DYNAMIC ANALYSIS OF FOLDED STRUCTURES BY THE BOUNDARY ELEMENT METHOD

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Abstract. *In this paper, a boundary element dynamic formulation is developed for the analysis of structures formed by three-dimensional association of plates. Initially, the boundary element formulations developed for plane elasticity isotropic problems and bending of isotropic thin plate problems are associated in one plane element with four degrees of freedom per node, given by normal, tangential and transverse displacements and normal rotations. Then, the formulation is extended in order to allow the spatial assembling of these structures. The technique of sub-regions is applied to combine macro-elements in three-dimensional space. The final system is obtained by assuming each individual plane structural element as a sub-domain. After the necessary transformation of these equations, they can be combined in a single system by using displacement compatibilities and traction equilibrium conditions. Numerical examples are presented and their results are compared to results available in literature.*

Keywords: *Boundary Element Method, DRM, Transient Analysis, Plates.*

1 Introduction

The Boundary Element Method (BEM) is an advanced numerical computational method developed for the solution of engineering and science problems. Its main characteristic is the boundary only discretization that reduces the modelling in one dimension. Because of this, when compared with the Finite Element Method (FEM) and Sladek, the size of influence matrices is very reduced. However, different from FEM where only polynomials are integrated, in BEM, integration is not a trivial task. Some techniques have been proposed to integrate the influence terms of the BEM, especially when the integral is singular or near-singular. The aim of numerical techniques used to integrate functions is always to obtain a value that is very near the exact or analytical value of the integral. In fact, analytical integration has been purposed in literature for 2D and 3D problems.

The boundary element method (BEM) has been applied successfully to solve the thin plate bending problem since late 1970s and early 1980s. Many researchers derived the direct boundary integral equation (BIE) for both linear and nonlinear responses have been analysed by [Stern(1979)], [Morjaria and Mukherjee(1980)], [Tanaka(1982)], [Kamiya and Sawaki(1982)], [Ye and Liu(1985)]. Using the Rayleigh-Green identity and the fundamental solution, the biharmonic governing equation of Kirchhoff thin plate theory can be transformed into a direct BIE where there are four boundary variables, that is, the deflection, rotation, bending moment, and Kirchhoff equivalent shear force. One major problem in plate bending formulation, where governing equations is a system of elliptic partial differential equations of fourth order, is the many hypersingular integrals within. The use of Gauss quadrature is not suitable to compute non-polynomial functions, especially for near, weak or strong singular integrals. Serious problems arises when calculating results at domain points near the boundary or when computing any results at very slender structures. Generally, two of them are given from the boundary conditions (BCs) and the other two are to be determined. Therefore, two BIEs are required for the thin plate bending problem: the displacement (deflection) BIE and the rotation (normal derivative) BIE. The first BIE is strongly singular, while the second is hypersingular.

[Brebbia(1992)] evaluated analytically the integral to compute the $\mathbf{G}$ matrix terms based on the two-dimensional constant and continuous linear elements with the source point located at the element, i.e., singular integrals. [Almeida and Pina(1993)] also developed analytical integration formulas to calculate the $\mathbf{G}$ matrix terms based on constant and linear elements with the source point on-element, for the two-dimensional Laplace equation. [Banerjee(1994)] calculated analytically the terms for the $\mathbf{G}$ matrix considering constant elements in BEM to solve the two-dimensional Laplace equation. [Fratantonio and Rencis(2000)], present the analytical solutions of the boundary element integral coefficients for the $\mathbf{H}$ and $\mathbf{G}$ influence matrices. [Ghadimi et al.(2010)Ghadimi, Dashtimanesh, and Hosseinzadeh] solve the two-dimensional Poisson equation by BEM based on Galerkin vector method, where the integrals that appear in BEM are computed analytically for constant and linear elements.

The development of BEM for dynamic problems has been a subject of research since the 1970s. In the early 1980s, there was a major development in the BEM known as the dual reciprocity method (DRM) developed by [Nardini and Brebbia(1982)]. The DRM has been used to solve potential, fluid mechanics and heat transfer problems, but there has been very few publications applying the method to dynamic problems. Among these publications are the work of [Bridges and Wrobel(1994)] where spline functions is used to model free vibration problems. [J. P. Agnantiaris(1996)] compared the behaviour of polynomial RBF against thin plate splines. [Rashed(2002)] extended the DRM to dynamics using Gaussian functions. [J. P. Agnantiaris(2001)] recently used the same formulation with polynomial and multi-quadratic functions 3D and axisymmetric structures. The formulation also applied to non-linear applications by [J. C. F. Telles(2001)] and [H. B. Coda(2000)]. Extension of the formulation to the Galerkin-type collocation is studied by [J. J. Perez-Gavilan(2000)]. Additional work for soil-structure interaction and fracture mechanics can be found in the textbook of [Dominguez(1993)].

The RIM approximates the body forces b as a sum of M products of approximation functions f^m and coefficients to be determined γ^m. Then writing another integral representation of these body forces. Both the original and the body force integral formulation use Green reciprocity theory; therefore the formulation is said to be dual reciprocity formulation.

This paper presents a dynamic formulation of the boundary element method to calculate the displacement of isotropic thin plates. The purpose is to reduce the processing time to solve plate bending problems as well as to avoid drawbacks that come from near singular integrals. The results obtained by numerical integration are compared to results obtained with the Sladek and FEM.

