A THEORETICAL AND COMPUTATIONAL FRAMEWORK FOR ISOTROPIC DAMAGE MODELS BASED ON A GENERALIZED MICRO-POLAR CONTINUUM THEORY

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Abstract. *Damage models represent a common approach to the physically non-linear analysis of quasi-brittle materials like concrete. Due to their strain-softening behavior, they may suffer for strain localization problems, hence leading to a set of pathological phenomena that affects the quality of the simulations. The micro-polar continuum theory has been applied in the past to the analysis of localization problems in elasto-plasticity, and can provide an alternative to mitigate such problems also for damage models. This paper investigates the application to isotropic damage models of the micro-polar continuum theory. The formulation of the micro-polar theory is presented in a tensorial generalized form, allowing its inclusion in a theoretical and computational unified framework for constitutive models. The computational implementation of such models in the numerical core of the INSANE computer system (INteractive Structural ANalysis Environment) is discussed, and is presented in a format that makes it independent on the peculiar numerical method adopted for the solution of the problem. Numerical simulations are presented in order to illustrate the proposed models.*

Keywords: *Micro-polar continua, Isotropic damage, Strain localization, Unified constitutive models formulation*

1 INTRODUCTION

One of the first applications of the *continuum damage* theory to the modeling of concrete fracture was proposed by Mazars and Cabot (1989), as an alternative to *discrete* and *smearing* crack models. Damage models represent now a common approach to the physically non-linear analysis of quasi-brittle materials like concrete. With this approach, the degradation of the material properties is described making use of a set of damage variables based on a thermodynamical formulation. Among the possible damage models, a simple approach is offered by the ones based on an isotropic damage with a single damage variable.

Despite the advantages offered by the continuum approach, due to their *strain softening* behavior, damage models may suffer for *strain localization* effects that, in a finite element analysis, lead to a mesh dependency (de Borst et al., 1993). This pathological behavior is due to the loss of ellipticity of the equilibrium equations at a certain load level, corresponding to a singular strain rate. The ill-posedness of the problem corresponds to an infinite set of solutions, from which the numerical method selects the one corresponding to the smallest energy dissipation. This approximated solution strongly depends on the mesh; at mesh refinement it tends to a failure with zero energy dissipation, and then to a non-physical behavior. The strain-softening model leads also to others pathological effects, as premature crack initiation and instantaneous perfectly-brittle fracture (Peerlings et al., 2002).

The problem of strain localization has been investigated by many authors in the past, with a peculiar attention on plasticity problems. It has been pointed out that the pathological behaviors obtained in the numerical simulations are due to the *local* representation offered by the classic continuum theory, in contrast with the *non-local* nature of phenomena like damage and plasticity (Bazant, 1991). The main aim of the proposed solutions to this problem is the introduction of an *intrinsic length* in the continuum model, allowing to recover the non-local character of the phenomenon. Among the possible methods, the *micro-polar* continuum model has been successfully applied in the past to regularize the solution in strain-softening elasto-plasticity (de Borst and Sluys, 1991; de Borst, 1993; Lages, 1997; Iordache and Willam, 1998; Fuina, 2009), while its application to continuum damage models has not been investigated yet.

Damage models have been shown to belong to the more general class of elastic-degrading models, and can then take advantage of a proper unified formulation. In the last years, a lot of efforts have been made in order to define a unified constitutive models formulation, ables to encompass various kinds of elasto-plastic, elastic-degrading and damage models. As pointed out by many authors, such a unified formulation provides a common basis, that makes possible an easy comparison between different models. The fact that different models may derive from a common formulation allows them to share common theoretical and computational resources. One of the first comprehensive attempts of unification can be found in Carol and Willam (1994), where the authors present a unified formulation of elasto-plasticity and elastic-degradation, based on a single loading function, and where the inclusion in the unified framework of existent isotropic damage models is explicitly illustrated. Such a unified formulation has been recently expanded by Penna (2011), in order to create a theoretical framework for constitutive models ables to enclose a large amount of constitutive models (elasto-plastic, isotropic, orthotropic and anisotropic elastic-degrading) based on multiple loading functions. In the same work, such a constitutive models framework has been successfully implemented in the numerical core of the software *INSANE*, an object-oriented open-source software for structural analysis, developed by the Structural Engineering Department of the Federal University of Minas Gerais (Alves et al., 2013). The peculiarity of the proposed theoretical and computational framework for constitutive models is the use of a *tensorial* format instead of a vectorial-matricial one, allowing the implementation of different constitutive models independently on the peculiar numerical method adopted. The use of a unified formulation allows also to optimize the implementation of new constitutive models, making possible to share common computational resources.

This paper presents, to the authors knowledge, one of the first attempts in the application of the micro-polar continuum theory to damage mechanics problems, with peculiar attention to isotropic damage models. The main aim of the paper is the inclusion in the aforementioned theoretical and computational framework of isotropic damage models based on a *generalized* micro-polar continuum theory. The first part of the paper resumes the basic aspects of the framework for constitutive models elaborated for the classic continuum formulation, focusing on isotropic damage models. The second part recalls the basic equations of the micro-polar theory, and presents a formulation for isotropic damage models extended to the micro-polar continuum. Such a formulation is then reformulated in a *generalized* form, to make possible its inclusion in the aforementioned constitutive models framework. Two damage models for the micro-polar continuum are introduced, as extensions of the *Mazars-Lemaitre* (Mazars and Lemaitre, 1984) and *De Vree* (de Vree et al., 1995; de Borst and Gutiérrez, 1999) models. The aspects on the implementation in the numerical core of the *INSANE* system are also discussed. Finally, numerical simulations are presented in order to illustrate the new models and their implementation.

