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Abstract. *In linear elastic fracture mechanics, the rate of crack propagation is proportional to the range of stress intensity factors. The most popular model relating these quantities is the Forman law. Crack growth computation is an initial value problem whose solution cannot be obtained in closed form, like stress intensity factors, hence crack growth rates depend on the accumulated growth. Complex geometries numerically evaluate stress intensity factors, and crack growth computations can become computationally intensive. This study presents a theoretical result, which establishes upper and lower bounds for crack size function for any number of cycles. The bounds are very narrow, therefore only two evaluations of stress factors can obtain accurate crack size approximations. This leads to a huge gain in computational effort for numerical crack growth computations. Here we use two examples to explore the accuracy and efficiency of the proposed solution for the initial problem of crack growth.*

Keywords: Fracture mechanics: Crack size; Forman law; Initial value problem.

1. INTRODUCTION

In linear elastic fracture mechanics, the rate of crack propagation (*da/dN*) is proportional to the variation in the intensity factor, (ΔK).

Several mathematical models try to predict the evolution of the size of a crack due to fatigue, such as the Paris-Erdogan, Elber, Huang, Priddle, Collipriest, Forman and others (ZHAN, 2014).

Figure 1 represents the three regions of the crack growth rate as a function of stress variation. Region I refers to the beginning of the crack formation and propagation, which is 10^{-6} mm / cycle order. Grain size, the average stress generated by the applied load and room temperature strongly influence this phase. An important fact of this region is the existence of a numerical value for the stress intensity factor below which cracks do not propagate. This is known as ΔK_{th} and it is determined experimentally. In the region II the plastic zone in front of the crack is long when compared to the grain size. In this region the interface behavior log (*da/dN*) x log (ΔK) is approximately linear. The crack growth rate ranges from 10^{-6} to 10^{-3} mm / cycle. It is a stable crack growth. In the region III we observe very high rates due to rapid crack growth. The curve log (*da/dN*) x log (ΔK) becomes steep approaching its asymptote which is defined by K_t tenacity to fracture. In this region the behavior is predominantly unstable (BEDEN, 2009). The Forman crack propagation equation describes crack growth in the regions II and III (FORMAN, 1967). However, we define the Forman equation as an ordinary differential, non-linear and autonomous equation of first order.

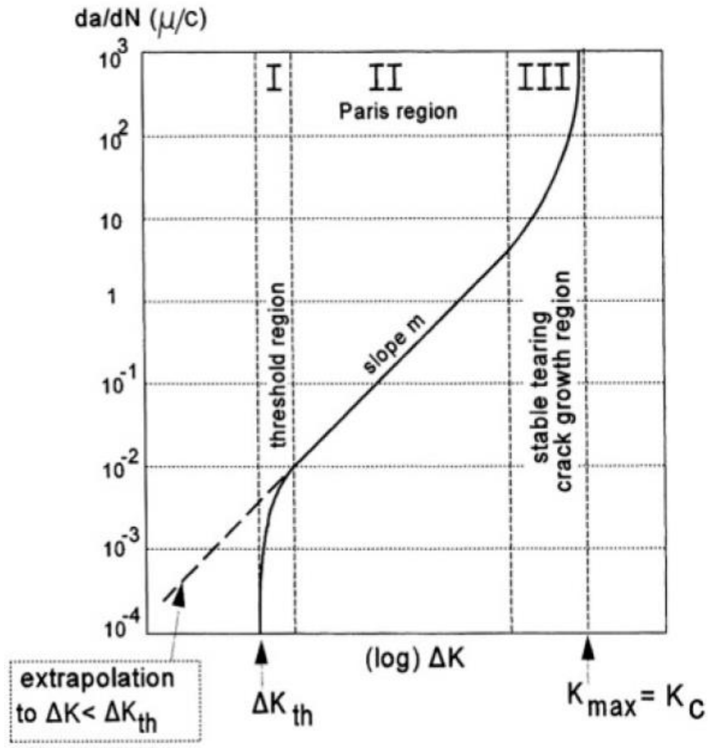


Figure 1 – Three regions of the crack growth rate as a function of ΔK

Source: Schivje (2001)

The models that gave rise to this diagram are the Initial Value Problem (IVP) and they do not have an exact solution. The propagation of a crack is proportional to the range of the stress intensity factor, the function load applied, the geometry of the material and the stress intensity correction factor, $f(.)$.