2 Theory of bending of thin plates

A plate is a structural element defined by two flat parallel surfaces (Fig. 1), where loads are applied transversely. The distance between these two surfaces defines the thickness of the plate, which is small compared with the its other dimensions. Depending on its material properties, a plate can be either anisotropic, with different properties in different directions, or isotropic, with equal properties in all directions. Depending on its thickness, a plate can be considered as either a thin or a thick plate. In this work, formulations will be developed for isotropic thin plates.

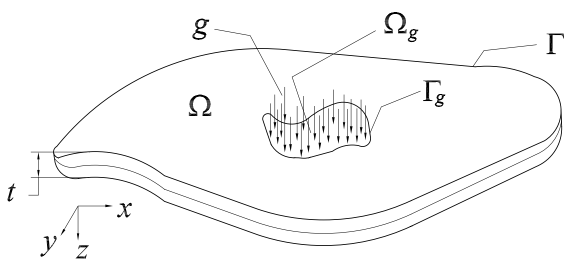

Figure 1: Thin plate.

The theory of bending of thin plates is based on the following assumptions (Lekhnitskii, 1968):

1. Straight sections which are normal to the middle surface in the undeformed state remain straight and normal to the deformed middle surface after loading.
2. The normal stresses σ_z in cross-sections parallel to the middle plane are small compared with the stresses in the transverse cross-section, σ_x, σ_y, and τ_{xy}.

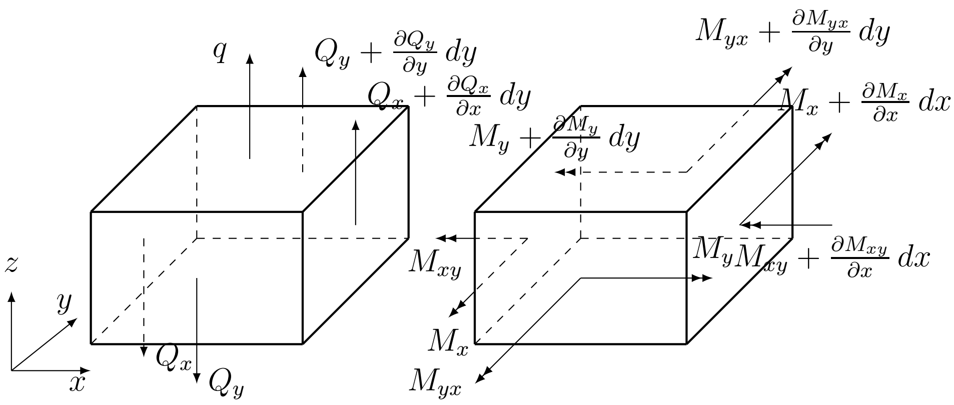

Figure 2: In-plane normal forces, bending moments, shear force, twisting moment and out of plane shearing forces.

3 Governing equation for deflection of plates

Consider equilibrium of an element $dx \times dy$ of the plate subject to a vertical distributed load of intensity $q(x, y)$ applied to an upper surface of the plate, as shown in Fig. 2. For the system of forces and moments shown in Fig. 2, the following three independent conditions of equilibrium may be set up:

1. The force summation in z axis gives

$$\frac{\partial Q_x}{\partial x} dxdy + \frac{\partial Q_y}{\partial y} dxdy + qdxdy = 0, \tag{1}$$

   from which

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0, \tag{2}$$

2. The moment summation about x axis leads to

$$\frac{\partial M_{xy}}{\partial x} dxdy + \frac{\partial M_y}{\partial y} dxdy - Q_y dxdy = 0, \tag{3}$$

   or

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0, \tag{4}$$

   Note that products of infinitesimal terms, have been omit in Eq. 4.

3. The moment summation about y axis results in

$$\frac{\partial M_{yx}}{\partial y} + \frac{\partial M_x}{\partial x} - Q_x = 0. \tag{5}$$

It follows from Eq. 4 and Eq. 5, that shear forces Q_x and Q_y can be expressed in terms of the moments, as follows:

$$\begin{aligned} Q_x &= \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}, \\ Q_y &= \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y}. \end{aligned} \tag{6}$$

4 Basic relations for isotropic plates

Considering a plate following the assumptions defined in the previous section. Knowing the moments M_x, M_y and M_{xy} are given by

$$
\begin{aligned}
M_x &= -D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right), \\
M_y &= -D\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right), \\
M_{xy} &= D(1-\nu)\left(\frac{\partial^2 w}{\partial x^2\partial y^2}\right). \qquad (7)
\end{aligned}
$$

where

$$D=\left(\frac{Et^3}{12(1-\nu^2)}\right) \qquad (8)$$

is the flexural rigidity of the plate, E elastic modulus and w denotes small deflections of the plate in the z direction. As shown by [Fernandes(1974)], derivative of this equations are given by

$$
\begin{aligned}
\frac{\partial M_x}{\partial x} &= -D\left(\frac{\partial^3 w}{\partial x^3}+\nu\frac{\partial^3 w}{\partial x\partial y^2}\right), \qquad (9)\\
\frac{\partial M_y}{\partial y} &= -D\left(\frac{\partial^3 w}{\partial y^3}+\nu\frac{\partial^3 w}{\partial x^2\partial y}\right), \qquad (10)\\
\frac{\partial M_{xy}}{\partial x} &= D(1-\nu)\left(\frac{\partial^3 w}{\partial x^2\partial y}\right), \qquad (11)\\
\frac{\partial M_{xy}}{\partial y} &= D(1-\nu)\left(\frac{\partial^3 w}{\partial x\partial y^2}\right). \qquad (12)
\end{aligned}
$$