2 A FRAMEWORK FOR ISOTROPIC DAMAGE MODELS

Damage models have been shown to belong to the more general class of elastic-degrading models; they can then take advantage of a *unified formulation*, incorporating also elasto-plastic models. Such a unified formulation for constitutive models can be used as a basis for the definition of a proper theoretical and computational framework, ables to represent a large class of constitutive models. The main characteristics of such a unified formulation are resumed in the following section, omitting the whole derivation, for which one can refer to Carol and Willam (1994).

2.1 Unified formulation for constitutive models

In the context of the classic continuum theory, constitutive models based on a unified formulation are characterized, in their general representation, by a total stress-strain relation

$$\underline{\sigma} = \hat{\mathbf{E}}^S \cdot \underline{\varepsilon} \tag{1}$$

where $\hat{\mathbf{E}}^S$ is the a *secant* constitutive operator.

Despite the fact that the unified formulation can be presented in two different formats, a *stress-based* representation and a *strain-based* one, here the attention is focused on the latter approach, that is obtained introducing an additive decomposition of the stress rate tensor, into an elastic part $\dot{\underline{\sigma}}^e$ and an inelastic one $\dot{\underline{\sigma}}^d$

$$\dot{\underline{\sigma}}^e = \dot{\underline{\sigma}} - \dot{\underline{\sigma}}^d \tag{2}$$

where the elastic part is such that

$$\dot{\underline{\sigma}}^e = \hat{\mathbf{E}}^S \cdot \dot{\underline{\varepsilon}} \tag{3}$$

The fundamentals elements of the strain-based approach to the unified formulation are resumed by the following set of equations

$$f(\underline{\varepsilon}, \lambda) \leq 0, \quad \dot{\underline{\sigma}}^d = \dot{\lambda}\, \underline{m}^*, \quad \underline{n}^* = \left.\frac{\partial f}{\partial \underline{\varepsilon}}\right|_\lambda, \quad H^* = -\left.\frac{\partial f}{\partial \lambda}\right|_{\underline{\varepsilon}}, \quad \hat{\mathbf{E}}^t = \hat{\mathbf{E}}^S + \frac{1}{H^*}\, \underline{m}^* \otimes \underline{n}^* \tag{4}$$

where are represented: the loading function, depending on the strain tensor and on the inelastic multiplier, the degradation rule for the inelastic stress rate, where the second-order tensor $\underline{m}^*$ represents the direction of the inelastic degradation, the gradient of the loading function, the hardening-softening modulus, and the tangent constitutive operator.

A damage model is obtained once the secant operator appearing in ”Eq. (3)” is assumed to depend on a set of *damage variables*, that completely defines the state of degradation, or damage, of the material during the loading process. The main advantage of this approach is that a reduced number of parameters is used instead of the 21 independent components of the constitutive operator, allowing a reduction in the complexity of the problem.

Isotropic damage models

A simple approach to damage is represented by *isotropic damage models*, characterized by a single damage variable, for which the secant constitutive operator is represented by the expression

$$\hat{\mathbf{E}}^S(D) = (1 - D)\, \hat{\mathbf{E}}^0 \tag{5}$$

where the damage variable D is assumed to varies from 0 (undamaged material) to 1 (completely damaged material), and where $\hat{\mathbf{E}}^0$ is the initial elastic operator.

Taking into account the strain-based formulation, the loading function appearing in ”Eq. (4)”, can be represented as

$$f(\underline{\varepsilon}, D) = g(\underline{\varepsilon}) - \kappa(D) \leq 0 \tag{6}$$

where $g(\underline{\varepsilon})$ is a function depending only on the deformation, that represents the loading condition of the continuum and that is a characteristic of each peculiar model, while $\kappa(D)$ is an historical parameter that depends on the damage variable. The historical parameter is representative of the maximum level of deformation reached during the loading process of the model.

Using ”Eq. (5)” and ”Eq. (6)”, the equations previously defined for an elastic-degrading model can be rewritten in the following way: for a generic isotropic damage model the degradation rule is represented by

$$\dot{\underline{\sigma}}^d = -\dot{\lambda}\, \hat{\mathbf{E}}^0 \cdot \underline{\varepsilon} = -\dot{\lambda}\, \underline{\sigma}^0 \tag{7}$$

and the hardening-softening modulus by

$$H = \frac{\partial \kappa}{\partial D} \tag{8}$$

Finally, the tangent constitutive operator assumes the form

$$\hat{\mathbf{E}}^t = (1-D)\hat{\mathbf{E}}^0 - \frac{1}{\frac{\partial \kappa}{\partial D}}\, \underline{\sigma}^0 \otimes \left.\frac{\partial f}{\partial \underline{\varepsilon}}\right|_{\lambda} \tag{9}$$

This set of equations must be complemented with an evolution law for the damage variable, that can be chosen independently on the peculiar damage model adopted.

2.2 INSANE constitutive models framework

The unified formulation described in the previous section, has been used as a theoretical basis for the creation of the constitutive models framework of the software *INSANE* (Penna, 2011), ables to enclose a large amount of elasto-plastic and elastic-degrading models. One of the main advantages in the use of such a formulation is that it allows different constitutive models to share common theoretical and numerical resources, leading to an optimized implementation and to minimum efforts for the framework expansion.

A peculiarity of the implementation of the framework in the software is that it preserves the tensorial nature of the equations of the unified formulation, without converting them into a vectorial-matricial format. The main quality of such a tensor-based format is that it leads to the implementation of different constitutive models *independently* on the peculiar numerical method used, allowing to take fully advantage of the object-oriented paradigm.