The objective of this study is to propose functions based on the Forman model to develop upper and lower bounds. The second-order Taylor expansion with Lagrange's rest determine the development of these bounds. The bounds will be compared to the numerical solution using the fourth-order Runge-Kutta method (ASCHER, 1998).

2. FORMAN CRACK PROPAGATING MODEL

Using the Forman crack propagation model (FORMAN, 1967), the crack growth problem can be written as:

$$\begin{cases} Find\ a \in C^1(N_0, N_1)\ such\ that: \\ \dfrac{da}{dN} = \dfrac{C_F(\Delta K)^{m_F}}{(1-R)K_c - \Delta K} \\ a(N_0) = a_0 \end{cases} \tag{1}$$

where C_F and m_F are material constants and N represents number of load cycles. Term $\Delta K(\cdot)$ is the range of stress intensity factors and is defined as:

$$\begin{cases} \Delta K\big(a(N)\big) = (K_{max} - K_{min})\big(a(N)\big) = \sqrt{\pi a(N)} f\big(a(N)\big)(\sigma_{max} - \sigma_{min}) \\ = \sqrt{\pi a(N)} f\big(a(N)\big)\Delta\sigma \end{cases} \tag{2}$$

In Eq. (2), $f(\cdot)$ is the geometry function and $\Delta\sigma$ is the stress range. The problem stated in Eq. (1) is classified as an initial value problem (IVP). Replacing Eq. (2) in Eq. (1), one obtains:

$$\begin{cases} Find\ a \in C^1(N_0, N_1)\ such\ that: \\ \dfrac{da}{dN} = \dfrac{C_F\left(\Delta\sigma\sqrt{\pi a(N)} f\big(a(N)\big)\right)^{m_F}}{(1-R)K_c - \Delta\sigma\sqrt{\pi a(N)} f\big(a(N)\big)} \\ a(N_0) = a_0 \end{cases} \tag{3}$$

Eq. (3) is an IVP, defined by an ordinary, first order, non-linear autonomous differential equation. More generally, Eq. (3) can be classified as a Cauchy problem, which consists in finding the trajectories that satisfy the IVP differential equation and the initial value, $\big(a(N_0) = a_0\big)$. Interval (N_0, N_1) corresponds to load cycle $(\Delta\sigma)$ numbers, which are within region II in Fig. 1.

3. DEDUCTION OF LOWER AND UPPER BOUNDS

Lower and upper bounds for the crack size are determined in this section based on the following hypotheses about the loading and the geometry function:

$$H1: \Delta\sigma(N) = \Delta\sigma_0, \forall N \in [N_0, N_1];$$
$$H2: \begin{cases} f \in C^1([a_0, a_1]; \mathbb{R}^+\backslash\{0\}); \\ 0 < f(a_0) \le f(x) \le f(y), x \le y, \forall x, y \in [a_0, a_1]; \\ f'^{(a_0)} \le f'(x) \le f'(y), x \le y, \forall x, y \in [a_0, a_1]; \end{cases} \tag{4}$$
$$H3: m_f = 1.$$

where $a_0 = a(N_0)$ and $a_1 = a(N_1)$ in H1 assume constant loading. In H2 the correction function of the stress intensity factor and its derivatives are increasingly monotonous, and H3 sets the value for the constant model of the material. These conditions are equal for most common geometry functions, since they represent an intrinsic characteristic of the crack propagation problem. Hypotheses (H1), (H2) and (H3) form the basis for the following theorem, which defines upper and lower bounds for the crack size.

If $f(\cdot)$ and $\Delta\sigma(\cdot)$ are functions that satisfy hypothesis, respectively, and $a^* \in [a_0, a_1]$, the following lower and upper bounds are valid:

$$\begin{cases} a_{LB} = a_0 + \left\{ \begin{array}{c} \dfrac{C_F(\Delta k(a_0))^{m_F}}{(1-R)k_c - \Delta k(a_0)} + \dfrac{1}{2}\left[\dfrac{C_F(\Delta k(a_0))^{m_F}}{(1-R)k_c - \Delta k(a_0)}\right]^2 \\ \left[m_F + \dfrac{1}{\dfrac{(1-R)k_c}{\Delta k(a_0)}} - 1\right]\left[\dfrac{1}{2a} + \left(\dfrac{f'}{f}\right)(a_0)\right](N - N_0) \end{array} \right\}(N - N_0) \\ a_{UB} = a_0 + \left\{ \begin{array}{c} \dfrac{C_F(\Delta k(a_0))^{m_F}}{(1-R)k_c - \Delta k(a_0)} + \dfrac{1}{2}\left[\dfrac{C_F(\Delta k(a^*))^{m_F}}{(1-R)k_c - \Delta k(a^*)}\right]^2 \\ \left[m_F + \dfrac{1}{\dfrac{(1-R)k_c}{\Delta k(a^*)}} - 1\right]\left[\dfrac{1}{2a^*} + \left(\dfrac{f'}{f}\right)(a^*)\right](N - N_0) \end{array} \right\}(N - N_0) \end{cases} \tag{5}$$