Substituting Eq. 9 to Eq. 12 in Eq. 4 and Eq. 5, Q_x and Q_y are obtained as a function of displacement w, given by

$$
\begin{aligned}
Q_x &= -D\left(\frac{\partial^3 w}{\partial x^3}+\frac{\partial^3 w}{\partial x\partial y^2}\right), \\
Q_y &= -D\left(\frac{\partial^3 w}{\partial y^3}+\frac{\partial^3 w}{\partial x^2\partial y}\right), \qquad (13)
\end{aligned}
$$

Into account that $M_{xy}=M_{yx}$. Substituting Eqs. 13 into Eq. 2, one finds the following:

$$\frac{\partial^2 M_x}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_y}{\partial y^2}=-q(x,y). \qquad (14)$$

Finally, introduction of the expressions for M_x, M_y and M_{xy} from Eq. 7 into Eq. 14 yields

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}=\frac{q}{D}. \qquad (15)$$

This is the *governing differential equation for the deflections* of thin plate bending analysis based on Kirchhoff assumptions. This equation was obtained by Lagrange in 1811. Mathematically, the differential equation 15 can be classified as a linear partial differential equation of the fourth order having constant coefficients [12-13].

Equation 15 may be rewritten, as follows:

$$\bigtriangledown^2(\bigtriangledown^2 w) = \bigtriangledown^4 w = \frac{q}{D}, \tag{16}$$

where

$$\bigtriangledown^4(\,) = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}. \tag{17}$$

is commonly called the *biharmonic operator*.

5 Calculation of flexural rigidity in arbitrary directions

Consider a vertical section of the plate whose normal n makes an angle α with the x axis Fig. 3. Take the origin of the Cartesian coordinate system n (outward normal to the edge) and t (tangent to the edge) at some point of the edge. In a general case, the normal bending M_n and twisting M_{nt} moments, as well the transverse shear force Q_n will act at this section. The above moments at any point of the skew edge are

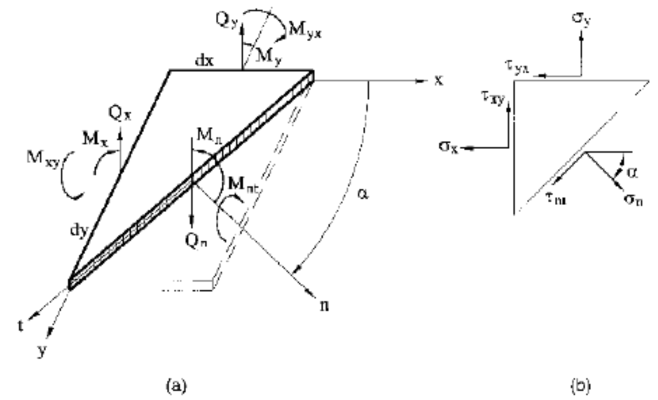

Figure 3: Element of plate subjected to moments in arbitrary directions.

$$M_n = \int_{-h/2}^{h/2} \sigma_n z dz, \quad M_{nt} = \int_{-h/2}^{h/2} \tau_{nt} z dz \tag{18}$$

The relationship between the stress components shown in Fig. 3 is given by

$$\begin{aligned} \sigma_n &= \sigma_x \cos^2\alpha + \sigma_y \sin^2\alpha + 2\tau_{xy}\cos\alpha\sin\alpha, \\ \tau_{nt} &= \tau_{xy}\cos 2\alpha - (\sigma_x - \sigma_y)\sin\alpha\cos\alpha. \end{aligned} \tag{19}$$

Substituting the Eq. 19 into Eq. 18, integrating over the plate thickness and taking into account that

$$\left\{ \begin{array}{c} M_x \\ M_y \\ M_{xy} \end{array} \right\} = \int_{-h/2}^{h/2} \left\{ \begin{array}{c} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{array} \right\} z dz, \tag{20}$$

one obtains the following relationship

$$\begin{aligned} M_n &= M_x \cos^2 \alpha + M_y \sin^2 \alpha + M_{xy} \sin 2\alpha, \\ M_{nt} &= M_{xy} \cos 2\alpha - (M_x - M_y) \sin \alpha \cos \alpha. \end{aligned} \tag{21}$$

The shear force Q_n acting at a section of the edge having a normal n can be evaluated from condition of equilibrium of the plate element, as follows:

$$\begin{aligned} Q_n ds - Q_x dy - Q_y dx &= 0, \\ Q_n &= Q_x \cos \alpha + Q_y \sin \alpha \end{aligned} \tag{22}$$

The effective shear force at a point of the skew edge can be determined as

$$V_n = Q_n + \frac{\partial M_{nt}}{\partial t} \tag{23}$$

where $\frac{\partial}{\partial t}$ denotes a differentiation with respect to the tangent of the skew edge.

6 Boundary integral equation for bending problems of isotropic plates

The equilibrium Eq. 17 can be transformed into a boundary integral equation given by

$$\begin{aligned} & Kw(Q) + \int_\Gamma \left[V_n^*(Q,P) w(P) - m_n^*(Q,P) \frac{\partial w(P)}{\partial n} \right] d\Gamma(P) \\ & + \sum_{i=1}^{N_c} R_{c_i}^*(Q,P) w_{c_i}(P) \\ & = \sum_{i=1}^{N_c} R_{c_i}(P) w_{c_i}^*(Q,P) + \int_\Omega q(P) w^*(Q,P)\, d\Omega \\ & + \int_\Gamma \left[V_n(P) w^*(Q,P) - m_n(P) \frac{\partial w^*}{\partial n}(Q,P) \right] d\Gamma(P), \end{aligned} \tag{24}$$

where $\frac{\partial ()}{\partial n}$ is the derivative in the direction of the outward vector $\mathbf{n}$ that is normal to the boundary Γ; m_n and V_n are, respectively, the normal bending moment and the Kirchhoff equivalent shear force on the boundary Γ; R_c is the thin-plate reaction of the corners; w_c is the transverse displacement of the corners; P is the field point; Q is the source point; and an asterisk denotes a fundamental solution. The constant K is introduced in order to take account of the possibilities that the point Q can be placed in the domain, on the boundary, or outside the domain. If the point Q is on a smooth boundary, then $K = 1/2$.