Despite the large amount of constitutive models represented by the framework, they are mainly based on a classic continuum theory. The objective of the paper is then the expansion of the framework, in order to include isotropic damage models based on the micro-polar continuum theory. In order to take advantage of the theoretical resources of the unified formulation already defined for the classic continuum theory, and of the numerical methods already implemented in the *INSANE* system, a proper micro-polar continuum formulation will be introduced, in such a way to guarantee its compatibility with the tensorial format of the unified formulation.

3 MICRO-POLAR CONTINUA

Micro-polar continua belong to the family of *multi-field* continua, e.i. continuum bodies which configuration is defined by two descriptors, the *motion* of the material points of the body and a *morphological descriptor* (see, e.g., Mariano and Stazi (2005) for further details). In the peculiar case of the micro-polar continuum, such a morphological descriptor corresponds to a *rigid rotation* of the material points of the continuum.

In a geometrically linear approach, at each point of the continuum, a *displacement field* $\bar{u}$ and a *micro-rotation field* $\bar{\varphi}$ can be defined, leading to the following strain measures

$$\underline{\gamma} = \mathrm{grad}(\bar{u})^T - \mathsf{e} \cdot \bar{\varphi} = (u_{j,i} - \mathsf{e}_{ijk}\, \varphi_k)\ \bar{e}_i \otimes \bar{e}_j \tag{10}$$

$$\underline{\kappa} = \mathrm{grad}(\bar{\varphi})^T = \varphi_{j,i}\, \bar{e}_i \otimes \bar{e}_j \tag{11}$$

which are referred to, respectively, as *strain tensor* and *micro-curvature tensor*, and where e represents the standard *Levi-Civita* symbol with three indexes and $(\bar{e}_i)$, with i = 1,2,3, is an orthonormal basis. To these strain measures correspond, respectively, the stress tensor $\underline{\sigma}$ and the couple-stress tensor $\underline{\mu}$, defined, in the field of elasticity, by the constitutive laws

$$\underline{\sigma} = \hat{\mathbf{A}}^0 \cdot \underline{\gamma} \tag{12}$$

$$\underline{\mu} = \hat{\mathbf{C}}^0 \cdot \underline{\kappa} \tag{13}$$

where $\hat{\mathbf{A}}^0$ and $\hat{\mathbf{C}}^0$ are the elastic constitutive operators for the micro-polar continuum theory.

It should be noted that the components of the strain tensor $\underline{\gamma}$ and of the micro-curvature tensor $\underline{\kappa}$, as the ones of the stress tensor $\underline{\sigma}$ and the couple-stress $\underline{\mu}$, are not characterized by the same units of measure. When dealing with a micro-polar model, it is common to scale the micro-curvature tensor and the couple-stress tensor using the characteristic lengths of the micro-polar theory (see e.g. Lages (1997) and Fuina (2009)), in order to obtain a dimensional compatibility with, respectively, the strain tensor and the stress tensor. In the following of the paper, in order to maintain a simple notation, we suppose to use the symbols $\underline{\kappa}$ and $\underline{\mu}$ to indicate, respectively, the scaled micro-curvature and couple-stress tensors.

3.1 Micro-polar isotropic damage models

In the context of the classic continuum theory, a damage model is characterized by a total stress-strain relation ("Eq. (3)"). Proceeding in an analogous way, the following equations can be defined for the micro-polar continua

$$\underline{\sigma} = \hat{\mathbf{A}}^S \cdot \underline{\gamma} \tag{14}$$

$$\underline{\mu} = \hat{\mathbf{C}}^S \cdot \underline{\kappa} \tag{15}$$

where the secant constitutive operators $\hat{\mathbf{A}}^S$ and $\hat{\mathbf{C}}^S$ have been introduced.

The definition of a damage model in the context of a unfied formulation requires the introduction of a proper *loading function*. In the case of micro-polar continua, two possible approaches can be followed. As pointed out by Forest and Sievert (2003) for the case of elasto-plasticity, for a micro-polar continuum both a *single* and a *multi-criterion* plasticity can be adopted. The first approach is based on a single loading function, depending, for the strain based formulation, on both the strain and micro-curvature tensors, while the second one is based on two different loading functions, each one defined, respectively, on the strain space and on the micro-curvature space. In the present paper, the extension of isotropic damage models to the micro-polar continuum is based on the single criterion approach. The choice of such an approach is based on the fact that, as it will be shown in the following, it best fits the purpose of the paper, allowing for a straightforward inclusion of the micro-polar continuum theory in the existent unified formulation for constitutive models.

The single loading function for the micro-polar continuum is then defined as

$$f(\underline{\gamma}, \underline{\kappa}, \lambda) \leq 0 \tag{16}$$

that in the specific case of an isotropic damage model assumes the expression

$$f(\underline{\gamma}, \underline{\kappa}, \lambda) = g(\underline{\gamma}, \underline{\kappa}) - \kappa(D) \leq 0 \tag{17}$$

analogous to the one of the classic continuum ("Eq. (6)"). The choice of a single-criterion approach reflects also on the damage variable definition; indeed, such an approach corresponds to the adoption of a single inelastic multiplier, and hence, for an isotropic damage model, to the adoption of a single damage variable. It then follows that, for an isotropic damage model, the secant constitutive operators of "Eq. (14)" and "Eq. (15)" are defined by

$$\hat{\mathbf{A}}^S(D) = (1-D)\,\hat{\mathbf{A}}^0 \tag{18}$$

$$\hat{\mathbf{C}}^S(D) = (1-D)\,\hat{\mathbf{C}}^0 \tag{19}$$

As for the classic continuum approach, also for the micro-polar model a degradation rule must be introduced, both for the inelastic stress and for the inelastic couple-stress; for an isotropic damage model, the degradation rules assume the form

$$\dot{\underline{\sigma}}^d = -\dot{\lambda}\,\hat{\mathbf{A}}^0 \cdot \underline{\gamma} = -\dot{\lambda}\,\underline{\sigma}^0 \tag{20}$$

$$\dot{\underline{\mu}}^d = -\dot{\lambda}\,\hat{\mathbf{C}}^0 \cdot \underline{\kappa} = -\dot{\lambda}\,\underline{\mu}^0 \tag{21}$$

