A FINITE ELEMENT IN EQUILIBRIUM IN A WEAK SENSE FOR THE STATIC THEOREM

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Abstract. *A three-node triangular finite element developed for the static theorem of limit analysis is evaluated considering the von Mises material in plane stress condition. This element satisfies the equilibrium equation and the mechanical boundary condition only in a weak sense. The equilibrium equation, the mechanical boundary condition and the yield criterion, expressed in terms of nodal stresses, are dealt with as constraints of a nonlinear optimization problem whose objective function is defined by the loading. Numerical results of some benchmark tests illustrate that the proposed element can provide reliable lower bound collapse load, and that not satisfying the equilibrium equation and the mechanical boundary condition rigorously is far from being a severe handicap.*

Keywords: *Finite Element, Limit Analysis, Optimization*

1 INTRODUCTION

The limit analysis is a branch of the plasticity analysis which plays an important role in structural design. The static and kinematic theorems, from which the analysis is based, do not consider the evolutive elasto-plastic computations but focus on determining directly the upper or lower bound loads which cause the collapse of the structure (Drucker *et al.*, 1952). Coupling these theorems with the finite element method, as originally proposed by Hayes and Marçal (1967), Lysmer (1970) and Belytschko and Hodge (1970), gives rise to large-scale constrained optimization problems which can be solved by means of linear or nonlinear programming techniques.

The static theorem of limit analysis requires stress fields satisfying the equilibrium equation, the mechanical boundary condition and the yield criterion everywhere (Chen, 1975). The three-node triangular finite element proposed in this paper provides stress fields satisfying the equilibrium equation and the mechanical boundary condition only in a weak sense. The discrete plane stress problem derived is dealt with as a nonlinear programming without violating the von Mises yield criterion. The solution is carried out using MATLAB (2015) and the nonlinear programming solver GAMS/CONOPT (Drud, 1996; Brooke *et al.*, 1998). Numerical tests demonstrate that not satisfying the equilibrium equation and the mechanical boundary condition exactly is far from being a severe handicap for this simple element.

2 STATIC THEOREM

The limit analysis relies on the assumption of an elastic-perfectly plastic material with a flow rule associated to a convex yield surface, and also on small displacement gradients so that the solid does not undergo large deformation at collapse. The lower bound approach follows the static theorem, which requires that the assumed stress field must satisfy the equilibrium equation, the mechanical boundary condition and the yield criterion everywhere. Under these idealized conditions, the computed limit load is a lower bound on the true collapse load (Chen, 1975).

2.1 Equilibrium equation

Let Ω be the region occupied by a solid subjected to the body force $\boldsymbol{b} = \lfloor b_x \quad b_y \rfloor^T$, expressed in an orthogonal Cartesian coordinate system xy used to describe the state of plane stress involved. The stress field $\boldsymbol{\sigma} = \lfloor \sigma_x \quad \sigma_y \quad \tau_{xy} \rfloor^T$ must satisfy the equilibrium equation

$$\boldsymbol{D\sigma} + \boldsymbol{b} = \boldsymbol{0} \tag{1}$$

throughout the domain Ω, where the differential operator

$$\boldsymbol{D} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}. \tag{2}$$

2.2 Mechanical boundary condition

Let Γ be the boundary of the domain Ω, with unit outward normal vector denoted by $\boldsymbol{n} = \lfloor n_x \quad n_y \rfloor^T$. In addition to (1), the stress field must also satisfy the mechanical boundary condition

$$\boldsymbol{t} = \boldsymbol{N\sigma} = \bar{\boldsymbol{t}} \tag{3}$$

on the portion Γ_t of Γ on which the traction (stress vector) $\boldsymbol{t} = \lfloor t_x \quad t_y \rfloor^T$ is prescribed as being $\bar{\boldsymbol{t}}$. Matrix

$$\boldsymbol{N} = \begin{bmatrix} n_x & 0 & n_y \\ 0 & n_y & n_x \end{bmatrix} \tag{4}$$

contains the components of $\boldsymbol{n}$.

2.3 Yield criterion

The stress field should satisfy the yield criterion

$$f(\boldsymbol{\sigma}) \leq 0 \tag{5}$$

everywhere. In a state of plane stress parallel to the xy-plane, the von Mises function is given by

$$f(\boldsymbol{\sigma}) = \left(\sigma_x - \sigma_y\right)^2 + \sigma_x\sigma_y + 3\sigma_{xy}^2 - \sigma_o^2 \tag{6}$$

with σ_o being the yield stress in simple tension.

