LOCKING-FREE HYBRIDIZED FEM FOR INCOMPRESSIBLE LINEAR ELASTICITY PROBLEM

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Abstract. *In this paper a primal hybrid finite element method for the nearly incompressible linear elasticity problem is shown. This method consists of coupling local discontinuous Galerkin problems to the primal variable with a global problem for the Lagrange multiplier, which is identified as the trace of the displacement field. Next, we propose an local post-processing of both stress tensor and displacement field, which allows us to treat the incompressible elasticity problems completely. Numerical studies show that the method is locking free for uniform or distorted meshes and optimal convergence rates are obtained.*

Keywords: *Linear elasticity, Discontinuous Galerkin, Hybridization*

1 INTRODUCTION

Classical displacement based finite element methods for the elasticity problem consider the displacement field as primal variable and compute stress approximations by post-processing. When applied to compressible elasticity problems these classical Galerkin finite element methods lead to good approximations for the displacement fields and a poorer approximations for the stress fields. Additionally, it is well known the inability of displacement based finite element formulations to deal with incompressible or nearly incompressible fields. This nonrobustness of the FEM is usually termed "locking". To avoid locking, several mixed finite element methods in stress and displacement formulations have been developed over the years, such as Arnold and Winther (2002, 2003); Arnold et al. (1984, 2007) Brezzi and Fortin (1991); Franca et al. (1988); Franca and Stenberg (1991), the nonconforming methods proposed in Kouhia and Stenberg (1995), the higher-order methods in Vogelius (1983) and the Discontinuous Galerkin (DG) methods in Rivière and Wheeler (2000); Hansbo and Larson (2002); Wihler (2006); Chen et al. (2010). All these methods have been quite extensively studied within an a priori context.

Discontinuous Galerkin (DG) methods are naturally a suitable alternative for solving linear elasticity problems. Robustness, local conservation and flexibility for implementing h and p-adaptivity strategies are well known advantages of DG methods stemming from the use of finite element spaces consisting of discontinuous piecewise polynomials. A natural connection between DG formulations and hybrid methods have been exploited successfully in many problems (Cockburn et al., 2010, 2008, 2009a,b; Arruda et al., 2013; Faria et al., 2014) and they are still being developed. These hybrid formulations have improved stability, robustness and flexibility of the DG methods with reduced complexity and computational cost. Hybrid finite element methods are characterized by the introduction of new unknowns variables, the Lagrange multipliers, defined an the edges of the elements to weakly impose continuity on the element interfaces. Their reduced complexity and computational cost are obtained by eliminating all degrees of freedom of the primal variables at the element level resulting a global system only with freedom-degrees of the multipliers.

In this paper, we present a primal Stabilized Hybrid DG (SHDG) method, based in Faria et al. (2014, 2013), for the linear elasticity problem considering uniform and distorted meshes. The multiplier, identified with the trace of the displacement field, is interpolated discontinuously. The method is stable for any order of interpolation of the displacement field and the multiplier. Stress approximations are recovered by a local post-processing for both displacement and stress field using the multiplier approximation obtained by the SHDG method. The residual form of the equilibrium equation at the element level is added in this local post-processing leading to optimal rates of convergence of the stress approximations in H(div) norm for compressible and nearly incompressible elasticity problems with uniform and distorted meshes.

The remainder of the paper is organized as follows. The linear elasticity problem is presented in Section 2. The primal SHDG formulation is introduced and discussed in Section 3. In Section 4, numerical strategies for solving the coupled system of linear equations associated with the SHDG formulation and a local post-processing is introduced to recover stress approximations more accurate. Numerical results on convergence studies are presented in Section 5. Concluding remarks are drawn in Section 6.

2 THE MODEL PROBLEM

Let Ω in $\mathbb{R}^2$ an open bounded domain with piecewise Lipschitz boundary $\Gamma_D = \partial\Omega$ of an elastic body subjected to external force $\mathbf{f} \in [L^2(\Omega)]^2$ where Γ_D is the Dirichlet boundary. The kinematical model of linear elasticity problem or the *pure displacement problem* in two dimensions consists in finding a displacement vector field $\mathbf{u}$ satisfying

$$\begin{aligned} -\operatorname{div}\boldsymbol{\sigma}(\mathbf{u}) &= \mathbf{f} && \text{in } \Omega, \\ \boldsymbol{\sigma}(\mathbf{u}) &= \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}) && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{g} && \text{on } \Gamma_D, \end{aligned} \tag{1}$$

where $\boldsymbol{\sigma}(\mathbf{u})$ is the symmetric Cauchy stress tensor, $\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}(\operatorname{grad}\mathbf{u} + \operatorname{grad}\mathbf{u}^T)$ is the linear strain tensor, $\mathbf{g}$ is a given boundary displacement.

For linear, homogeneous and isotropic material $\boldsymbol{\sigma}(\mathbf{u})$ is given by

$$\boldsymbol{\sigma}(\mathbf{u}) = \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}) = 2\mu\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda(\operatorname{tr}\boldsymbol{\varepsilon}(\mathbf{u}))\mathbb{I}, \tag{2}$$

where $\operatorname{tr}\boldsymbol{\varepsilon}(\mathbf{u}) = \operatorname{div}\mathbf{u}$, $\mathbb{I}$ is the identity tensor and λ and μ are called the Lamé parameters which are given in terms of elasticity modulus E and Poisson ratio ν by

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \mu = \frac{E}{2(1+\nu)}.$$

Nearly incompressible materials are modeled by Poisson ratio $\nu \to 1/2$. It is well known that in this limit classical Galerkin FEM approximations lead to volume locking.

3 THE SHDG FORMULATION

In this section we present the Stabilized Hybrid Discontinuous Galerkin (SHDG) formulation for the linear elasticity problem based in Faria et al. (2014). In this primal hybrid formulation the multiplier $\boldsymbol{\lambda}$ is identified as the trace of the displacement field. That is, $\mathbf{u}$: $\boldsymbol{\lambda} = \mathbf{u}|_e$ on each edge $e \in \mathcal{E}_h$.

To construct the primal hybrid formulation we introduce the following broken function space for the displacement field

$$\mathbf{V}_h^k = \{\mathbf{v}_h \in \mathbf{L}^2(\Omega) \ : \ \mathbf{v}_h|_K \in [S_k(K)]^2 \quad \forall K \in \mathcal{T}_h\}, \tag{3}$$

where $S_k(K) = P_k(K)$ (the space of polynomial functions of degree at most k in both variables) or $S_k(K) = Q_k(K)$ (the space of polynomial functions of degree at most k in any variable). The finite dimension space $\mathbf{V}_h^k$ is usually associated with triangular or quadrilateral elements, however the proposed hybrid formulation can be naturally applied to any finite element partition consisting of general polygons.To approximate the multiplier we consider the space

$$\mathbf{M}_h^l = \{\boldsymbol{\lambda} \in \mathbf{L}^2(\mathcal{E}_h) : \boldsymbol{\lambda}|_e = [p_l(e)]^2, \ \forall e \in \mathcal{E}_h^0\}, \tag{4}$$

of polynomial $p_l(e)$, of degree at most l, on each edge e.