In order to have an equal number of equations and unknown variables, it is necessary to write an integral equation corresponding to the derivative of the displacement $w(Q)$ with respect to a Cartesian coordinate system fixed to the source point. The axis directions of this coordinate system are coincident with the directions of two vectors that are normal and tangential to the boundary at the source point. For the particular case where the source point Q is placed on a smooth boundary, the boundary equation is given by [Paiva(1987)]

$$\begin{aligned}
&\frac{1}{2}\frac{\partial w(Q)}{\partial n_1} + \int_\Gamma \left[\frac{\partial V^*}{\partial n_1}(Q,P)w(P) - \frac{\partial m_n^*}{\partial n_1}(Q,P)\frac{\partial w}{\partial n}(P)\right] d\Gamma(P) \\
&+ \sum_{i=1}^{N_c} \frac{\partial R_{c_i}^*}{\partial n_1}(Q,P)w_{c_i}(P) \\
&\qquad = \sum_{i=1}^{N_c} R_{c_i}(P)\frac{\partial w_{c_i}^*}{\partial n_1}(Q,P) + \int_\Omega q(P)\frac{\partial w^*}{\partial n_1}(Q,P)\, d\Omega \\
&\qquad + \int_\Gamma \left\{V_n(P)\frac{\partial w^*}{\partial n_1}(Q,P) - m_n(P)\frac{\partial}{\partial n_1}\left[\frac{\partial w^*}{\partial n}(Q,P)\right]\right\} d\Gamma(P),
\end{aligned} \tag{25}$$

where $\frac{\partial()}{\partial n_1}$ is the derivative in the direction of the outward vector $\mathbf{n}_1$ normal to the boundary Γ at the source point Q. It is important to say that it is possible to use only equation (24) in a boundary element formulation, by using the boundary nodes and an equal number of points external to the domain of the problem as source points.

7 Fundamental solutions for bending problems of isotropic plates

The fundamental solution for the transverse displacement in plate bending is computed by setting the non-homogeneous term of the differential Eq. 15as equal to a concentrated force given by a Dirac delta function $\delta(Q,P)$), i.e.,

$$w^* = \frac{1}{8\pi D_{22}} r^2[\ln r], \tag{26}$$

where $r^2 = [(x-x_0)^2 + (y-y_0)^2]$ is the distance between source and field point.

The boundary quantities $\frac{\partial w^*}{\partial n^*}$, m_n^* and v_n^* at a regular point on Γ are expressed in terms of the deflection w by the relations (21) - (23) and Eq. 7. We obtain the following:

$$V_n^* = \frac{1}{4\pi\rho}(\mathbf{nr})[-3+\nu+2(1-\nu)(\mathbf{sr})^2], \tag{27}$$

$$m_n^* = -\frac{1}{8\pi}[2(\mathbf{nr})^2(1-\nu)+2\nu+(1+\nu)(1+2\ln r)], \tag{28}$$

$$\frac{\partial w^*}{\partial n} = \frac{1}{8\pi D_{22}}[(\mathbf{nr})r(1+2\ln r)], \tag{29}$$

$$\frac{\partial w^*}{\partial m} = -\frac{1}{8\pi D_{22}}[(\mathbf{nr})r(1+2\ln r], \tag{30}$$

$$\begin{aligned}\frac{\partial v_n^*}{\partial m} &= -\frac{1}{4\pi r^2}[(-\mathbf{mn}+2(\mathbf{nr})(\mathbf{nr}))(3-\nu)+ \\ &\quad 2(1-\nu)\mathbf{sr}(2\mathbf{ms}(\mathbf{nr})+\mathbf{mn}(\mathbf{sr})-4(\mathbf{nr})(\mathbf{nr})(\mathbf{sr}))],\end{aligned} \tag{31}$$

$$\frac{\partial m_n^*}{\partial m} = \frac{1}{4\pi r}[-2(\mathbf{nr})(-\mathbf{mn}+\mathbf{nr}(\mathbf{nr}))(1-\nu)+\mathbf{nr}(1+\nu)], \tag{32}$$

$$\frac{\partial^2 w^*}{\partial n \partial m} = -\frac{1}{4\pi D_{22}}[\mathbf{nr}(\mathbf{nr})+(\mathbf{mn})\ln r]. \tag{33}$$

In Equations above, the following relations were used:

$$\mathbf{nr} = d/r \tag{34}$$

$$\mathbf{sr} = d\tan(\theta)/r \tag{35}$$

$$\mathbf{nr} = m_1'\cos(\theta)+m_2'\sin(\theta) \tag{36}$$

where m_1' and m_2' are the coordinates of the normal vector at source point at local system (x', y').

8 Transformation of domain integrals into boundary integrals in the isotropic plate bending problem

As can be seen in Eqs. (24) and (25), there are domain integrals in the formulation due to the distributed load in domain. These integrals can be computed for the domain by direct integration over the area Ω_g (see Fig. 1). However, the boundary element formulation loses its main feature, i.e., the boundary only discretization. In this work, domain integrals which arise from distributed loads are transformed into boundary integrals by Dual Reciprocity Method.