3 FINITE ELEMENT FORMULATION

Suppose that the solid is divided into a number of elements and treated as an assembly of them. To apply the above equations to a typical element Ω_e shown in Fig. 1, the definition of the boundary should be extended to include the traction continuity on the interelement portion Γ_i:

$$(\mathbf{t})^+ + (\mathbf{t})^- = \mathbf{0} \tag{7}$$

where the superscripts "+" and "–" denote the two sides of Γ_i.

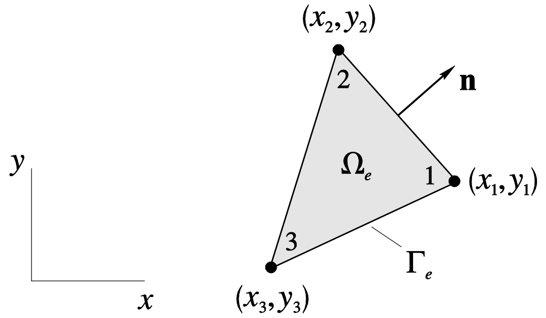

Figure 1. The element Ω_e with the unit normal vector n on its boundary Γ_e.

Equations (1), (3) and (7) can be enforced to be satisfied on average over the element by means of

$$\int_{\Omega_e} \boldsymbol{w}^T (\boldsymbol{D\sigma} + \boldsymbol{b})\, dx\, dy - \int_{\Gamma_t} \boldsymbol{w}^T (\boldsymbol{t} - \bar{\boldsymbol{t}})\, ds - \int_{\Gamma_i} \boldsymbol{w}^T \boldsymbol{t}\, ds = 0, \tag{8}$$

where $\boldsymbol{w} = \lfloor w_x \quad w_y \rfloor^T$ is an arbitrary weight function that is continuous across the element interfaces. The last integral, when considered jointly with those of the neighborhood elements, enforces Eq. (7).

In view of the divergence theorem

$$\int_{\Omega_e} \mathbf{w}^T (\mathbf{D}\boldsymbol{\sigma})\, dx\, dy = \int_{\Gamma} \mathbf{w}^T \mathbf{t}\, ds - \int_{\Omega_e} (\mathbf{D}^T \mathbf{w})^T \boldsymbol{\sigma}\, dx\, dy, \tag{9}$$

Eq. (8) reduces to

$$\int_{\Gamma_u} \mathbf{w}^T \mathbf{t}\, ds + \int_{\Gamma_t} \mathbf{w}^T \bar{\mathbf{t}}\, ds + \int_{\Omega_e} \mathbf{w}^T \mathbf{b}\, dx\, dy - \int_{\Omega_e} (\mathbf{D}^T \mathbf{w})^T \boldsymbol{\sigma}\, dx\, dy = 0. \tag{10}$$

The portion Γ_u of the element boundary Γ_e falls on the solid boundary with prescribed displacement.

The stress field is linearly approximated over the element by

$$\boldsymbol{\sigma} = N_1 \boldsymbol{\sigma}_1 + N_2 \boldsymbol{\sigma}_2 + N_3 \boldsymbol{\sigma}_3 \tag{11}$$

where

$$\boldsymbol{\sigma}_i = \lfloor \sigma_{xi} \quad \sigma_{yi} \quad \tau_{xyi} \rfloor^T \qquad N_i = \frac{1}{2A}(\alpha_i + \beta_i x + \gamma_i y) \qquad i = 1, 2, 3 \tag{12}$$

are the stress nodal values and the shape functions, respectively, with A standing for the triangle area. To evaluate

$$\alpha_i = x_j y_k - x_k y_j \qquad \beta_i = y_j - y_k \qquad \gamma_i = x_k - x_j \tag{13}$$

from the node coordinates, the indices i, j, k should be permuted in a natural order ($i \neq j \neq k$).

The linearly varying weight function

$$\boldsymbol{w} = N_1\boldsymbol{w}_1 + N_2\boldsymbol{w}_2 + N_3\boldsymbol{w}_3 \tag{14}$$

with nodal values

$$\mathbf{w}_i = \lfloor w_{xi} \quad w_{yi} \rfloor^T \qquad i = 1, 2, 3, \tag{15}$$

is the simplest choice to be used in Eq. (10).