In order to obtain a complete set of equations like the one presented in "Eq. (4)", a tangent operator, ables to link the rate of stress with the rate of strain, must be introduced. It should be noted that an approach analogous to the one followed for the classic continuum cannot be directly applied to the micro-polar theory. As it will be shown in the following section, the coupling between the rate equations of the micro-polar formulation doesn't allow to introduce two different tangent operators, one for the rate of stress and one for the rate of couple-stress. It will be shown that a formulation for the damage in micro-polar continua analogous to the one of the classic continuum theory can be obtained introducing a proper *generalized formulation* for the micro-polar continuum theory.

4 GENERALIZED FORMULATION FOR MICRO-POLAR CONTINUA

The rate of stress and the rate of couple-stress that can be obtained for a micro-polar continuum from "Eq. (14)" and "Eq. (15)" are represented, in the context of isotropic damage models, by the equations

$$\dot{\underline{\sigma}} = \left((1-D)\,\hat{\mathbf{A}}^0 - \frac{1}{\frac{\partial \kappa}{\partial D}}\,\underline{\sigma}^0 \otimes \left.\frac{\partial f}{\partial \underline{\gamma}}\right|_{\lambda,\underline{\kappa}} \right) \cdot \dot{\underline{\gamma}} - \left(\frac{1}{\frac{\partial \kappa}{\partial D}}\,\underline{\sigma}^0 \otimes \left.\frac{\partial f}{\partial \underline{\kappa}}\right|_{\lambda,\underline{\gamma}} \right) \cdot \dot{\underline{\kappa}} \tag{22}$$

$$\dot{\underline{\mu}} = \left((1-D)\,\hat{\mathbf{C}}^0 - \frac{1}{\frac{\partial \kappa}{\partial D}}\,\underline{\mu}^0 \otimes \left.\frac{\partial f}{\partial \underline{\kappa}}\right|_{\lambda,\underline{\gamma}} \right) \cdot \dot{\underline{\kappa}} - \left(\frac{1}{\frac{\partial \kappa}{\partial D}}\,\underline{\mu}^0 \otimes \left.\frac{\partial f}{\partial \underline{\gamma}}\right|_{\lambda,\underline{\kappa}} \right) \cdot \dot{\underline{\gamma}} \tag{23}$$

It should be noted that the above equations are *coupled*, since each one depends on both the strain and the micro-curvature rates. Due to such a coupling it is not possible to introduce in the micro-polar formulation of isotropic damage a tangent operator for each one of the equations, like the one defined in "Eq. (4)". Hence, the damage model defined for the micro-polar continuum is not explicitly compatible with the tensorial format of the unified formulation for constitutive models.

A *generalized formulation* for the micro-polar theory is now introduced, in order to extend the compatibility of the micro-polar model to the tensorial format of the unified formulation, and to obtain a proper tangent operator like the one of "Eq. (4)". The generalized formulation is introduced condensing both the stress and couple-stress rates of "Eq. (22)" and "Eq. (23)" in a single equation, represented by the following rate constitutive equation

$$\dot{\underline{\Sigma}} = \hat{\mathcal{E}}^t \cdot \dot{\underline{\Gamma}} \tag{24}$$

where the *generalized stress-rate operator* $\dot{\underline{\Sigma}}$ and the *generalized strain-rate operator* $\dot{\underline{\Gamma}}$, both represented by second-order tensors with dimension six, are defined by

$$\dot{\underline{\Sigma}} = \begin{pmatrix} \dot{\underline{\sigma}} & \underline{0} \\ \underline{0} & \dot{\underline{\mu}} \end{pmatrix}, \qquad \dot{\underline{\Gamma}} = \begin{pmatrix} \dot{\underline{\gamma}} & \underline{0} \\ \underline{0} & \dot{\underline{\kappa}} \end{pmatrix} \tag{25}$$

and where the *generalized tangent operator* $\hat{\mathcal{E}}^t$, represented by a fourth-order tensor with dimension six, is defined by

$$\hat{\mathcal{E}}^t = (1 - D)\,\hat{\mathcal{E}}^0 - \frac{1}{\frac{\partial \kappa}{\partial D}}\,\underline{\Sigma}^0 \otimes \left.\frac{\partial f}{\partial \underline{\Gamma}}\right|_{\lambda} \tag{26}$$

The equation representing the generalized tangent operator makes use of the elastic operator $\hat{\mathcal{E}}^0$, represented by a fourth-order tensor with dimension six that contains the components of both $\hat{\mathbf{A}}^0$ and $\hat{\mathbf{C}}^0$, and of the following operators

$$\underline{\Sigma}^0 = \begin{pmatrix} \underline{\sigma}^0 & \underline{0} \\ \underline{0} & \underline{\mu}^0 \end{pmatrix}, \qquad \left.\frac{\partial f}{\partial \underline{\Gamma}}\right|_{\lambda} = \begin{pmatrix} \left.\frac{\partial f}{\partial \underline{\gamma}}\right|_{\lambda,\underline{\kappa}} & \underline{0} \\ \underline{0} & \left.\frac{\partial f}{\partial \underline{\kappa}}\right|_{\lambda,\underline{\gamma}} \end{pmatrix} \tag{27}$$

The generalized formulation for isotropic damage models based on the micro-polar theory defined in this section is based on an *enhanced* tensorial format, that leads to a tensorial representation of the micro-polar model that is completely analogous to the one of the unified formulation for the classic continuum theory ("Eq. (4)"). Having introduced this formulation, one of the main objectives of the paper, e.i. the expansion of the constitutive models framework of the software *INSANE* with isotropic damage models based on the micro-polar theory, can be easily reached. Indeed, due to the completely analogy between the enhanced tensorial formulation of the micro-polar model and the one of the classic continuum, damage models based on the micro-polar theory can be implemented in the software taking advantage of the existent implementations of classic continuum damage models, requiring only minor modifications of the code, without altering its basic structure.

