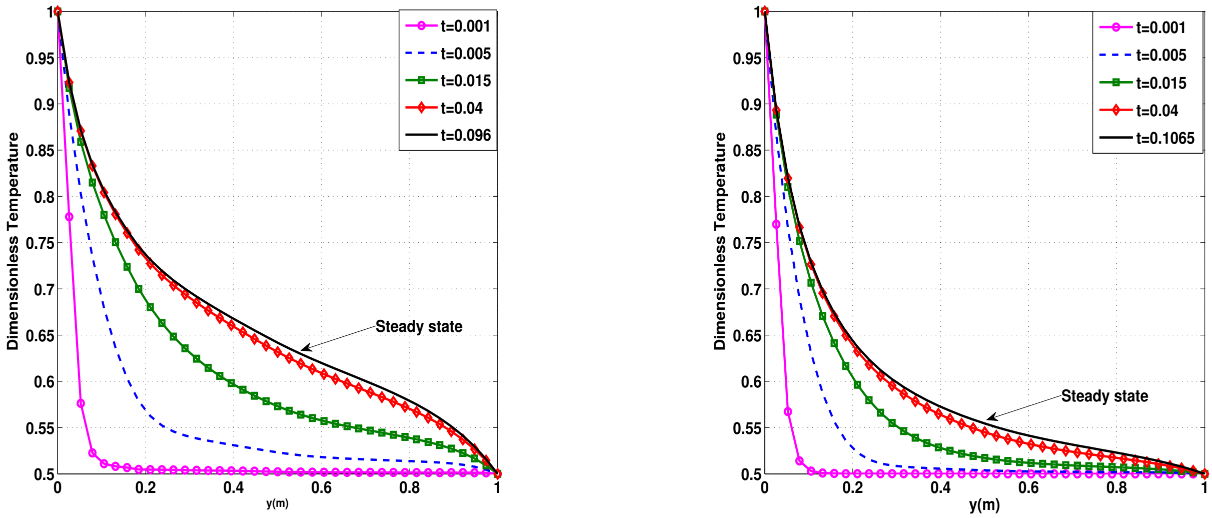
(a) $N_s = 1$

(b) $N_s = 0.1$

Figure 2: Dimensionless temperature along the centerline position at different dimensionless times and for two different values of the conduction-radiation parameter.

4.2 Combined black and diffuse reflection opaque surfaces.

In this second test case, the hot (south) surface was assumed to be opaque with diffuse reflection and the other three surfaces were black. We use the same mesh given in the section 4.1. The number N_s was set at 0.01. Figure 3 show the dimensionless temperature distributions along the centerline position for two values of the wall emissivity ϵ in the boundary condition (4), $\epsilon = 0.5$ and $\epsilon = 0.1$ respectively. These also show that the physical time required to obtain a converged solution decreases with ϵ. A steady state was reached as dimensionless time $t = 0.096$ for $\epsilon = 0.5$ and $t = 0.1065$ for $\epsilon = 0.1$. Our results were found to comply with those of Mishra et al. (2003).



(a) $\epsilon = 0.5$ (b) $\epsilon = 0.1$

Figure 3: Dimensionless temperature along the centerline position at different dimensionless times and for two different values of the emissivity coefficient.

4.3 Opaque surfaces with combined specular and diffuse reflection

The hot (south) surface was assumed to be opaque with specular reflection and the other three surfaces to be opaque with diffuse reflection. To our knowledge, this test case does not exist in literature in this area of research. In Figure 4 the dimensionless temperature distribution is presented for three values of the wall emissivity ϵ on boundary condition (5), ($\epsilon = 0.1$, $\epsilon = 0.5$ and $\epsilon = 0.9$) for all surfaces and with the conduction radiation number N_s equal to 1. The decrease in the emissivity coefficient was found to delay the steady state. The convergence was reached as $t = 0.086$ for $\epsilon = 0.9$, $t = 0.1065$ for $\epsilon = 0.5$ and $t = 0.0998$ for $\epsilon = 0.1$.

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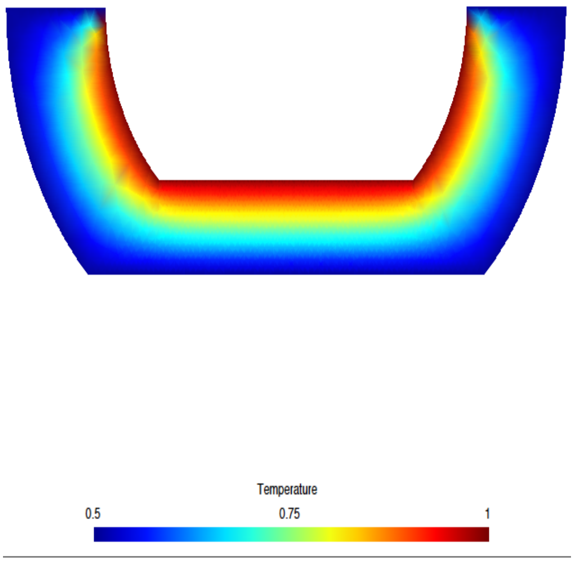
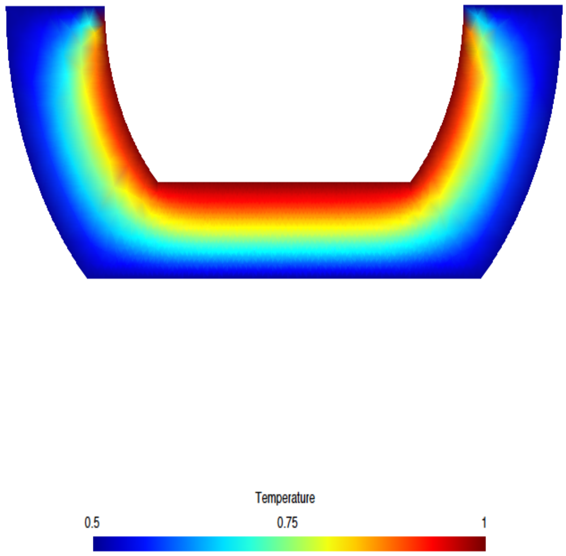
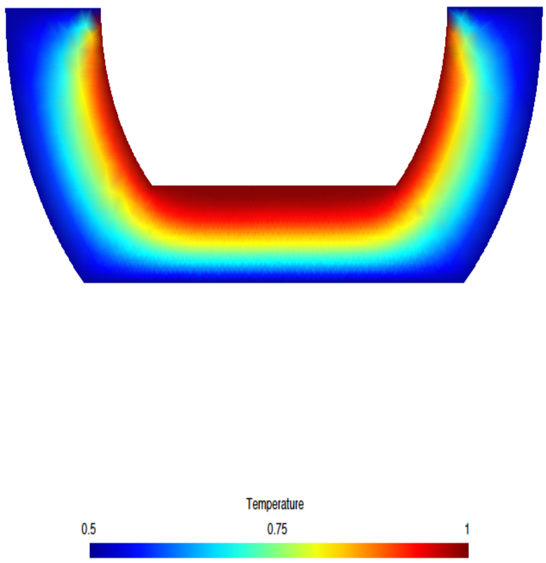


(a) $\epsilon = 0.1$



(b) $\epsilon = 0.5$



(c) $\epsilon = 0.9$

Figure 4: The dimensionless temperature distribution for three different values of the emissivity coefficient using a non-structured mesh refined in high-temperature region composed of 1513 nodes.

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