Substitution of Eq. (11) and Eq. (14) into Eq. (10) yields

$$\begin{Bmatrix} \boldsymbol{w}_1 \\ \boldsymbol{w}_2 \\ \boldsymbol{w}_3 \end{Bmatrix}^T \left(\boldsymbol{F} - [\boldsymbol{G} \quad \boldsymbol{G} \quad \boldsymbol{G}] \begin{Bmatrix} \boldsymbol{\sigma}_1 \\ \boldsymbol{\sigma}_2 \\ \boldsymbol{\sigma}_3 \end{Bmatrix} \right) = 0, \tag{16}$$

where

$$\boldsymbol{G} = \frac{1}{6} \begin{bmatrix} y_2 - y_3 & 0 & x_3 - x_2 \\ 0 & x_3 - x_2 & y_2 - y_3 \\ y_3 - y_1 & 0 & x_1 - x_3 \\ 0 & x_1 - x_3 & y_3 - y_1 \\ y_1 - y_2 & 0 & x_2 - x_1 \\ 0 & x_2 - x_1 & y_1 - y_2 \end{bmatrix}$$

$$\mathbf{F} = \int_{\Gamma_t} [\boldsymbol{N}_1 \quad \boldsymbol{N}_2 \quad \boldsymbol{N}_3]^T \bar{\boldsymbol{t}}\, ds + \int_{\Omega_e} [\boldsymbol{N}_1 \quad \boldsymbol{N}_2 \quad \boldsymbol{N}_3]^T \boldsymbol{b}\ dx\, dy \tag{17}$$

and

$$\mathbf{N}_i = \begin{bmatrix} N_i & 0 \\ 0 & N_i \end{bmatrix}. \tag{18}$$

Since Eq. (16) holds for any arbitrary weight function, it follows that

$$[\boldsymbol{G} \quad \boldsymbol{G} \quad \boldsymbol{G}] \begin{Bmatrix} \boldsymbol{\sigma}_1 \\ \boldsymbol{\sigma}_2 \\ \boldsymbol{\sigma}_3 \end{Bmatrix} = \boldsymbol{F}. \tag{19}$$

The integral over Γ_u is zero because $\boldsymbol{w}$ will be chosen to be null over there in order to eliminate reaction force. The element enforces the equilibrium equation Eq. (1) and the mechanical boundary condition Eq. (3) to be satisfied by the discrete equation Eq. (19) on average according to Eq. (10).

The conversion of the forces $\bar{\boldsymbol{t}}$ and $\boldsymbol{b}$ into the element nodal force $\boldsymbol{F}$ is identical to that for the well-known constant strain triangular (CST) element (Reddy, 2006). For a constant prescribed traction $\bar{\boldsymbol{t}} = \lfloor \bar{t}_x \quad \bar{t}_y \rfloor^T$ acting on an element edge of length L,

$$\int_{\Gamma_t} [\boldsymbol{N}_1 \quad \boldsymbol{N}_2 \quad \boldsymbol{N}_3]^T \bar{\boldsymbol{t}}\, ds = \frac{L}{2} \lfloor \bar{t}_x \quad \bar{t}_y \quad \bar{t}_x \quad \bar{t}_y \quad 0 \quad 0 \rfloor^T. \tag{20}$$

For a constant body force $\boldsymbol{b} = \lfloor b_x \quad b_y \rfloor^T$,

$$\int_{\Omega_e} [\boldsymbol{N}_1 \quad \boldsymbol{N}_2 \quad \boldsymbol{N}_3]^T \boldsymbol{b}\, dx\, dy = \frac{A}{3} \lfloor b_x \quad b_y \quad b_x \quad b_y \quad b_x \quad b_y \rfloor^T. \tag{21}$$

If $\boldsymbol{w}$ is viewed as the virtual displacement field, $\boldsymbol{D}^T\boldsymbol{w}$ is the virtual strain and Eq. (8) becomes the principle of virtual displacements adopted by Silva *et al.* (1999). The first formulation of the static theorem which satisfies the equilibrium equation and the mechanical boundary condition according to the principle of virtual displacements seems to have been proposed by Anderheggen and Knöpfel (1972). Their formulation was addressed to plate problems.

4 PLASTIC ADMISSIBILITY

It can be shown that the yield criterion $f(\boldsymbol{\sigma}) \leq 0$ is satisfied throughout the element as long as it is satisfied at the element nodes. Fig. 2 depicts the plane defined by the ends of the vectors $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$ and $\boldsymbol{\sigma}_3$ in the stress space.