It should be noted that the compatibility between the enhanced tensorial formulation and the one of the classic continuum theory has been made possible by the adoption of a single-criterion formulation for the damage ("Eq. (16)"). If the multi-criterion formulation had been adopted, instead of "Eq. (22)" and "Eq. (23)", one could have obtained two similar but completely uncoupled equations, allowing the direct definition of two tangent operators. Despite the apparent simplicity of this formulation if compared to the single-criterion one, it must be emphasized that it would have not allowed a direct correspondence of the micro-polar tensorial format with the one of the classic continuum, requiring much more efforts for its implementation.

With the generalized formulation presented in this section and with the concepts introduced in the previous one, a general theoretical basis for damage models based on the micro-polar theory has been introduced. Specific models can be obtained specifying the function $g(\underline{\gamma}, \underline{\kappa})$ appearing in "Eq. (17)", that is a characteristic of each model. In the following, two damage models are introduced, as extensions of the *Mazars-Lemaitre* and *De Vree* damage models already defined for the classic continuum theory.

4.1 Mazars-Lemaitre micro-polar model

In the Mazars-Lemaitre damage model of the classic continuum (Mazars and Lemaitre, 1984), the function $g(\underline{\varepsilon})$ appearing in "Eq. (6)" is represented by an *equivalent strain*, depending only on the strain tensor $\underline{\varepsilon}$. Extending the classic formulation, a version of the Mazars-Lemaitre model for the micro-polar theory is introduced, defining the function $g(\underline{\gamma}, \underline{\kappa})$ as the following *generalized equivalent strain*

$$\Gamma^{eq} = \sqrt{\underline{\Gamma} \cdot \underline{\Gamma}} = \sqrt{\underline{\gamma} \cdot \underline{\gamma} + \underline{\kappa} \cdot \underline{\kappa}} \tag{28}$$

to which corresponds the generalized gradient of the loading function

$$\left.\frac{\partial f}{\partial \underline{\Gamma}}\right|_{\lambda} = \begin{pmatrix} \frac{\underline{\gamma}}{\Gamma^{eq}} & \underline{0} \\ \underline{0} & \frac{\underline{\kappa}}{\Gamma^{eq}} \end{pmatrix} \tag{29}$$

that completes the formulation of the Mazars-Lemaitre damage model for the micro-polar theory.

4.2 De Vree micro-polar model

As for the Mazars-Lemaitre model, extending the classic formulation of the De Vree damage model (de Vree et al., 1995), a version for the micro-polar theory can be introduced. Such a model is introduced defining the function $g(\underline{\gamma}, \underline{\kappa})$ as the following *generalized equivalent strain*

$$\Gamma^{eq} = \frac{k-1}{2k(1-2\,\nu)}\,\mathrm{I}_1^{\Gamma} + \frac{1}{2k}\sqrt{\left(\frac{k-1}{1-2\,\nu}\,\mathrm{I}_1^{\Gamma}\right)^2 + \frac{12\,k}{(1+\nu)^2}\,\mathrm{J}_2^{\Gamma}} \tag{30}$$

where the parameter k allows to take into account the different behavior of the material for a compression and a tension state. Such a parameter is assumed equal to the ratio of the compressive uniaxial strength and the tensile uniaxial strength, $k = f_c/f_t$. In "Eq. (30)", the first and second invariants of the generalized strain operator are defined as

$$\mathrm{I}_1^{\Gamma} = \mathrm{I}_1^{\gamma} + \mathrm{I}_1^{\kappa} \tag{31}$$

$$\mathrm{J}_2^{\Gamma} = \mathrm{J}_2^{\gamma} + \mathrm{J}_2^{\kappa} \tag{32}$$

Such quantities can be calculated, for a micro-polar model in a plane state, as

$$\mathrm{I}_1^{\gamma} = \mathrm{tr}(\underline{\gamma}) \tag{33}$$

$$\mathrm{I}_1^{\kappa} = \mathrm{tr}(\underline{\kappa}) \tag{34}$$

$$\mathrm{J}_2^{\gamma} = \frac{1}{4}\left(\underline{\gamma}^{dev} \cdot \underline{\gamma}^{dev} + \underline{\gamma}^{dev} \cdot \underline{\gamma}^{devT}\right) \tag{35}$$

$$\mathrm{J}_2^{\kappa} = \frac{1}{2}\left(\underline{\kappa}^{dev} \cdot \underline{\kappa}^{dev}\right) \tag{36}$$

where with the superscript *dev* it is indicated the *deviatoric* part of a tensor. To the equivalent strain defined in "Eq. (30)", corresponds the following components of the generalized gradient of the loading function