Consider that the term b_i of the Equation (24) is approached throughout the domain as a set of unknown coefientes α_i^m and a class of functions denoted by $f^m(x)$. The approximated $b_i(x)$ is given by

$$b_i(x) = \sum_{m=1}^{M} \alpha_i^m f^m(x) \tag{37}$$

Thus, the integral over the area in Eq. (24) can be written as

$$\int_\Omega b_i U_{ik} d\Omega = \sum_{m=1}^{M} \alpha_i^m \int_\Omega U_{ik} f^m d\Omega \tag{38}$$

Particular solutions $\hat{u}^j$ can be found in order to satisfy equilibrium equations

$$C_{mnrs}\hat{u}^j_{rk,ns} = f^j_{mk} \tag{39}$$

The reciprocal relationship between fundamental solution and particular solution can be written as

$$\hat{u}^m_{ln} + \int_\Gamma T_{li}\hat{u}^m_{in} d\Gamma = -\int_\Omega f^m_{in} U_{li} d\Omega + \int_\Gamma U_{li}\hat{t}^m_{in} d\Gamma \tag{40}$$

Substituting Eq. (40) in Eq. (38), gives

$$\int_\Omega U_{li} p_i d\Omega = \sum_{m=1}^{M} \alpha_n^m \left\{ -\hat{u}^m_{ln} + \int_\Gamma \hat{t}^m_{in} U_{li} d\Gamma - \int_\Gamma T_{li}\hat{u}^m_{in} d\Gamma \right\} \tag{41}$$

Now, substituting Eq. (41) in the Eq. (24), you have

$$u_l + \int_\Gamma T_{li} u_i d\Gamma = \int_\Gamma U_{li} t_i d\Gamma + \sum_{m=1}^{M} \alpha_n^m \left\{ \hat{u}_{ln}^m - \int_\Gamma \hat{t}_{in}^m U_{li} d\Gamma + \int_\Gamma T_{li} \hat{u}_{in}^m d\Gamma \right\} \tag{42}$$

Equation (42) will be used as basic equation for the dual reciprocity boundary element. It may be noted the absence of domain integrals.

In order to obtain a solution for the elastic-dynamic problem, boundary is divided into elements. Thus:

$$\begin{aligned} \mathbf{u} &= \phi \mathbf{u}^{(i)} \\ \mathbf{t} &= \phi \mathbf{t}^{(i)} \\ \hat{\mathbf{u}} &= \phi \hat{\mathbf{u}}^{(i)} \\ \hat{\mathbf{t}} &= \phi \hat{\mathbf{t}}^{(i)} \end{aligned} \tag{43}$$

where variables represented in bold letters are vectors of size $2N$ where N is the number of nodes, $\mathbf{u}^{(i)}$ and $\mathbf{t}^{(i)}$ are displacements and tractions nodal values, respectively, ϕ is the shape function matrix, $\mathbf{u}$ and $\mathbf{t}$ represent displacements and stresses along the element and $\hat{\mathbf{u}}^{(i)}$ and $\hat{\mathbf{t}}^{(i)}$ are vectors and nodal displacements and tractions, particular solutions respectively. Thus:

$$\begin{aligned} &\mathbf{c}^l \mathbf{u}^l + \sum_{j=1}^{NE} \left\{ \fint_{\Gamma_j} \mathbf{T} \phi d\Gamma \right\} \mathbf{u}^l = \sum_{j=1}^{NE} \left\{ \int_{\Gamma_j} \mathbf{U} \phi d\Omega \right\} \mathbf{t}^j + \\ &\sum_{m=1}^{M} \left[\mathbf{c}^l \hat{\mathbf{u}}^l - \sum_{j=1}^{NE} \left\{ \int_{\Gamma_j} \mathbf{U} \phi d\Gamma \right\} \hat{\mathbf{T}}^{jm} + \sum_{j=1}^{NE} \left\{ \fint_{\Gamma_j} \mathbf{T} \phi d\Gamma \right\} \hat{\mathbf{U}}^{jm} \right] \alpha^m \end{aligned} \tag{44}$$

considering that,

$$\int_\Gamma \mathbf{U} \phi d\Gamma = \mathbf{G} \tag{45}$$

and

$$\fint_{\Gamma_j} \mathbf{T} \phi d\Gamma = \mathbf{H} \tag{46}$$

it is possible towrite Eq. 44 as:

$$\sum_{j=1}^{N} H^{lj} u^j = \sum_{j=1}^{N} G^{lj} t^j + \sum_{m=1}^{M} \sum_{j=1}^{N} \left(H^{lj} \hat{u}^{jm} - G^{lj} \hat{t}^{jm} \right) \alpha^m \tag{47}$$

The M vectors α $(2\text{x}1)$ corresponding to the different approximatting functions f^m, can be grouped into a vector α $(2M\text{x}1)$ and related to the body forces vector$\mathbf{p}^m$ $(2M\text{x}1)$ containing body forces for all we outline. Thus, equation (8) can be written as

$$\mathbf{b} = \mathbf{F} \alpha \tag{48}$$

where elements of matrix $\mathbf{F}$ are values of functions $f^m(x)$ at all nodes. Reversing the matrix $\mathbf{F}$

can be written

$$\alpha = \mathbf{Ep} \tag{49}$$

where $\mathbf{E} = \mathbf{F^{-1}}$.