In the above defined finite dimension spaces, the SHDG method is formulated as:
Find the pair $[\mathbf{u}_h, \boldsymbol{\lambda}_h] \in \mathbf{V}_h^k \times \mathbf{M}_h^l$ such that, for all $[\mathbf{v}_h, \boldsymbol{\mu}_h] \in \mathbf{V}_h^k \times \mathbf{M}_h^l$

$$\begin{aligned}
&\sum_{K\in\mathcal{T}_h} \int_K \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h) : \boldsymbol{\varepsilon}(\mathbf{v}_h) dx - \sum_{K\in\mathcal{T}_h} \int_{\partial K} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h)\mathbf{n}_K \cdot (\mathbf{v}_h - \boldsymbol{\mu}_h) ds \\
&\quad - \sum_{K\in\mathcal{T}_h} \int_{\partial K} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{v}_h)\mathbf{n}_K \cdot (\mathbf{u}_h - \boldsymbol{\lambda}_h) ds + 2\mu \sum_{K\in\mathcal{T}_h} \beta \int_{\partial K} (\mathbf{u}_h - \boldsymbol{\lambda}_h) \cdot (\mathbf{v}_h - \boldsymbol{\mu}_h) ds \\
&\quad = \sum_{K\in\mathcal{T}_h} \int_K \mathbf{f} \cdot \mathbf{v}_h dx, \qquad (5)
\end{aligned}$$

where the stabilization parameter β depends on the mesh parameter h. Here we have considered the following definition for this stabilization parameter

$$\beta = \frac{\beta_0}{h}, \quad \forall e \in \mathcal{E}_h \quad \text{with} \quad \beta_0 > 0.$$

The influence of β_0 on the stability and accuracy of this primal hybrid formulation is analyzed in Faria et al. (2014) for compressible elasticity problems.

4 COMPUTATIONAL IMPLEMENTATION

Considering that $\mathbf{v}_h$, belonging to the broken function space $\mathbf{V}_h^k$, is defined independently on each element $K \in \mathcal{T}_h$, we observe that equation (5) can be split into a set of local problems defined on each element K coupled to the global problem defined on $\mathcal{E}_h$, as follow:
Local problems: Find $\mathbf{u}_h|_K \in \mathbf{V}_h^k(K) = \mathbf{V}_h^k|_K$, such that, for all $\mathbf{v}_h|_K \in \mathbf{V}_h^k(K)$,

$$\begin{aligned}
&\int_K \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h) : \boldsymbol{\varepsilon}(\mathbf{v}_h) dx - \int_{\partial K} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h)\mathbf{n}_K \cdot \mathbf{v}_h ds - \int_{\partial K} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{v}_h)\mathbf{n}_K \cdot (\mathbf{u}_h - \boldsymbol{\lambda}_h) ds \\
&\quad + 2\mu \int_{\partial K} \beta(\mathbf{u}_h - \boldsymbol{\lambda}_h) \cdot \mathbf{v}_h ds = \int_K \mathbf{f} \cdot \mathbf{v}_h dx, \qquad (6)
\end{aligned}$$

Global Problem: Find $\boldsymbol{\lambda}_h \in \mathbf{M}_h^l$, such that, for all $\boldsymbol{\mu}_h \in \mathbf{M}_h^l$,

$$\sum_{K\in\mathcal{T}_h} \int_{\partial K} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h)\mathbf{n}_K \cdot \boldsymbol{\mu}_h ds - \sum_{K\in\mathcal{T}_h} 2\mu \int_{\partial K} \beta(\mathbf{u}_h - \boldsymbol{\lambda}_h) \cdot \boldsymbol{\mu}_h ds = 0, \ \forall \boldsymbol{\mu}_h \in \mathbf{M}_h^l. \qquad (7)$$

Given that the multiplier of the proposed hybrid formulation is identified with the trace of the primal variable $\mathbf{u}$ on the element edges, for appropriate choices of β, we can always eliminate the degrees-of-freedom of the primal variable $\mathbf{u}_h$ at the element level in favor of the degrees-of-freedom of the multiplier leading to a global system in the multiplier only. Note that we can adopt any (l) order of continuous or discontinuous interpolation functions to Lagrange multiplier $\boldsymbol{\lambda}_h$ independently of the (k) order adopted for the primal variable $\mathbf{u}_h$. Here, we consider only discontinuous interpolations for the Lagrange multiplier. Defining the local bilinear forms:

$$\begin{aligned}
a_K(\mathbf{u}_h, \mathbf{v}_h) \ &:= \ \int_K \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h) : \boldsymbol{\varepsilon}(\mathbf{v}_h) dx - \int_{\partial K} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h)\mathbf{n}_K \cdot \mathbf{v}_h ds - \int_{\partial K} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{v}_h)\mathbf{n}_K \cdot \mathbf{u}_h ds \\
&+ \ 2\mu \int_{\partial K} \beta\, \mathbf{u}_h \cdot \mathbf{v}_h ds \qquad (8) \\
b_K(\boldsymbol{\lambda}_h, \mathbf{v}_h) \ &:= \ \int_{\partial K} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{v}_h)\mathbf{n} \cdot \boldsymbol{\lambda}_h ds - 2\mu \int_{\partial K} \beta \boldsymbol{\lambda}_h \cdot \mathbf{v}_h ds \qquad (9)
\end{aligned}$$

and the linear functional

$$f_K(\mathbf{v}_h) := \int_K \mathbf{f} \cdot \mathbf{v}_h dx, \tag{10}$$

the system (6)-(7) can be presented as:
Find $\mathbf{u}_h|_K \in \mathbf{V}_h^k(K)$, for each $K \in \mathcal{T}_h$, and $\boldsymbol{\lambda}_h \in M_h^l$ such that

$$a_K(\mathbf{u}_h, \mathbf{v}_h) + b_K(\boldsymbol{\lambda}_h, \mathbf{v}_h) = f_K(\mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h^k(K), \tag{11}$$

$$\sum_{K \in \mathcal{T}_h} b_K^T(\mathbf{u}_h, \boldsymbol{\mu}_h) + \sum_{K \in \mathcal{T}_h} c_K(\boldsymbol{\lambda}_h, \boldsymbol{\mu}_h) = \mathbf{0}, \quad \forall \boldsymbol{\mu}_h \in \mathbf{M}_h^l, \tag{12}$$

or in matrix form:

$$\mathbf{A}_K \mathbf{U} + \mathbf{B}_K \boldsymbol{\Lambda} = \mathbf{F}_K, \quad \forall K \in \mathcal{T}_h \tag{13}$$

$$\sum_{K \in \mathcal{T}_h} \mathbf{B}_K^T \mathbf{U} + \sum_{K \in \mathcal{T}_h} \mathbf{C}_K \boldsymbol{\Lambda} = \mathbf{0}. \tag{14}$$

