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Applying the BEM to evaluate cross-section properties of 3D (spatial) frames elements

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Abstract. *In this paper, algorithms based on boundary-integral formulations are employed to calculate geometrical section properties as areas, 1st and 2nd moments of area, torsional moment of inertia, shear-stress shape factors, useful for constructing 3D frame element stiffness matrices. Particularly for determining the torsional constant and the torsion center, the BEM is employed to solve the Laplace equation involved. Herein, specific integration algorithms that allow the modeling of elements with thin-walled cross-sections are considered. After that, taking advantage of existing boundary-element meshes, other section geometrical characteristics are calculated. In the applications, section properties are calculated by means of BE-based routines included into a computational code for carrying out the analysis of spatial frameworks by using the Direct Stiffness Method (DSM).*

Keywords: *section properties, 3D frame element, BEM, Direct Stiffness Method*

1 INTRODUCTION

The Direct Stiffness Method (DSM) has been largely employed in structural engineering since the 1950s when the bases for the today's matrix formulations of framed structures might well be first presented in the seminal papers (1954-1955) by Argyris, afterwards collected in the book entitled ”Energy theorems and structural analysis”, Argyris and Kelsey (1960). Following Argyris' fundamental steps, important classical books on the DSM formulation and the matrix structural analysis (MSA) in general were published in the 1960s, Pestel and Leckie (1963), Przemieniecki (1968), Gere and Weaver(2012). In fact, the DSM is the most simple way to implement the Finite Element Method (FEM), and was around the starting point from which the whole FEM evolved. Not mentioned yet, but not less relevant in this scenario concerning the formulations for the MSA, was, of course, the developments achieved in computer hardware, which allowed converting these formulations in practical computer codes for structural analysis as NASTRAN (1968), ANSYS (1970), and SAP2000 (1975). Indeed, these codes are not specifically DSM-based codes but general FEA-based ones not restricted to structural analysis.

In this paper, we propose a strategy to enlarge the analysis capability of DSM-based codes by including a option of calculating all of the geometric properties of general cross-sections by employing, when needed, the Boundary Element Method (BEM) or just the existing BE mesh. Particularly for determining the torsional constant, J, and the torsion center, the BEM has been employed to calculate the warping function, ψ. Herein, special care has been taken concerning thin-walled sections, which demands special algorithms to deal with the nearly-singular integrals, and so to accurately evaluate the torsional constant. This problem will typically take place in 3D steel structures, and in the code, the Telles coordinate transformation,Telles(1987), and an analytical process over linear elements (Araújo and Gray (2008), Hillesheim (2013)) are available to evaluate the nearly-singular integrals. In fact, the special integration processes are activated only if nearly-singular are identified according a certain criterion. For calculating the other cross-section geometrical properties as areas, 1st and 2nd moments of area, and shear-stress shape factor, the involved domain integrals are converted into boundary integrals and standard Gauss quadrature is applied to evaluate them. Herein, the existing BE mesh used to evaluate the torsional constant and torsion center is used to calculate these boundary integrals, which makes the computational modules for determining all the element cross-section properties especially efficient.

The BE-based process for evaluating cross-section properties is then eventually incorporated into a DSM-based code for 3D (spatial) frames, and results obtained for special sections are compared to the SAP2000 ones to show correctness and suitability of the process.

2 Geometrical section properties for 3D frame elements

In fact, our major objective with the strategies proposed in this paper is, at the end, to construct DSM-based codes to solve spatial (3D) frames with elements whose geometrical cross section may be of any generic form, and additionally may generically vary along their centroidal axis. For this class of structures (spatial frameworks), the structural element is that shown in Fig. 1, with 12 degrees of freedom, whose stiffness coefficients k_{ij} correspond to the element actions for unit prescribed node displacements $u_j = \delta_{ij}, j = 1, 2, ..., 12$ along the degrees of freedom, while the equivalent nodal loads associated to the external element loading

$q_1, m_1, q_{2p}, m_{2p}, q_{3p}, m_{3p}$ (which abstractly denote concentrated and distributed loads) are obtained by solving the element under homogeneous boundary conditions $u_j = 0, j = 1, 2, ..., 12$ (i.e. clamped at either end).