$$\left.\frac{\partial f}{\partial \underline{\gamma}}\right|_{\lambda,\underline{\kappa}} = \frac{k-1}{2k(1-2\,\nu)}\,\frac{\partial \mathrm{I}_1^\Gamma}{\partial \underline{\gamma}} + \frac{1}{4k}\,\frac{2\left(\frac{k-1}{1-2\nu}\right)^2 \mathrm{I}_1^\Gamma \frac{\partial \mathrm{I}_1^\Gamma}{\partial \underline{\gamma}} + \frac{12\,k}{(1+\nu)^2}\,\frac{\partial \mathrm{J}_2^\Gamma}{\partial \underline{\gamma}}}{\sqrt{\left(\frac{k-1}{1-2\,\nu}\,\mathrm{I}_1^\Gamma\right)^2 + \frac{12\,k}{(1+\nu)^2}\,\mathrm{J}_2^\Gamma}} \tag{37}$$

$$\left.\frac{\partial f}{\partial \underline{\kappa}}\right|_{\lambda,\underline{\gamma}} = \frac{k-1}{2k(1-2\,\nu)}\,\frac{\partial \mathrm{I}_1^\Gamma}{\partial \underline{\kappa}} + \frac{1}{4k}\,\frac{2\left(\frac{k-1}{1-2\nu}\right)^2 \mathrm{I}_1^\Gamma \frac{\partial \mathrm{I}_1^\Gamma}{\partial \underline{\kappa}} + \frac{12\,k}{(1+\nu)^2}\,\frac{\partial \mathrm{J}_2^\Gamma}{\partial \underline{\kappa}}}{\sqrt{\left(\frac{k-1}{1-2\,\nu}\,\mathrm{I}_1^\Gamma\right)^2 + \frac{12\,k}{(1+\nu)^2}\,\mathrm{J}_2^\Gamma}} \tag{38}$$

Taking into account "Eq. (31)", the derivatives of the first invariant are represented by

$$\frac{\partial \mathrm{I}_1^\Gamma}{\partial \underline{\gamma}} = \frac{\partial \mathrm{I}_1^\gamma}{\partial \underline{\gamma}} = \frac{\partial \mathrm{tr}(\underline{\gamma})}{\partial \underline{\gamma}} = \underline{id} \tag{39}$$

$$\frac{\partial \mathrm{I}_1^\Gamma}{\partial \underline{\kappa}} = \frac{\partial \mathrm{I}_1^\kappa}{\partial \underline{\kappa}} = \frac{\partial \mathrm{tr}(\underline{\kappa})}{\partial \underline{\kappa}} = \underline{id} \tag{40}$$

while taking into account "Eq. (32)", the derivatives of the first invariant, for a micro-polar model in a plane state, are represented by

$$\frac{\partial \mathrm{J}_2^\Gamma}{\partial \underline{\gamma}} = \frac{\partial \mathrm{J}_2^\gamma}{\partial \underline{\gamma}} = \frac{1}{2}\,\left(\underline{\gamma}^{dev} + \underline{\gamma}^{devT}\right) \tag{41}$$

$$\frac{\partial \mathrm{J}_2^\Gamma}{\partial \underline{\kappa}} = \frac{\partial \mathrm{J}_2^\kappa}{\partial \underline{\kappa}} = \underline{\kappa}^{dev} \tag{42}$$

5 NUMERICAL SIMULATIONS

In order to illustrate the proposed damage models, together with their implementation in the framework of the software *INSANE*, two different finite element models are analyzed. For the two models, the numerical simulations are performed using both the Mazars-Lemaitre and the De Vree damage models. Both the models are analyzed considering an exponential damage law

$$D(\Gamma^{eq}) = 1 - \frac{k_0}{\Gamma^{eq}}\left(1 - \chi + \chi e^{-\beta(\Gamma^{eq} - k_0)}\right) \tag{43}$$

5.1 Three-point bending test

The three-point bending test of the notched beam analyzed by Petersson (1981) is considered. The geometric characteristics of the beam are represented in "Fig. 1", with measures in millimeters.

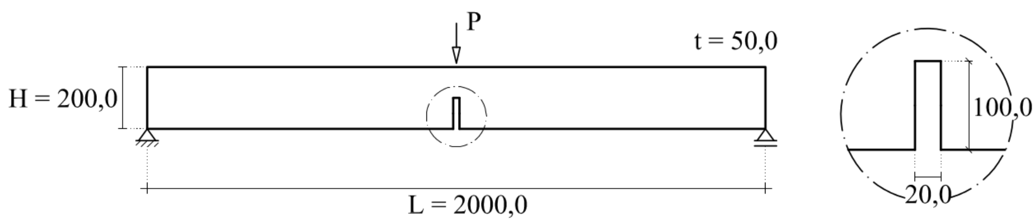


Figure 1: Three-point bending test - Beam geometry

The material is characterized by an elastic modulus E = 30000 N/mm^2, a Poisson's modulus ν = 0.2, a Cosserat's shear modulus α = 2000 N/mm^2 and a bending length L_f = 10 mm. The bending length of the micro-polar model is obtained considering the characteristic length of the material used in the experimental test that, as suggested by the author, is 40 mm. The tensile and compressive uniaxial strengths are assumed to be, respectively, f_t = 3.33 N/mm^2 and f_c = 33.30 N/mm^2, hence with a factor k = 10. The parameters of the damage law ("Eq. (43)") are χ = 0.999 and β = 1000. The threshold value k_0 is calibrated for the two damage models considering the stress state at the zone where the damage initiates and the tensile stress f_t; for the Mazars-Lemaitre model it has been evaluated as 0.000117, while for the De Vree model as 0.000143.

The finite element model ("Fig. 2") is obtained discretizing the beam with 4-nodes quadrilateral elements, in a plane-stress state. The loading process is driven by the *generalized displacement control method* (Yang and Shieh, 1990), assuming a reference load P = 800 N, a loading factor of 0.02 and a tollerance for the convergence in displacement of 1×10^{-4}.