Inverting the local matrix $\mathbf{A}_K$ we have

$$\mathbf{U} = \mathbf{A}_K^{-1}(\mathbf{F}_K - \mathbf{B}_K \boldsymbol{\Lambda}), \quad \forall K \in \mathcal{T}_h. \tag{15}$$

Replacing (15) in (14), we obtain the global system in the multiplier only:

$$\sum_{K \in \mathcal{T}_h} (\mathbf{C}_K - \mathbf{B}_K^T \mathbf{A}_K^{-1} \mathbf{B}_K) \boldsymbol{\Lambda} = \sum_{K \in \mathcal{T}_h} -\mathbf{B}_K^T \mathbf{A}_K^{-1} \mathbf{F}_K. \tag{16}$$

After solving the global system (16), the vector $\mathbf{U}$ is obtained from (15) with $\boldsymbol{\Lambda}$ given by (16).

4.1 Stress and displacement local post-processing

In most engineering applications, stresses are the variables of main interest. Classically, in displacement finite element formulation, stresses are computed indirectly using the displacement approximation and the constitutive equation only. With this classical approach, the stress approximation is given by

$$\boldsymbol{\sigma}_h = \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h) \tag{17}$$

which converges at best with the following rates in $\mathbf{L}^2$ and $\mathbf{H}(\mathrm{div})$ norms:

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{L}^2} = \|\mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}) - \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h)\|_{\mathbf{L}^2} = Ch^k, \tag{18}$$

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{H}(\mathrm{div})} = Ch^{k-1}. \tag{19}$$

As an improved alternative to compute stress approximations, we propose a local post-processing consisting in solving at each element $K \in \mathcal{T}_h$ the local problem:

$$\begin{aligned} -\operatorname{div} \boldsymbol{\sigma}(\mathbf{u}) &= \mathbf{f} && \text{in } K, \\ \mathbb{A}\boldsymbol{\sigma}(\mathbf{u}) &= \boldsymbol{\varepsilon}(\mathbf{u}) && \text{in } K, \\ \mathbf{u} &= \boldsymbol{\lambda}_h && \text{on } \partial K, \end{aligned} \tag{20}$$

in stress and displacement fields, in which $\mathbb{A} = \mathbb{D}^{-1}$ and $\boldsymbol{\lambda}_h$ is given by the solution of the global problem (16).

Stress and displacement approximations $[\boldsymbol{\sigma}_{pp}, \mathbf{u}_{pp}]$ for $[\boldsymbol{\sigma}, \mathbf{u}]$, solution of (20), are obtained in the finite dimension spaces

$$\mathbb{W}_h^k(K) = \{\tau_{i,j} \in S_k(K),\ \ \tau_{i,j} = \tau_{j,i},\ \ i,j = 1,2\} \tag{21}$$

and

$$\mathbf{V}_h^k(K) = \{\mathbf{v}_i \in S_k(K),\ \ i = 1,2\}, \tag{22}$$

respectively, considering the following residual form on each element $K \in \mathcal{T}_h$

$$\begin{aligned}\int_K \mathbb{A}\boldsymbol{\sigma}_{pp} : \boldsymbol{\tau}_h dx + \int_K \mathbf{u}_{pp} \cdot \mathrm{div}\boldsymbol{\tau}_h dx - \int_{\partial K} \boldsymbol{\lambda}_h \cdot \boldsymbol{\tau}_h \mathbf{n}_K ds + \int_K \mathrm{div}\boldsymbol{\sigma}_{pp} \cdot \mathbf{v}_h dx \\ + \int_K \mathbf{f} \cdot \mathbf{v}_h dx + \frac{\delta}{2\mu} \int_K (\mathrm{div}\boldsymbol{\sigma}_{pp} + \mathbf{f}) \cdot \mathrm{div}\boldsymbol{\tau}_h dx + 2\mu \int_{\partial K} \beta(\mathbf{u}_{pp} - \boldsymbol{\lambda}_h) \cdot \mathbf{v}_h ds = 0.\end{aligned} \tag{23}$$

The residual form (23) defines the local problem:

Given $\boldsymbol{\lambda}_h$, find $[\boldsymbol{\sigma}_{pp}|_K, \mathbf{u}_{pp}|_K] \in \mathbb{W}_h^k(K) \times \mathbf{V}_h^k(K)$, such that

$$a_{pp}([\boldsymbol{\sigma}_{pp}, \mathbf{u}_{pp}], [\boldsymbol{\tau}_h, \mathbf{v}_h]) = f_{pp}([\boldsymbol{\tau}_h, \mathbf{v}_h]) \quad \forall\, [\boldsymbol{\tau}_h|_K, \mathbf{v}_h|_K] \in \mathbb{W}_h^k(K) \times \mathbf{V}_h^k(K), \tag{24}$$

with

$$\begin{aligned}a_{pp}([\boldsymbol{\sigma}_{pp}, \mathbf{u}_{pp}], [\boldsymbol{\tau}_h, \mathbf{v}_h]) = \int_K \mathbb{A}\boldsymbol{\sigma}_{pp} : \boldsymbol{\tau}_h dx + \int_K \mathbf{u}_{pp} \cdot \mathrm{div}\boldsymbol{\tau}_h dx + \int_K \mathrm{div}\boldsymbol{\sigma}_{pp} \cdot \mathbf{v}_h dx \\ + \frac{\delta}{2\mu} \int_K \mathrm{div}\boldsymbol{\sigma}_{pp} \cdot \mathrm{div}\boldsymbol{\tau}_h dx + 2\mu \int_{\partial K} \beta\, \mathbf{u}_{pp} \cdot \mathbf{v}_h ds\end{aligned} \tag{25}$$

$$\begin{aligned}f_{pp}([\boldsymbol{\tau}_h, \mathbf{v}_h]) = \int_{\partial K} \boldsymbol{\lambda}_h \cdot \boldsymbol{\tau}_h \mathbf{n}_K ds - \frac{\delta}{2\mu} \int_K \mathbf{f} \cdot \mathrm{div}\boldsymbol{\tau}_h dx - \int_K \mathbf{f} \cdot \mathbf{v}_h dx \\ + 2\mu \int_{\partial K} \beta\, \boldsymbol{\lambda}_h \cdot \mathbf{v}_h ds\end{aligned} \tag{26}$$

For appropriate choices of the stabilization parameters δ, we have observed the following convergence rate for the post-processed stress:

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_{pp}\|_{\mathbf{H}(\mathrm{div})} = Ch^k \tag{27}$$

which is one order higher than that observed for $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{H}(\mathrm{div})}$.