Substituting equation (49) in the equation (47), you can write

$$\mathbf{Hu} = \mathbf{Gt} + \left[\mathbf{H\hat{U}} - \mathbf{G\hat{T}}\right]\mathbf{Ep} \tag{50}$$

and maning

$$\mathbf{M} = \left[\mathbf{G\hat{T}} - \mathbf{H\hat{U}}\right]\mathbf{E} \tag{51}$$

Equation (50) becomes

$$\mathbf{Mp} + \mathbf{Hu} = \mathbf{Gt} \tag{52}$$

which has the same aspect of the equilibrium equations of finite elements, namely, a mass matrix $\mathbf{M}$ a stiffness matrix $\mathbf{H}$ and force vector $\mathbf{Gt}$.

9 Matrix equations

In order to compute the unknown boundary variables, the boundary Γ is discretized into NE straight elements the boundary variables w, $\partial w/\partial n$, m_n, and V_n are assumed to be constant along each element. Taking a node d as the source point, Eqs. (24) and (25) can be written in matrix form as follows:

$$\begin{aligned}
\frac{1}{2}\left\{\begin{array}{c} w^{(d)} \\ \\ \frac{\partial w^{(d)}}{\partial n_1} \end{array}\right\} + \sum_{i=1}^{NE}\left(\left[\begin{array}{cc} H_{11}^{(i,d)} & H_{12}^{(i,d)} \\ H_{21}^{(i,d)} & H_{22}^{(i,d)} \end{array}\right]\left\{\begin{array}{c} w^{(i)} \\ \frac{\partial w^{(i)}}{\partial n} \end{array}\right\}\right) \\
= \sum_{i=1}^{NE}\left(\left[\begin{array}{cc} G_{11}^{(i,d)} & G_{12}^{(i,d)} \\ G_{21}^{(i,d)} & G_{22}^{(i,d)} \end{array}\right]\left\{\begin{array}{c} V_n^{(i)} \\ m_n^{(i)} \end{array}\right\}\right) + \sum_{i=1}^{NC}\left(\left\{\begin{array}{c} C_1^{(i,d)} \\ C_2^{(i,d)} \end{array}\right\} w_c^{(i)}\right) \\
+ \sum_{i=1}^{NC}\left(\left\{\begin{array}{c} F_1^{(i,d)} \\ F_2^{(i,d)} \end{array}\right\} R_c^{(i)}\right) + \left\{\begin{array}{c} P_1 \\ P_2 \end{array}\right\},
\end{aligned} \tag{53}$$

where NC stands for the number of corners. The terms of Eq. (53) are integrals, given by

$$H_{11}^{(i,d)} = \int_{\Gamma_i} V_n^* \, d\Gamma, \qquad H_{12}^{(i,d)} = -\int_{\Gamma_i} m_n^* \, d\Gamma, \tag{54}$$

$$H_{21}^{(i,d)} = \int_{\Gamma_i} \frac{\partial V_n^*}{\partial n_1} \, d\Gamma, \qquad H_{22}^{(i,d)} = -\int_{\Gamma_i} \frac{\partial m_n^*}{\partial n_1} \, d\Gamma, \tag{55}$$

$$G_{11}^{(i,d)} = \int_{\Gamma_i} w^* \, d\Gamma, \qquad G_{12}^{(i,d)} = -\int_{\Gamma_i} \frac{\partial w^*}{\partial n} \, d\Gamma, \tag{56}$$

$$G_{21}^{(i,d)} = \int_{\Gamma_i} \frac{\partial w^*}{\partial n_1}\, d\Gamma, \qquad G_{22}^{(i,d)} = -\int_{\Gamma_i} \frac{\partial}{\partial n_1} \frac{\partial m_n^*}{\partial n}\, d\Gamma, \tag{57}$$

$$C_1^{(i,d)} = R_c^*, \qquad C_2^{(i,d)} = \frac{\partial R_c^*}{\partial n_1}, \tag{58}$$

$$F_1^{(i,d)} = w_{ci}^*, \qquad F_2^{(i,d)} = \frac{\partial w_{ci}^*}{\partial n_1}, \tag{59}$$

$$P_1 = \int_{\Gamma_g} F^*\, d\Gamma, \qquad P_2 = \int_{\Gamma_g} G^*\, d\Gamma. \tag{60}$$

Some of the above integrals show weak singularities, strong singularities, or hypersingularities when the element that is being integrated contains the source point. In the work described in this paper, these integrals were treated analytically as described by [Paiva and Albuquerque(2003)].

The matrix Eq. (53) contains two equations and $2NE + NC$ variables. In order to obtain a solvable linear system, the source point is placed successively in each boundary node ($d = 1, \ldots, NE$) and in each corner node ($d = NE + 1, \ldots, NE + NC$). It is worth noting that while both of equations (24) and (25) are used for each boundary node (providing the first $2NE$ equations), only Eq. (24) is used for each corner (providing the other NC equations). So, the following matrix equation is obtained:

$$\begin{bmatrix} \mathbf{H} & \mathbf{C} \\ \mathbf{H}' & \mathbf{C}' \end{bmatrix} \begin{Bmatrix} \mathbf{w} \\ \mathbf{w_c} \end{Bmatrix} = \begin{bmatrix} \mathbf{G} & \mathbf{F} \\ \mathbf{G}' & \mathbf{F}' \end{bmatrix} \begin{Bmatrix} \mathbf{V} \\ \mathbf{V_c} \end{Bmatrix} + \begin{Bmatrix} \mathbf{P} \\ \mathbf{P_c} \end{Bmatrix}, \tag{61}$$