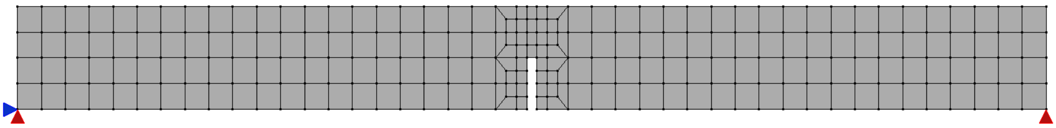


Figure 2: Three-point bending test - Finite elemet model

The results of the analysis are presented, for both the damage models, in "Fig. 3", where the relation between the vertical displacement of the point where the load is applied and the load factor is presented, and in "Fig. 4", where it is illustrated the distribution of the damage for the two models at a vertical displacement of -0.35 mm. As emphasized in "Fig. 4", the Mazars-Lemaitre model does not distinguish between tensile and compressive behavior, leading then to a damage initiation at both the sides of the beam model.

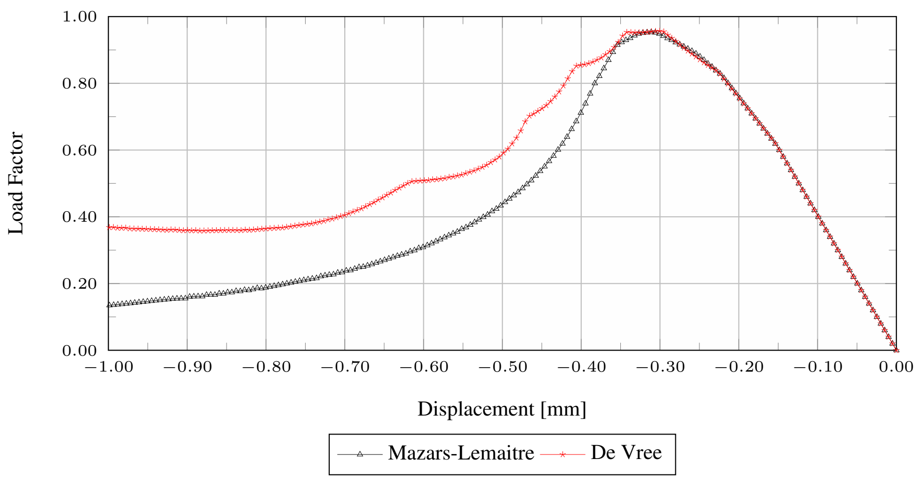
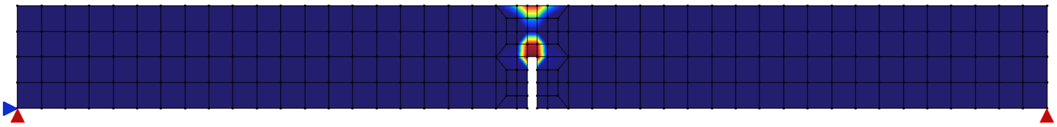


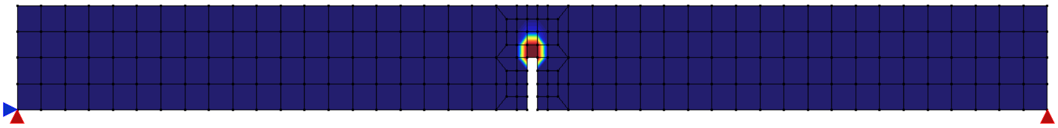
Figure 3: Three-point bending test - Equilibrium paths

5.2 L-shaped panel

The analysis of an L-shaped panel (Winkler et al., 2004) is considered. The geometric characteristics of the panel are illustrated in "Fig. 5", with measures in millimeters.



(a) Mazars-Lemaitre



(b) De Vree

Figure 4: Damage distribution at d$_y$ = -0.35 mm

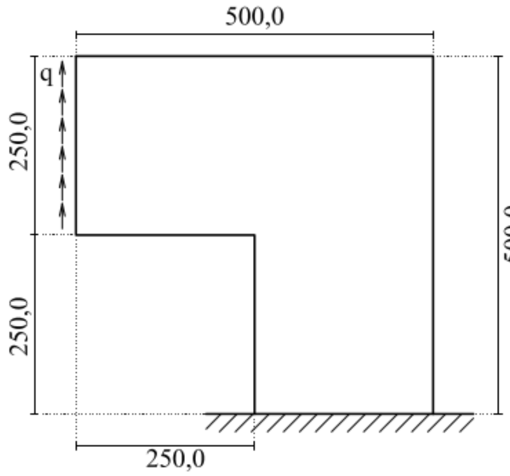


Figure 5: L-shaped panel - Geometry

The material is characterized by an elastic modulus E = 25850 N/mm^2, a Poisson's modulus ν = 0.18, a Cosserat's shear modulus α = 2000 N/mm^2 and a bending length L$_f$ = 5 mm. The bending length of the micro-polar model is obtained considering the characteristic length of the material used in the experimental test that, as suggested by the authors, is 28 mm. The tensile and compressive uniaxial strengths are assumed to be, respectively, f$_t$ = 2.70 N/mm^2 and f$_c$ = 31.00 N/mm^2, hence with a factor k = 11.5. The parameters of the damage law ("Eq. (43)") are χ = 0.999 and β = 1000. The threshold value k$_0$ is calibrated for the two damage models considering the stress state at the zone where the damage initiates and the tensile stress f$_t$; for the Mazars-Lemaitre model it has been evaluated as 0.000115, while for the De Vree model as 0.000161.