5 NUMERICAL RESULTS

In this section the behavior of the proposed formulation is tested on some numerical experiments, considering discontinuous interpolations for the multiplier. The approximate solutions have been obtained using triangular elements $P_k - p_l$ and quadrilateral elements $Q_k - p_l$, where

k and l denote the degree of Lagrangian polynomial space for the displacement and the Lagrange multiplier, respectively. The post-processed stress approximations are recovered with Lagrangian polynomials of degree k. The stabilized hybrid system of equations Eq.(6-7) is solved using the strategy presented in (15-16). The performance of the method is tested by presenting some results of convergence studies aiming at confirming the rates of convergence for a plane-strain problem, defined on square domain $\Omega = (0,1) \times (0,1)$ with homogeneous boundary conditions, considering the elasticity modulus $E = 1$ and forcing term:

$$f_1(x,y) = (2\nu(2\mu+\lambda) - (\mu+\lambda))\sin(\pi x)\cos(\pi y) \tag{28}$$

$$f_2(x,y) = (2\nu(2\mu+\lambda) - (3\mu+\lambda))\sin(\pi y)\cos(\pi x) \tag{29}$$

such that the exact solution is given by

$$u_1(x,y) = \frac{\nu}{\pi^2}\sin(\pi x)\cos(\pi y) \tag{30}$$

$$u_2(x,y) = \frac{(\nu-1)}{\pi^2}\cos(\pi x)\sin(\pi y). \tag{31}$$

Next, we present the incompressible elasticity problem, using uniform and distorted meshes, comparing the stabilized hybrid approximation (SHDG) $\mathbf{u}_{pp}$, obtained with polynomial degree k and $l = k$ for approximating the displacement and multiplier field, respectively, with the continuous interpolant $\mathbf{u}_I$. Also, we compare the stress approximations σ_{pp} obtained from the local post-processing (24), with σ_h, obtained by the constitutive equation (1).

5.1 h-convergence of displacement

Figures (1) and (2) present h-convergence comparisons of the SHDG approximation $\mathbf{u}_h$ $(l = k)$ with the interpolant $\mathbf{u}_I$ in L^2 norm for uniform and distorted meshes, respectively. In these experiments, we consider the parameters $\beta = 20$, $\nu = 0.49999$ (nearly incompressible) with triangular elements $(P_k - p_l)$ and quadrilateral elements $(Q_k - p_l)$, with $(k = 1,2,3)$. Optimal rates of convergence are observed in $L^2(\Omega)$ norm, with $\mathcal{O}(h^2)$ to $P_1 - p_1$ and $Q_1 - p_1$, $\mathcal{O}(h^3)$ to $P_2 - p_2$ and $Q_2 - p_2$, $\mathcal{O}(h^4)$ to $P_3 - p_3$ and $Q_3 - p_3$. For these parameter choices, all approximations $\mathbf{u}_h$ have accuracy close to the corresponding interpolant.

When $\nu \to 1/2$, in Figures (3) and (4) a h-convergence of the SHDG approximation $\mathbf{u}_h$ $(l = k)$ in L^2 norm for both meshes it is shown. We consider the same problem of the previous experiment and the following values for ν : $0.3,\ 0.49,\ 0.499,\ 0.4999,\ 0.49999$. Optimal convergence rates for all approximations $\mathbf{u}_h$ are observed using triangular and quadrilateral elements, for both uniform and distorted meshes. These results confirm the robustness of the proposed primal hybrid formulation. The accuracy of the approximations are independent of the Poisson ration. We have observed the same accuracies for $\nu = 0.49,\ 0.499,\ 0.4999,\ 0.49999$, which are also very close to $P_2 - p_2$, $P_3 - p_3$, $Q_2 - p_2$, $Q_3 - p_3$ accuracy for $\nu = 0.3$.

5.2 Convergence rates for the stress tensor

Figures (5) and (6) present the convergence rates in $H(div)$ norm for the stress tensor approximations when calculated using the constitutive equation σ_h or by the post-processing technique σ_{pp} presented in Section 4.1, for uniform and distorted mesh, respectively. In this study, we consider the parameters $\beta = 20$, $\nu = 0.49999$ (nearly incompressible) and the local stabilization parameter $\delta = 1$ for all elements $K \in \mathcal{T}_h$. In this study we use triangular elements $P_k - p_l$ and quadrilateral elements $Q_k - p_l$ with $(k = 1, 2, 3)$, adopting same order of interpolations for displacement and stress fields. No change in the behavior of the solution for uniform and distorted mesh has been observed. As expected for $k = 1$, σ_h, calculated from the constitutive equation does not converge in H(div) norm. For $k = 2$ and $k = 3$ converges with order $\mathcal{O}(h^1)$ and $\mathcal{O}(h^2)$, respectively. Improved rates of convergence $\mathcal{O}(h^1)$, for $k = 1$, $\mathcal{O}(h^2)$, for $k = 2$ and $\mathcal{O}(h^3)$, for $k = 3$ are obtained for σ_{pp}, calculated by the proposed post-processing technique.

Figures (7) and (8) present the convergence rate of the SHDG approximation σ_{pp} in $H(div)$ norm for uniform and distorted meshes, respectively, for $\nu : 0.3,\ 0.49,\ 0.499,\ 0.4999,\ 0.49999$. We have considered the same elements and parameters chosen as in the previous experiment. Optimal convergence rates are observed for all approximations σ_{pp}, independent of ν, using triangular and quadrilateral elements. Same accuracies are obtained for $\nu = 0.49,\ 0.499,\ 0.4999$, 0.49999 and very close to $\nu = 0.3$. No significant changes in accuracy and convergence rates of the approximations were observed with uniform or distorted meshes.

6 CONCLUDING REMARKS

We have presented a Stabilized Hybrid Discontinuous Galerkin finite element formulation for incompressible linear elasticity problems, based in Faria et al. (2014). The method was developed from a primal hybrid formulation, with the Lagrange multiplier identified with the trace of the displacement field on the edges of the elements, leading to a set of local problems defined at the element level and a global problem in the multiplier only. The formulation preserves the main properties of the corresponding DG method, but with reduced computational complexity. Numerical experiments confirm the locking free ability of the method for nearly incompressible problems. Accurate approximations and optimal rates of convergence in L_2 norm are obtained for both triangular and quadrilateral elements on uniform and distorted meshes.