where $\mathbf{w}$ contains the transverse displacement and the rotation of each boundary node, $\mathbf{V}$ contains the shear force and the twisting moment for each boundary node, $\mathbf{P}$ contains the integrals P_1 and P_2 for each boundary node, $\mathbf{w_c}$ contains the transverse displacement of each corner, $\mathbf{V_c}$ contains the corner reaction for each corner, and $\mathbf{P_c}$ contains the integral P_1 for each corner. The terms $\mathbf{H}$, $\mathbf{C}$, $\mathbf{G}$, and $\mathbf{F}$ are matrices which contain the respective terms of Eq. (53), written for the NE boundary nodes. The terms $\mathbf{H}'$, $\mathbf{C}'$, $\mathbf{G}'$, and $\mathbf{F}'$ are matrices which contain the respective first-line terms of Eq. (53), written for the NC corners. Applying boundary conditions, Eq. (61) can be rearranged as

$$\mathbf{Ax} = \mathbf{b}, \tag{62}$$

which can be solved by standard procedures for linear systems.

10 Transient Problems: Formulation in the time domain

That this the method is the suitable for use in the direct time integration along with the radial integration reciprocity boundary element method.

10.1 Matrix equations

Seeing that in this case, the only body forces present in body domain Ω and contour Γ are due to the acceleration field,

$$b_i = \rho \ddot{u}_i \tag{63}$$

matrix equation (52) may be written to a time step $\tau + \Delta\tau$ as

$$\rho \mathbf{M}\ddot{\mathbf{u}}_{\tau+\Delta\tau} + \mathbf{H}\mathbf{u}_{\tau+\Delta\tau} = \mathbf{G}\mathbf{t}_{\tau+\Delta\tau} \tag{64}$$

To carry out time integration for a period $\mathcal{T}$, this time period is divided into N equal intervals $\Delta\tau$ ($\mathcal{T} = N\Delta\tau$). Assuming that the solution of equation (52) is known $0, \Delta\tau, 2\Delta\tau, ..., \tau$ solution at time $\tau + \Delta\tau$ is found according to the approach proposed Houbolt (1950). The acceleration in $\tau + \Delta\tau$ is approximated by the expression of finite difference

$$\ddot{\mathbf{u}}_{\tau+\Delta\tau} = \frac{1}{\Delta\tau^2}\left(2\mathbf{u}_{\tau+\Delta\tau} - 5\mathbf{u}_{\tau} + 4\mathbf{u}_{\tau-\Delta\tau} - \mathbf{u}_{\tau-2\Delta\tau}\right) \tag{65}$$

An important feature of Houbolt method is that it has a high numerical damping.

Substituting equation (65) in the equation (64), you have

$$\left[\frac{2}{\Delta\tau^2}\rho\mathbf{M} + \mathbf{H}\right]\mathbf{u}_{\tau+\Delta\tau} = \mathbf{G}\mathbf{t}_{\tau+\Delta\tau} + \frac{1}{\Delta\tau^2}\rho\mathbf{M}\left(5\mathbf{u}_{\tau} - 4\mathbf{u}_{\tau-\Delta\tau} + \mathbf{u}_{\tau-2\Delta\tau}\right) \tag{66}$$

Since all unknown variables are passed to the left side, equation (66) can be written as

$$\mathbf{A}\mathbf{y}_{\tau+\Delta\tau} = \mathbf{w}_{\tau+\Delta\tau} \tag{67}$$

where $\mathbf{y}_{\tau+\Delta\tau}$ is the unknown variable vector and $\mathbf{w}_{\tau+\Delta\tau}$ is the known variables vector, and its elements are calculated from the values of $\mathbf{u}$ from the earlier times steps, and boundary conditions in time $\tau + \Delta\tau$.

11 Numerical results

The example chosen to evaluate the Boundary Element Method (BEM) performance in the sense of thin-shaped structures, as well as to analyse the effect of the size of the time step, is the elastodynamic problem. The problem corresponds to a plate under uniform load applied as a step function at time $t_0 = 0$, as defined in Figure 4. Dimensions of the plate are: length L = 1000 mm, width b = 1000 mm, thickness t = 8 mm. The Material properties are: elastic modulus E = 210 GPa, Poisson coefficient ν = 0.3 and density ρ = 1 Kg/m^3 . The plate is under a uniformly distributed load q = 100 N/m^2.

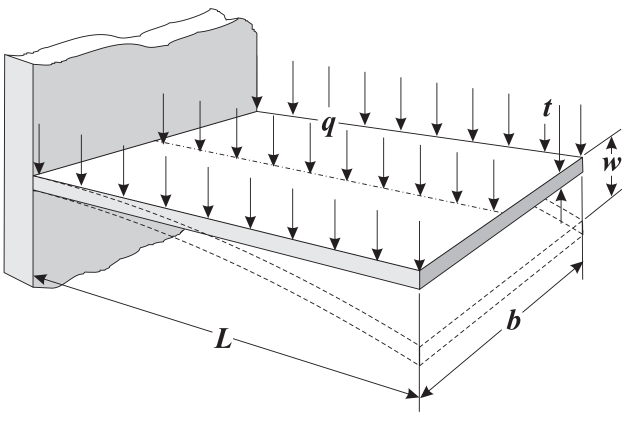

Figure 4: Cantilever slender plate.

The plate was discretized in 12 and 24 constant boundary elements of equal length and time steps $\Delta\tau$ = 5.5741 10^{-5} s. In this paper, we are interested in evaluating how the solution for vertical displacement obtained with BEM varies with the number of integration points used on the numerical integration and the Radial Integration Method. The problem was solved with a time step and result is shown in Figure 5.