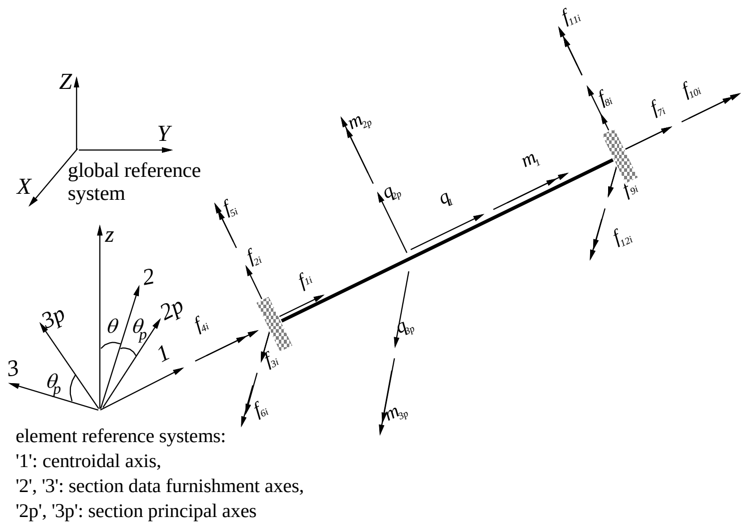


Figure 1: 3D frame element

In this paper, one is especially interested in presenting a strategy for calculating all the section properties needed to determine the stiffness matrix and equivalent nodal loads for 3D frame elements, namely, the cross-sectional area, A, centroidal point, the moments of inertia, I_{22} and I_{33}, the principal moments of inertia, $I_{2_p2_p}$ and $I_{3_p3_p}$, torsional constant, J, torsion center, x_s, y_s, and the shear-stress shape factor, χ_k, $k = 2_p$, 3_p along the principal axes (see Fig. 2). In the code, the geometrical section data are furnished referred to the 2-3 reference system (section cardinal reference system), which may be on their turn possibly rotated of some counterclockwise rotation angle θ around the 1 axis (element centroidal axis) from the 1-Z plane (see Fig. 2). Eventually, the element stiffness matrix and equivalent nodal loads associated to the external element load are determined referred to the $1, 2_p, 3_p$ axes (element analysis axes). Additionally, for generic sections, for that the centroidal point is not previously known, a translational coordinate transformation may be needed from the 2'-3' reference system to the 2-3 one. All coordination transformations involved in this process are carried out in a completely automated way, and are sketched in Fig. 2.

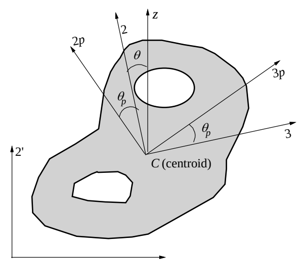


Figure 2: Section reference systems

It is worth mentioning that differently from Sapoutzakis and Mokos (2004), who employs a pure BEM formulation for constructing stiffness matrices for frame elements with varying section properties along their centroidal axis, the boundary integral formulations employed here are restricted to the section properties while the variation of these properties along the element axis is taken into account inside the DSM formulation considered. Details of this formulation are not addressed here, as it is out of the scope of this paper, focused specifically on the boundary integral formulations involved.