A local post-processing based on residual forms of the constitutive and equilibrium equations at the element level, and using the multiplier as a kinematic boundary condition, is presented to recover stress approximations with improved stability, accuracy and robustness. Numerical experiments show that, compared to the standard post-processing based on constitutive law only, it retrieves optimal convergence rates in $H(div)$ norm and improves accuracy, using both triangular and quadrilateral elements. We do not observe significant change in accuracy and convergence rates of displacement and stress approximations when using uniform or distorted meshes.

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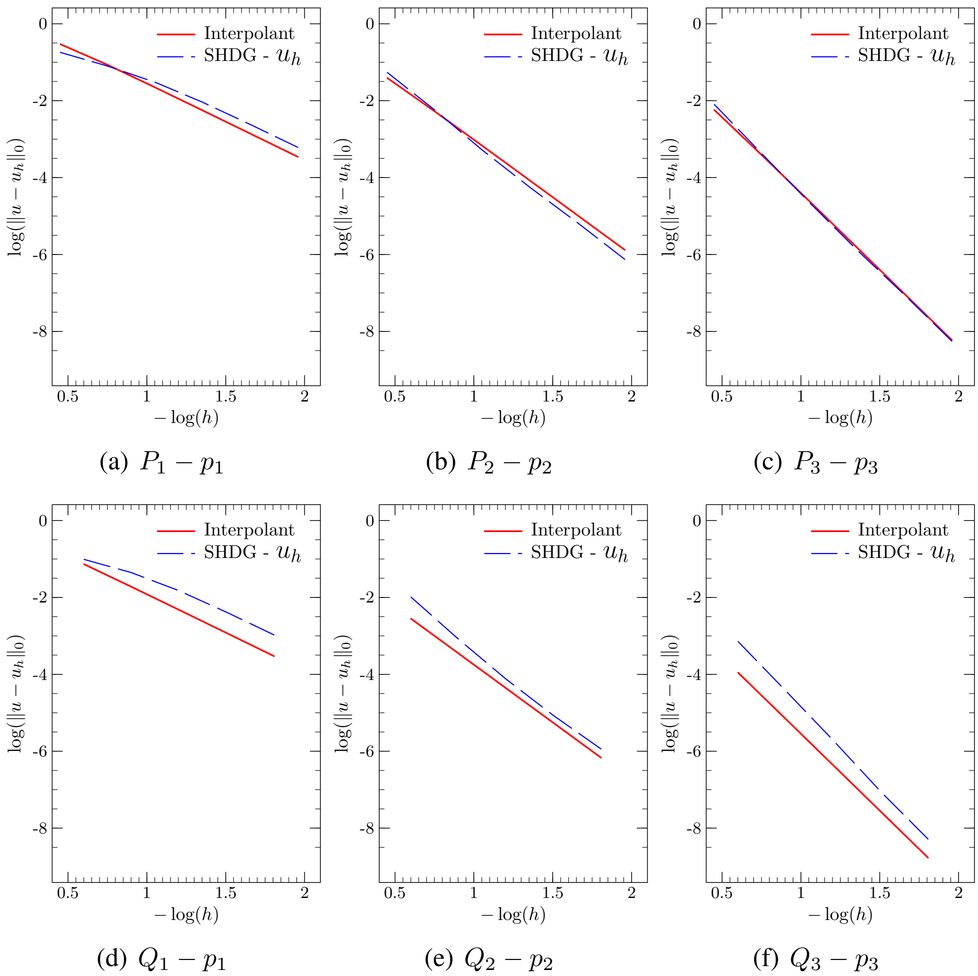
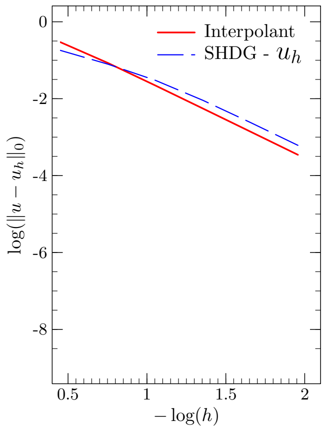
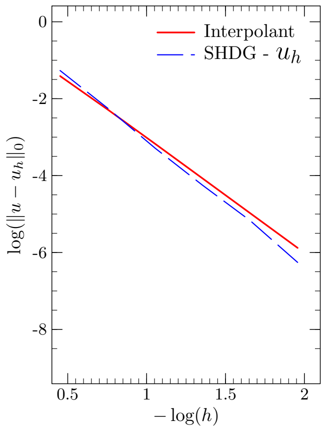


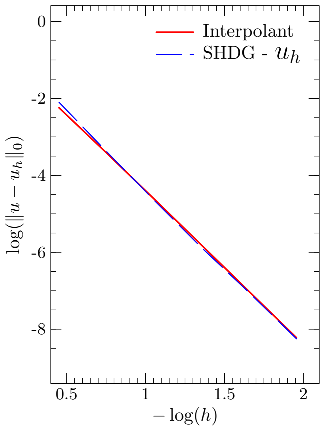
Figure 1: h-convergence study in L_2 norm for u_h of SHDG approximations compared to interpolant with $\beta = 20$, $\nu = 0.49999$ and uniform mesh.