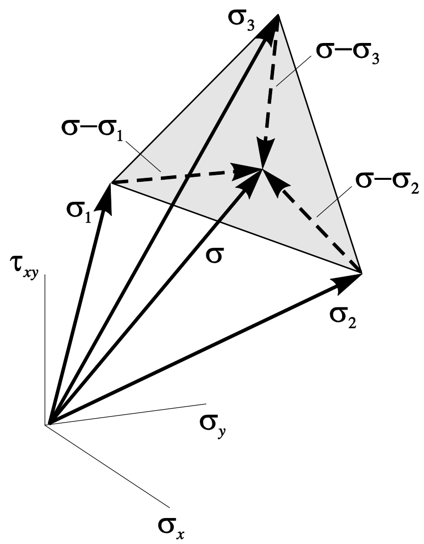

Figure 2. The linear stress approximation (11) makes the end of the vector σ to lie on the plane defined by the ends of the vector σ_1, σ_2 and σ_3 in the stress space.

If the end of $\boldsymbol{\sigma}$ lies on that same plane, the vectors $\boldsymbol{\sigma} - \boldsymbol{\sigma}_1$, $\boldsymbol{\sigma} - \boldsymbol{\sigma}_2$ and $\boldsymbol{\sigma} - \boldsymbol{\sigma}_3$ should be coplanar:

$$(\boldsymbol{\sigma} - \boldsymbol{\sigma}_1) \cdot [(\boldsymbol{\sigma} - \boldsymbol{\sigma}_2) \times (\boldsymbol{\sigma} - \boldsymbol{\sigma}_3)] = 0. \tag{22}$$

In view of the linear stress approximation (11), the condition (22) is always verified because it reduces to

$$\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \times \boldsymbol{\sigma}_3 (N_1 + N_2 + N_3 - 1) = 0 \tag{23}$$

and one knows that

$$N_1 + N_2 + N_3 - 1 = 0. \tag{24}$$

This coplanarity condition assures that satisfying the yield criterion at the element nodes,

$$f(\boldsymbol{\sigma}_1) \leq 0 \qquad f(\boldsymbol{\sigma}_2) \leq 0 \qquad f(\boldsymbol{\sigma}_3) \leq 0, \tag{25}$$

implies satisfying it throughout the element,

$$f(\boldsymbol{\sigma}) \leq 0, \tag{26}$$

because (6) is a convex yield surfaces in the stress space.

5 OPTIMIZATION PROBLEM

In the lower bound analysis, the equilibrium equation, the mechanical boundary condition and the yield criterion, expressed in terms of nodal stresses, are constraints of an optimization problem for the applied load maximization. The optimal solution

$$\lambda^* = \{max\, \lambda \mid \boldsymbol{l}\bar{\boldsymbol{\sigma}} = \lambda \boldsymbol{f}_1 + \boldsymbol{f}_2,\ \boldsymbol{g}(\bar{\boldsymbol{\sigma}}) \leq \mathbf{0}\} \tag{27}$$

identifies the collapse load, where the applied load has been splitted into two parts: $\lambda \boldsymbol{f}_1$ which is adjusted during the optimization by means of the load factor λ and $\boldsymbol{f}_2$ which is kept constant. The design variables are the nodal stresses $\bar{\boldsymbol{\sigma}}$.

The equality constraint

$$\boldsymbol{l}\bar{\boldsymbol{\sigma}} = \lambda \boldsymbol{f}_1 + \boldsymbol{f}_2 \tag{28}$$

arises from the assembly of (19). The inequality constraint

$$\mathbf{g}(\bar{\boldsymbol{\sigma}}) \leq \mathbf{0} \tag{29}$$

is a result of the evaluation at each element node of the yield criterion (6).

The optimization problem solution is carried following the steps: (a) formation of relevant quantities to set up the nonlinear programming problem in MATLAB; (b) production of an appropriate data file for use by a GAMS (General Algebraic Modeling System) model that implements the nonlinear programming formulation (Brooke *et al.*, 1998); (c) solution by one of the GAMS solvers, in case CONOPT (Drud, 1996). GAMS/CONOPT is well suited for models with very nonlinear constraints. This solver leads to very precise and fast results.

6 NUMERICAL EXAMPLE

The well-known reference case of a square plate with a central circular hole, as shown in Fig. 3a, is analyzed. The plate is submitted to a uniform tension of magnitude p at two opposite edges and has a ratio of 0.2 between the diameter of the hole and the side length ($R/L = 0.2$). The exact collapse solution is given by Gaydon and McCrum (1954) as being

$$p_{\lim} = \left(1 - \frac{R}{L}\right)\sigma_0 = 0.8\sigma_0 \tag{30}$$

where σ_0 indicates the yield stress in simple tension. Because we have two axes of symmetry, we need consider only one quadrant in the analysis. The plate was modeled with various meshes defined by parameters n_h and n_l shown in Fig. 3b.