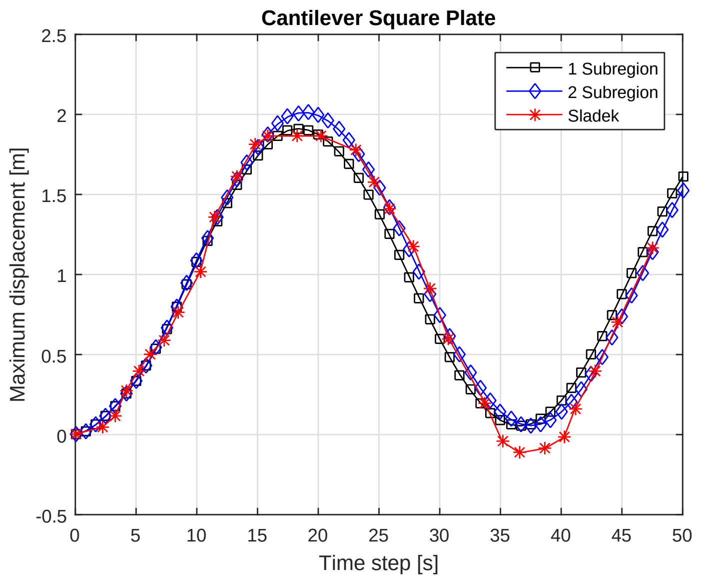

Figure 5: Maximum transversal displacement, w_{max}, computed with different number of integration points and .

In Figure 6 the model consists of two plates attached to form a L-shaped structure. Dimensions of the plate are: length $L_1 = 1\,\mathrm{m}$, $L_2 = 1\,\mathrm{m}$, $L_3 = 2\,\mathrm{m}$, $t_1 = 0,1\,\mathrm{m}$ and $t_2 = 0,1\,\mathrm{m}$. The Material properties are: elastic modulus E = 210 GPa, Poisson coefficient ν = 0.3 and desity ρ = 1 Kg/m^3 . The plate is under a uniformly distributed load q = 100 N/m.

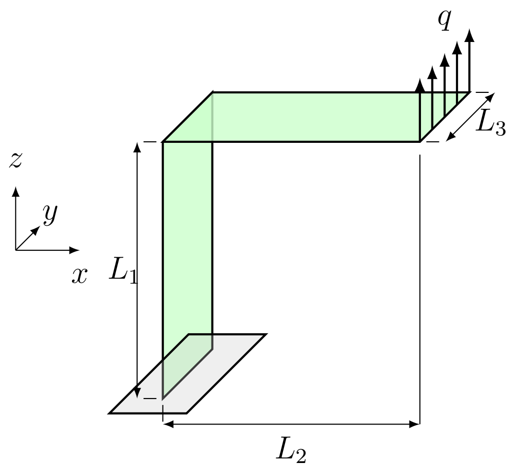

Figure 6: Dimensions and boundary conditions for the structure 'L'

Distribution of displacement can be seen in Figure 7.

As can be seen, the number of elements and time step influence in obtaining displacement versus time on dynamic analysis at the numerical integration to the situation in the first example.

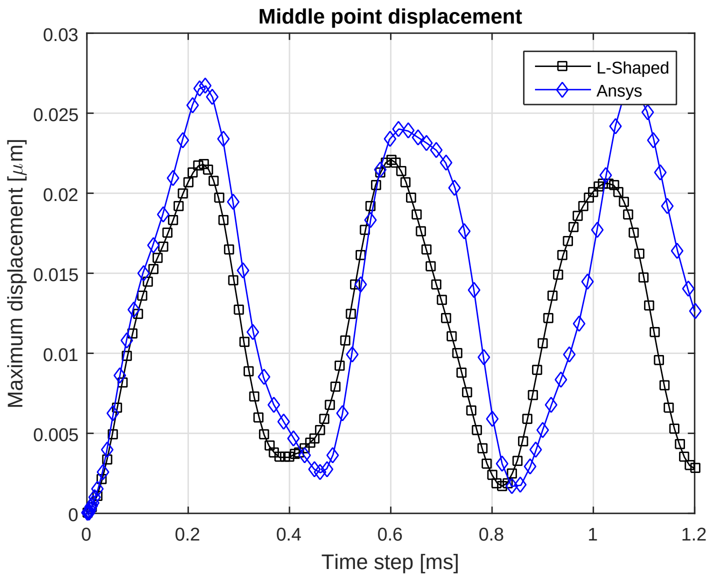

Figure 7: Distribution of the total displacement for the structure "L"

For this problem, at least 12 to 24 elements numbers are necessary in order to obtain an agreement with Sladek results. The second case shows an example of plates associated in space. It was noticed in both cases that the use of RIM may have generated a numerical damping. For this problem, was used 300 elements numbers are necessary in order to obtain an agreement with Ansys results.This behavior could be due to the transformation method used for the domain integral.

12 Conclusions

This paper presents numerical expressions for influence matrices and dynamic analysis using the radial integration method and direct method elements for isotropic thin plates. The proposed formulation has been applied to a thin plate and two plates associated subjected to uniformly distributed loads. It used the RIM method to perform the transformation from domain to boudary integral. It was observed that the use of constant boundary elements has shown a good agreement when compared with Ansys and Sladek results. Some efforts are still necessary to make adjustments and improve results. As a suggestion, give up the DRM and use the RIM. Indeed, the use of radial integration method shows better results for this application, despite the greater computational cost.

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