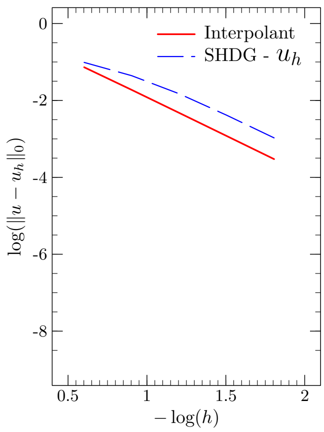
(a) $P_1 - p_1$



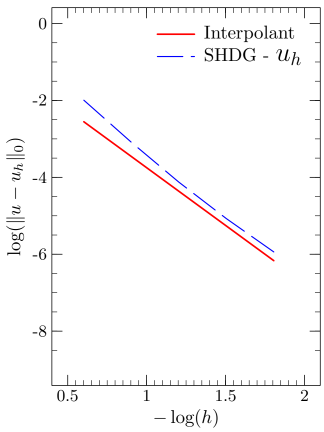
(b) $P_2 - p_2$



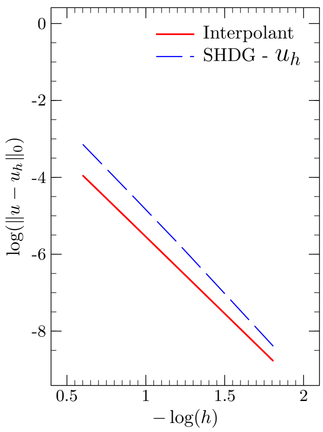
(c) $P_3 - p_3$



(d) $Q_1 - p_1$

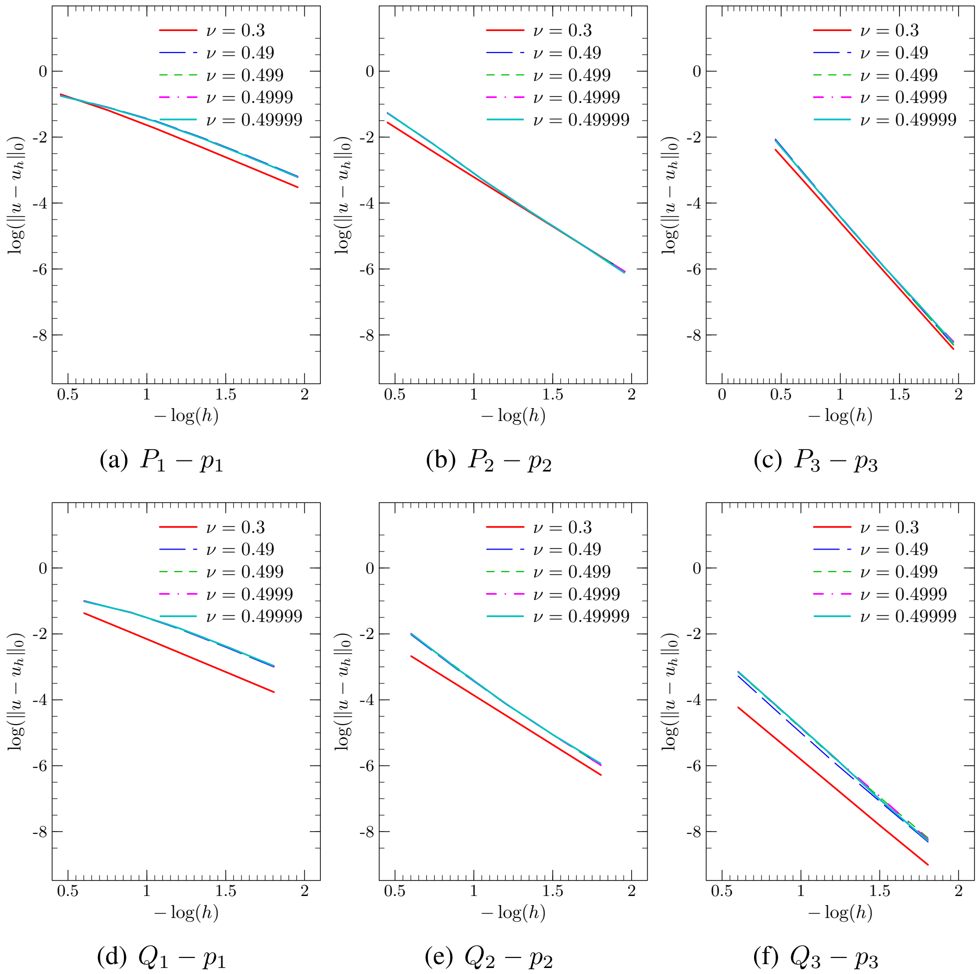


(e) $Q_2 - p_2$



(f) $Q_3 - p_3$

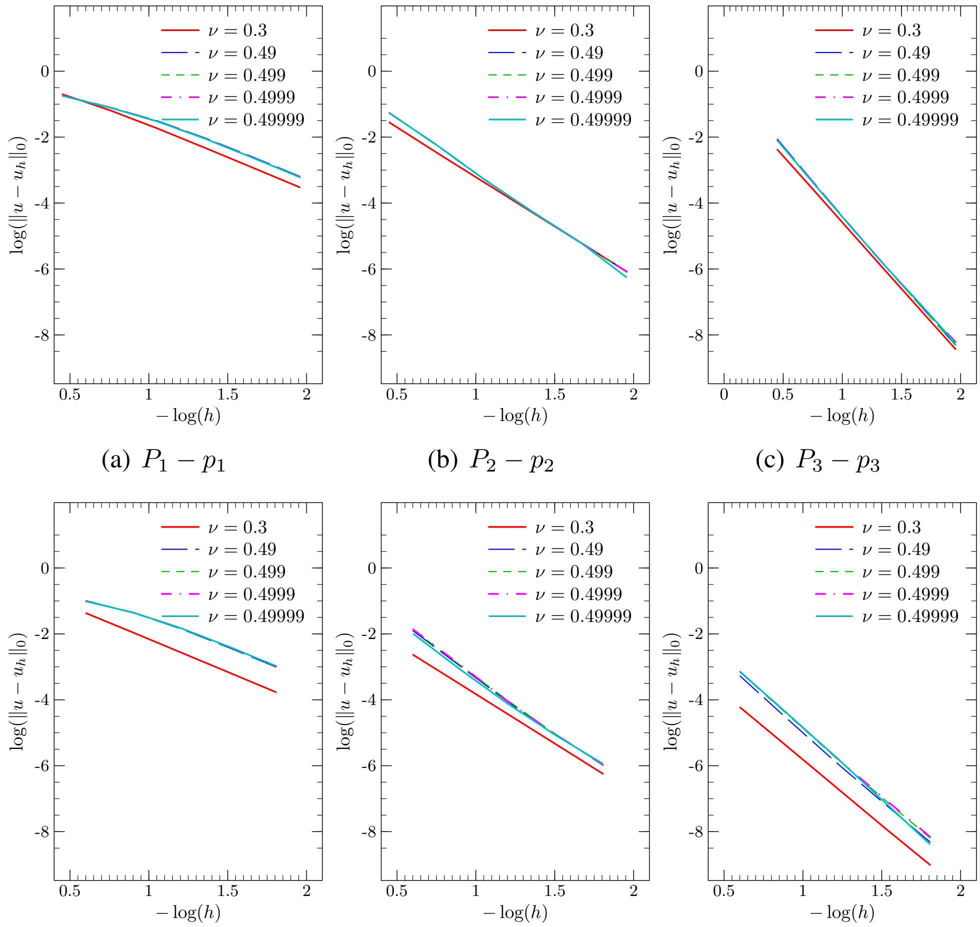
Figure 2: h-convergence study in L_2 norm for u_h of SHDG approximations compared to interpolant with $\beta = 20$, $\nu = 0.49999$ and distorted mesh.