The finite element model ("Fig. 6") is obtained discretizing the panel with 3-nodes triangular elements, in a plane-stress state, considering a thickness of 100 mm. The loading process is driven by the *generalized displacement control method* (Yang and Shieh, 1990), assuming a reference load q = 28 N/mm, a loading factor of 0.02 and a tollerance for the convergence in displacement of 1×10^{-4}.

The results of the analysis are presented, for both the damage models, in "Fig. 7", where the relation between the vertical displacement of the top left point and the load factor is presented, and in "Fig. 8", where it is illustrated the distribution of the damage for the two models at

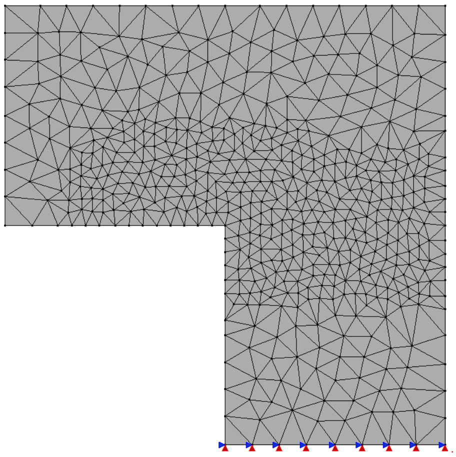


Figure 6: L-shaped panel - Finite elemet model

a vertical displacement of 0.25 mm. As in the case of the three-point bending test, also this

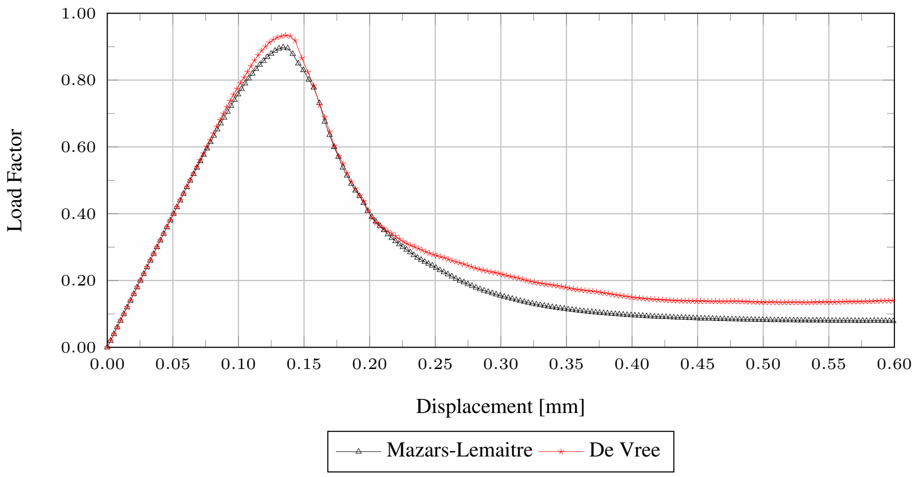
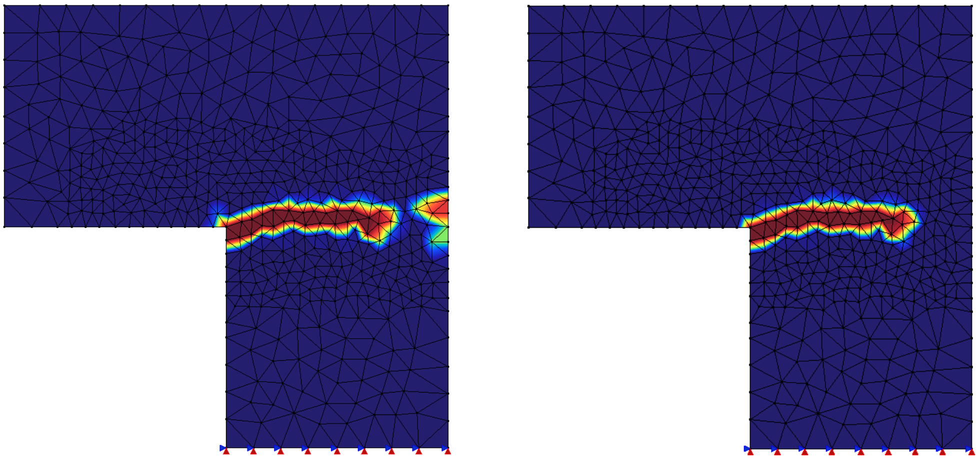


Figure 7: Three-point bending test - Equilibrium paths

simulation shows that the Mazars-Lemaitre model is not able to distinguish between tensile and compressive behavior, leading then to a damage initiation at both the sides of the panel model (“Fig. 8”).

6 CONCLUSIONS

A theoretical and computational framework for isotropic damage models based on a generalized micro-polar continuum theory has been presented. Peculiar attention has been devoted to the introduction of a proper *generalized* formulation of the micro-polar theory; an enhanced tensorial format ables to guarantee the compatibility of the micro-polar model with the unified formulation for constitutive models alredy defined for the classic continuum. Within this generalized formulation, a theoretical basis for isotropic damage models based on the micropolar continuum theory has been proposed. Two specific damage models have been presented, as extensions to the micro-polar theory of the *Mazars-Lemaitre* and *De Vree* models of the classic continnum theory. The proposed framework has been successfully implemented in the



(a) Mazars-Lemaitre (b) De Vree

Figure 8: Damage distribution at d_y = 0.25 mm

software *INSANE*, in a way to take advantage of the existent implementations of the classic continuum model, emphasizing the benefits of the *generalized* formulation. Finally, two numerical simulations have been performed, in order to show the proposed models together with their implementation.

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