4. NUMERICAL EXAMPLES

In order to demonstrate the efficiency of the work, the functions that define the bounds will be used for the two solution examples.

a. Example 1: Finite width plate with center crack:

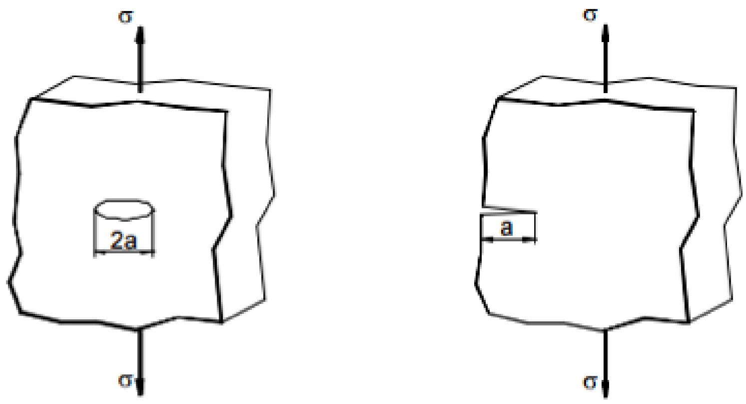
Example 1 represents a finite plate, with width *b*, and a center crack, a_0, as shown in Fig. 2a. For this example, the function of stress intensity factor correction is (BANNANTINE et al., 1989):

$$f(a(N)) = \sqrt{sec\left(\frac{\pi a(N)}{2b}\right)} \tag{6}$$

b. Example 2: Finite plate with a single edge crack:

Example 2 represents a finite plate, with width b, and an edge single crack, a_0, as shown in Fig. 2b. For this example, the function of stress intensity factor correction is (BANNANTINE et al., 1989):

$$f\big(a(N)\big) = 1.122 - 0.231\left(\frac{a(N)}{b}\right) + 10.55\left(\frac{a(N)}{b}\right)^2 - 21.72\left(\frac{a(N)}{b}\right)^3 + 30.39\left(\frac{a(N)}{b}\right)^4 \qquad (7)$$



a) **b)**

Figure 2: Fig. a- example 1 – finite width plate with center crack. Fig. b – example 2 – finite plate with a single edge crack.

For the resolution of the numerical examples, we defined the parameters used in the equation model, as shown Tab. 1. Barsom and Rolf (1999) employed these parameters and are applied to ferritic steels.

Table 1: parameters that adjust the model of crack propagation

Properties – Ferritic Steels				
Model	**C_f (m/ciclo)**	**m_f**	**R**	**K_c (Mpa.m$^{1/2}$)**
Forman	2.10^{-9}	2,9	0	250

The upper bound is a function of the parameter a^*, thus we used the specific value for a^* in this paper, that was $a^*=1.3\ a_0$. After the definition of a^* value, for the resolution of the examples, we used the following data: loading $\Delta\sigma$ = 70 Mpa, number of cycles, N, between [0,900000] cycles, initial crack size a = 1 mm and width b = 100 mm. Therefore, the Figs. 3 and 4 show through graphics *N x a(N)*, for Forman model.