(a) $P_1 - p_1$ (b) $P_2 - p_2$ (c) $P_3 - p_3$

(d) $Q_1 - p_1$ (e) $Q_2 - p_2$ (f) $Q_3 - p_3$

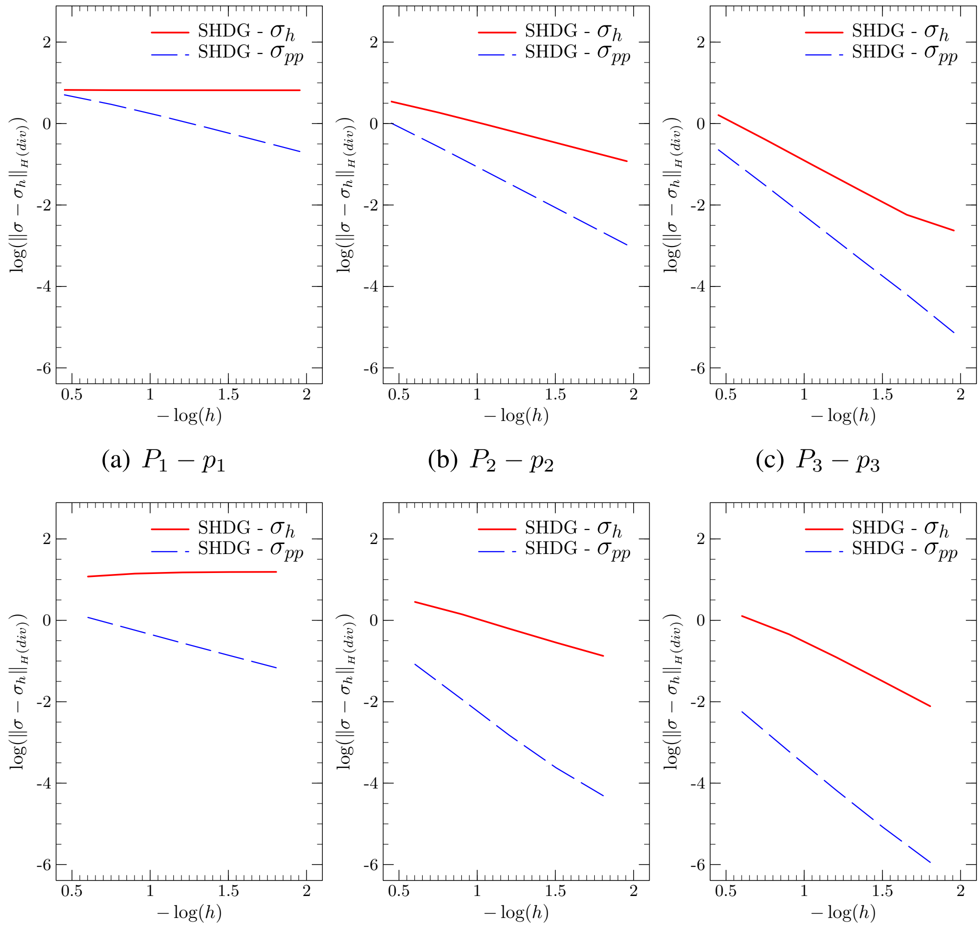
Figure 3: h-convergence study in L_2 norm for u_h of SHDG approximations with $\beta = 20$, $\nu = 0.3; 0.49; 0.499; 0.4999; 0.49999$ and uniform mesh.



(a) $P_1 - p_1$ (b) $P_2 - p_2$ (c) $P_3 - p_3$

(d) $Q_1 - p_1$ (e) $Q_2 - p_2$ (f) $Q_3 - p_3$

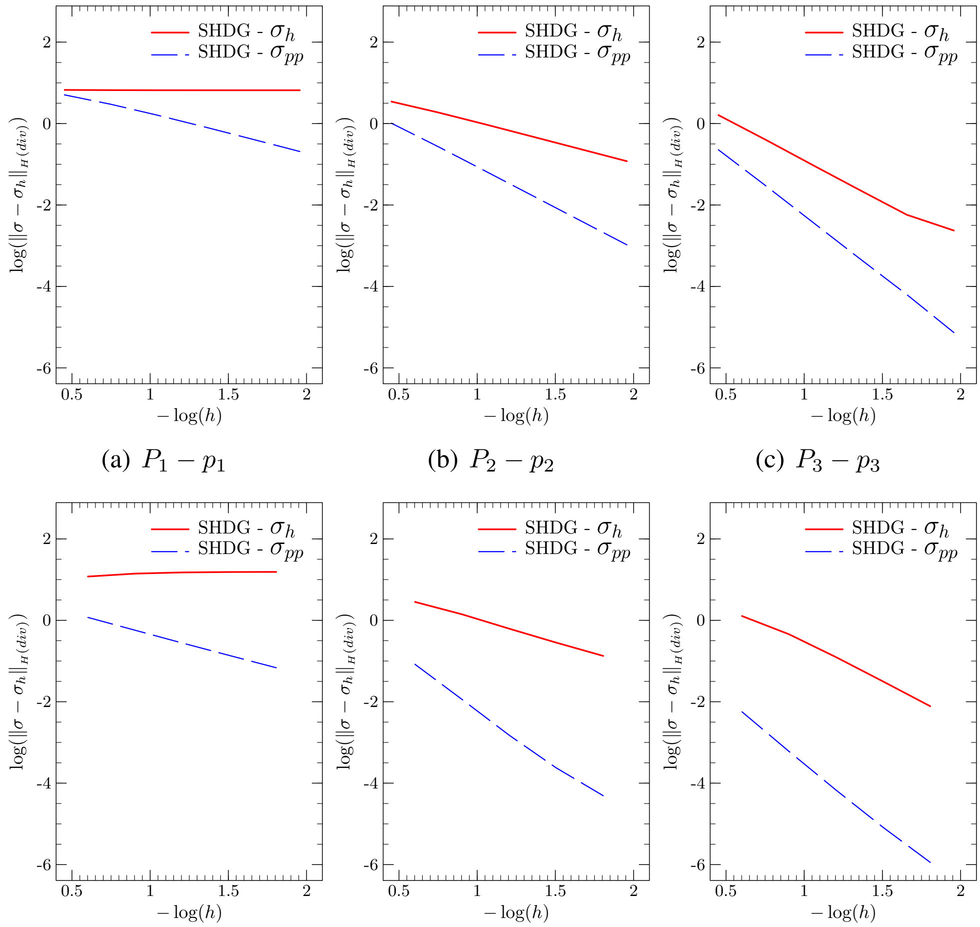
Figure 4: h-convergence study in L_2 norm for u_h of SHDG approximations with $\beta = 20$, $\nu = 0.3; 0.49; 0.499; 0.4999; 0.49999$ and distorted mesh.