Table 1 summarizes the results obtained for different numbers of elements E and nodes N. Some numbers related to the nonlinear programming problem (namely, degrees of freedom DOF, number of variables $nvar$, and number of constraints $ncon$) and an idea of the computational cost (namely, run time in seconds) are also given. The convergence toward the exact value of p/σ_o can be observed as the mesh is refined.

Table 2 compares our results with some reported lower bound on the limit load. Good agreement is generally observed. In particular, our solutions are very close to those of a node-based smoothed finite element formulation by Nguyen-Xuan *et al.* (2012), even though our model for $n_h = 8$ has only 260 DOF compared with 544 of them.

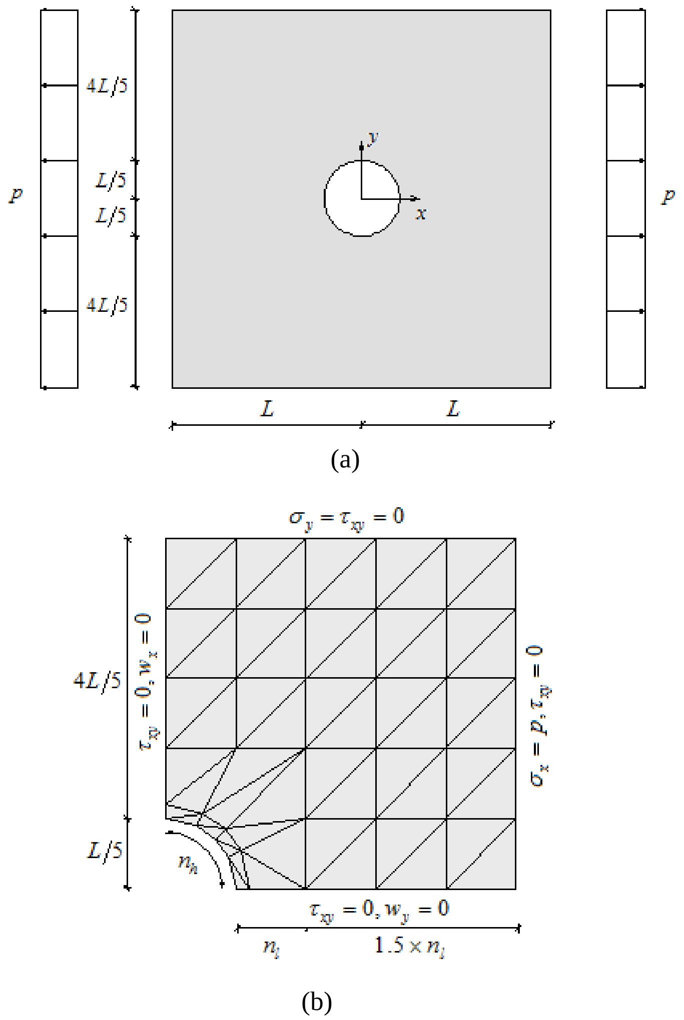

Figure 3. a) Square plate with central hole and loading; b) mesh for upper right quarter of plate with $n_h = 4$ and $n_l = 2$ and boundary conditions.

Table 1. Results for plate with a central hole.

n_h	n_l	DOF	E	N	$nvar$	$ncon$	p/σ_o (error %)	CPU (s)
4	2	72	58	42	127	114	0.810 (1.25)	2.1
8	4	260	232	141	424	401	0.803 (0.38)	8.1

16	8	984	928	513	1540	1497	0.801 (0.13)	120.1

Table 2. Comparison of p/σ_o values.

Reference	p/σ_o
Present work ($n_h = 16$)	0.801
Nguyen-Xuan *et al.* (2012), node-based smoothed finite element	0.802
Tin-Loi and Ngo (2003), p-version finite element	0.803
Belytschko (1972), triangular equilibrium element	0.780
Corradi and Zavelani (1974), triangular constant stress element	0.691
Exact value (Gaydon and McCrum, 1954)	0.800

7 CONCLUSIONS

The finite element here presented does not lead to strict lower bound solution, since the computed stress field does not satisfy all the requirements of the static theorem. This seems irrelevant in view of the element accuracy and low computational cost. It is believed that the combination of the element with a powerful adaptive mesh generator will be an excellent tool for finite element limit analysis.

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