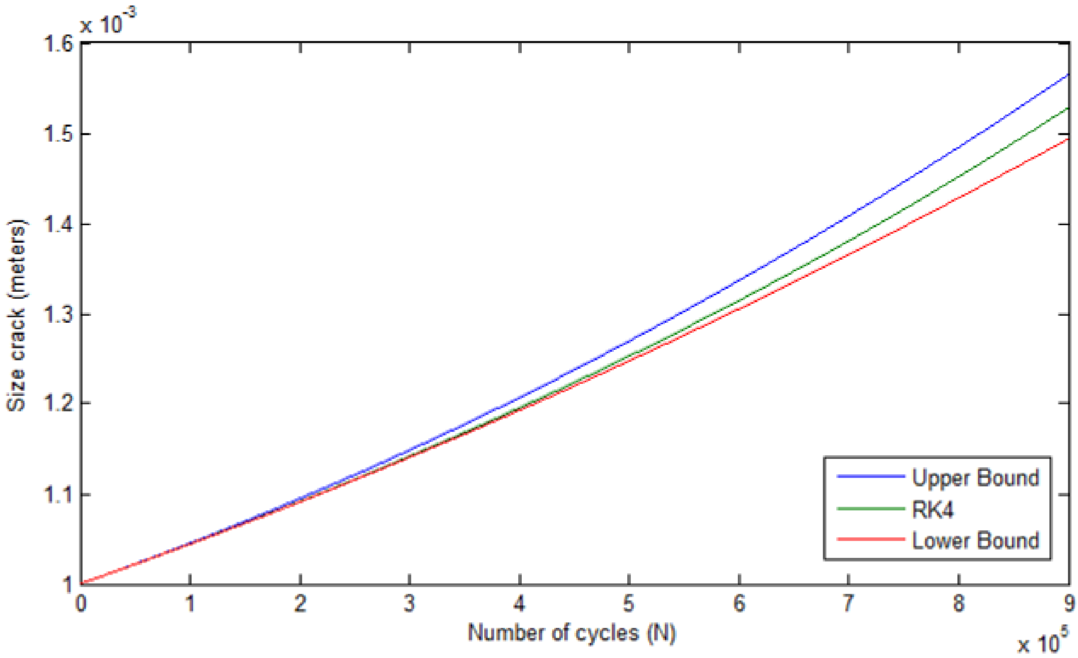


Figure 3: Upper and lower bounds and the approximate numerical solution (RK4), for example 1.

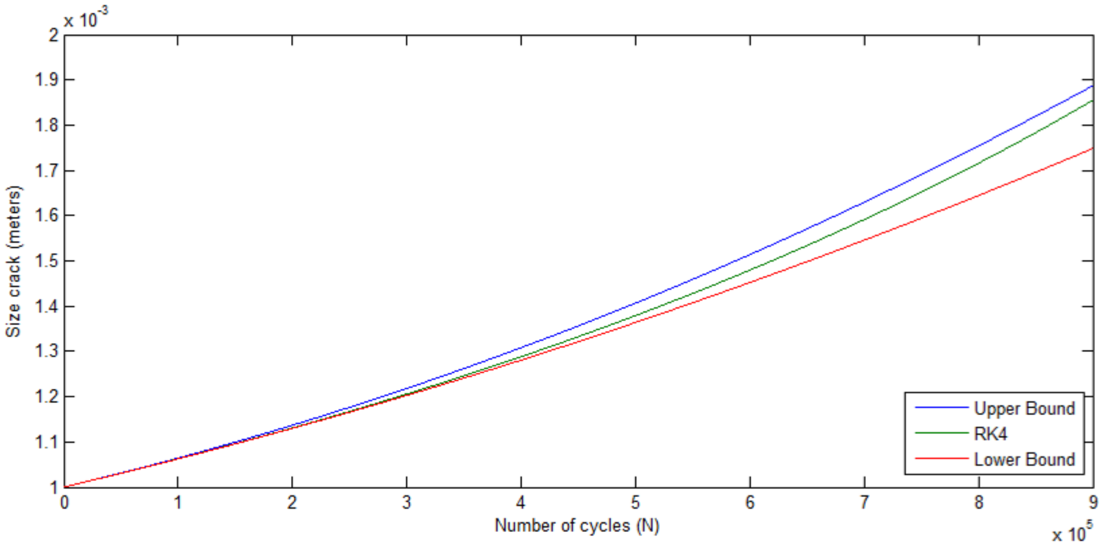


Figure 4: Upper and lower bounds and the approximate numerical solution (RK4), for example 2.

It is possible to verify through the graphs that the upper and lower bounds approached the exact solution, RK4 method. Another thing that is very important is the time that was computationally spent; in table 2 we did the comparison between the methods, and upper/lower bounds are very faster when compared to the RK4 method.

Table 2: Time (in seconds) computationally spent between the two method

Forman Model			
Examples	**RK4 (sec)**	**Upper/Lower Bounds (sec)**	**Efficient (ratio of RK4/Bounds)**
1	3,8079	0,01087	350
2	5,5081	0,01642	335

5. CONCLUSION

This study proposed an efficient and accurate method to analyze the evolution of initial crack, from the Forman model. Two functions for the upper/lower bounds were also proposed, but a set of hypotheses for the model and the use of the second-order Taylor expansion with Lagrange's rest was necessary.

In general, the upper and lower bounds had a shorter time, computationally, when compared to the RK4 method.

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