(a) $P_1 - p_1$ (b) $P_2 - p_2$ (c) $P_3 - p_3$

(d) $Q_1 - p_1$ (e) $Q_2 - p_2$ (f) $Q_3 - p_3$

Figure 5: Convergence study for σ_h in $H(div)$ norm of SHDG approximation using the constitutive equation and a post-processing technique, with $\nu = 0.49999$ and uniform mesh



(a) $P_1 - p_1$ (b) $P_2 - p_2$ (c) $P_3 - p_3$

(d) $Q_1 - p_1$ (e) $Q_2 - p_2$ (f) $Q_3 - p_3$

Figure 6: Convergence study for σ_h in $H(div)$ norm of SHDG approximation using the constitutive equation and a post-processing technique, with $\nu = 0.49999$ and distorted mesh

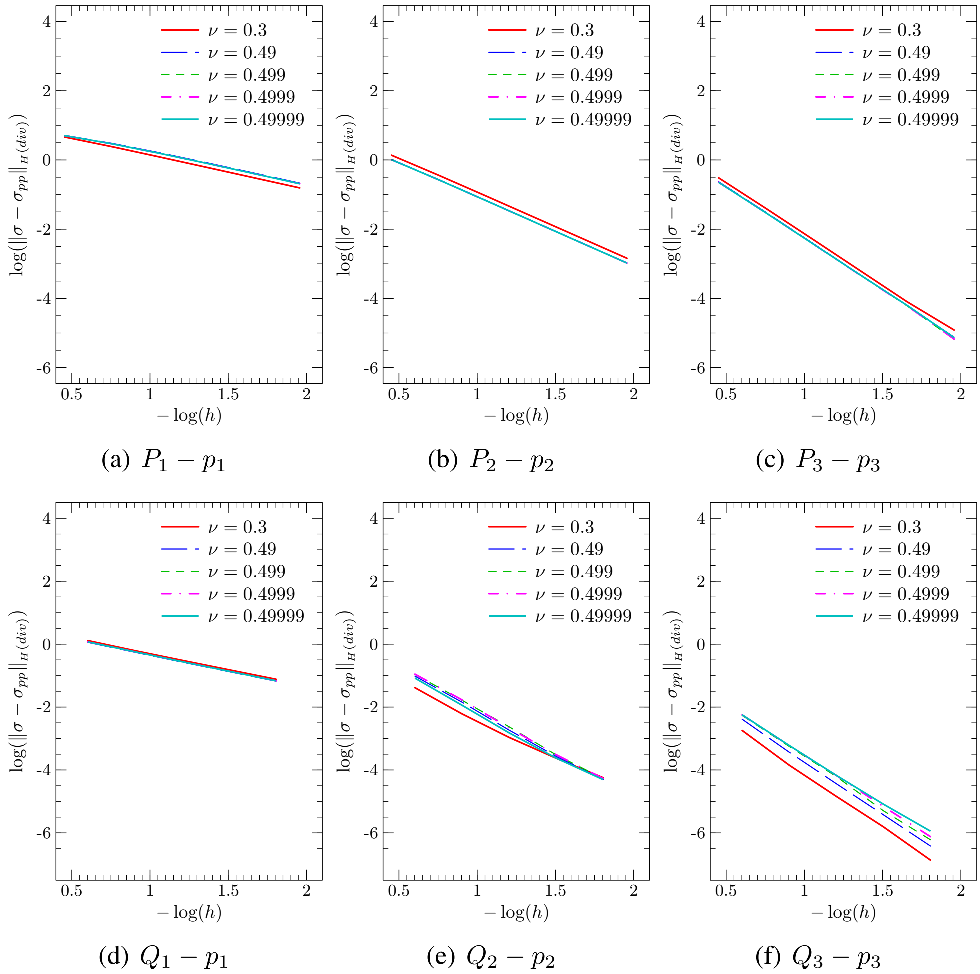
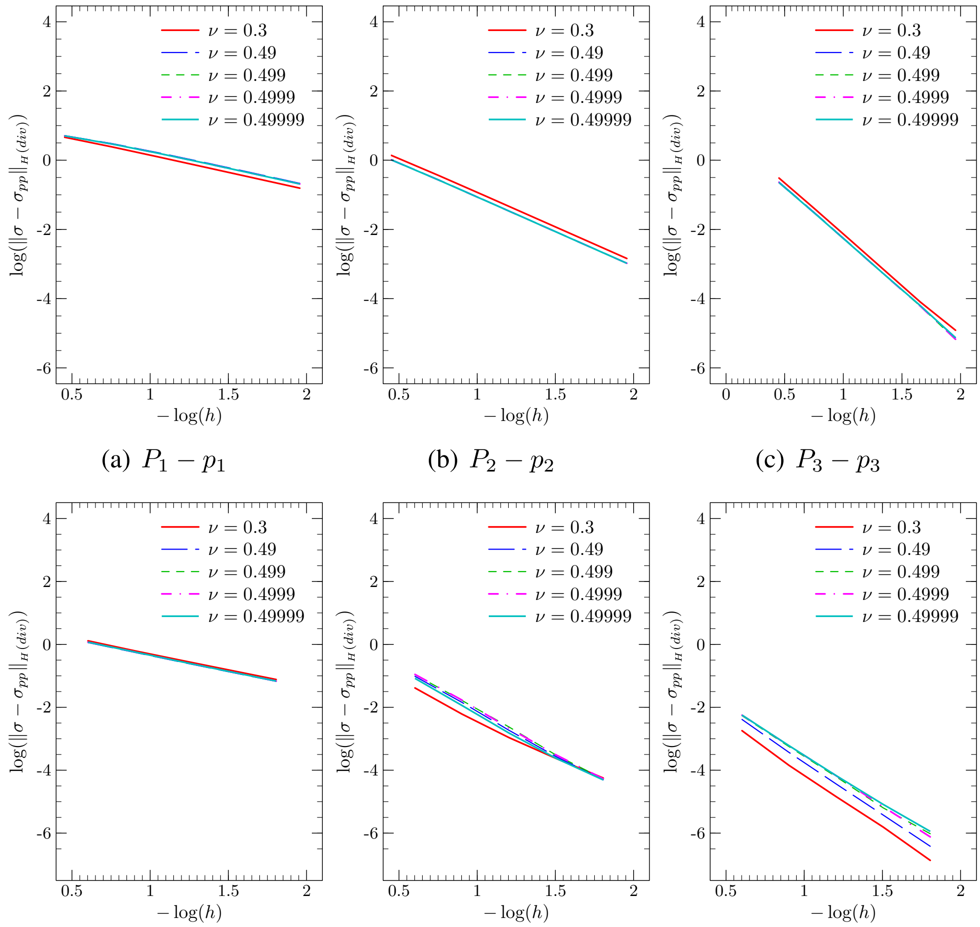


Figure 7: Convergence study for post-processing SHDG approximation σ_{pp} in $H(div)$ norm with $\nu = 0.3; 0.49; 0.499; 0.4999; 0.49999$ and uniform mesh.



(a) $P_1 - p_1$ (b) $P_2 - p_2$ (c) $P_3 - p_3$

(d) $Q_1 - p_1$ (e) $Q_2 - p_2$ (f) $Q_3 - p_3$

Figure 8: Convergence study for post-processing SHDG approximation σ_{pp} in $H(div)$ norm with $\nu = 0.3; 0.49; 0.499; 0.4999; 0.49999$ and distorted mesh.

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