2.1 Geometrical properties associated with torsion

Here, the classical Saint-Venant torsion problem is considered, which is governed by the Laplace equation under Neumann's boundary conditions. The following boundary-value problem (BVP) has to be solved:

$$\nabla^2\psi = 0, \quad \mathbf{x} \in \Omega, \tag{1}$$

$$p(\mathbf{x}) = \frac{\partial \psi}{\partial \mathbf{n}} = \mathbf{x} \cdot \mathbf{t}(\mathbf{x}), \quad \mathbf{x} \in \Gamma. \tag{2}$$

where ψ is the section warping function referred to x-y reference system (with origin at the torsion center), and $\mathbf{n}(\mathbf{x})$ and $\mathbf{t}(\mathbf{x})$ are, respectively, the outward normal and tangent unit vectors at point $\mathbf{x} \in \Gamma$. Once the BVP given by equations 1 and 2 is solved, all relevant physical quantities involved in the torsion problem as displacements, stresses, and the section torsional constant can be evaluated. Among others, the following expressions are useful here:

(i) Shear stresses, τ_{xz} and τ_{yz}:

$$\begin{cases} \tau_{xz} = G\theta_t'(\frac{\partial\psi}{\partial x} - y) \\ \\ \tau_{yz} = G\theta_t'(\frac{\partial\psi}{\partial y} + x) \end{cases} \tag{3}$$

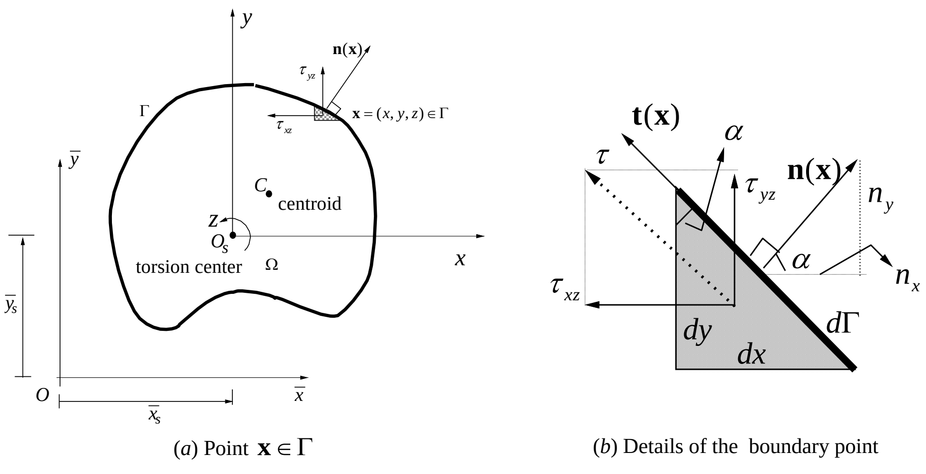


Figure 3: Torsion domain

Here, in case of using the BEM to solve the torsion BVP, singular boundary integrals will take place for evaluating the Cartesian derivatives, $\frac{\partial\psi}{\partial x}$ and $\frac{\partial\psi}{\partial y}$ for points at the boundary Γ. To avoid then dealing with that kind integrals, the expressions below can be used:

$$\begin{Bmatrix} \frac{\partial\psi}{\partial x} \\ \\ \frac{\partial\psi}{\partial y} \end{Bmatrix} = \begin{bmatrix} \mathbf{t}^T(\mathbf{x}) \\ \mathbf{n}^T(\mathbf{x}) \end{bmatrix}^{-1} \begin{Bmatrix} \frac{1}{J(\eta)}\frac{\partial\psi(\eta))}{\partial\eta)} \\ \\ \mathbf{x}\cdot\mathbf{t}(\mathbf{x}) \end{Bmatrix} \quad , \tag{4}$$

where η is the natural coordinate employed for the parametric representation of the boundary geometry.

<u>(ii) Torsional constant, J (by means exclusively of boundary integrals):</u>

$$J = -\int_\Gamma \psi(\mathbf{x})p(\mathbf{x})d\Gamma + \int_\Gamma xy[(y,x)\cdot(n_x,n_y)]d\Gamma \tag{5}$$

In equation 5, the warping function ψ is referred to the x, y system centered at the torsion center. However, as that point for a generic section is not known a priori, a basic step in the whole procedure is, of course, first, the determination of its position, $\bar{x}_s$, $\bar{y}_s$ (see Fig. 3). To do this, the BVP 1 and 2 is re-written referred to a $\bar{x}$, $\bar{y}$ reference system with arbitrary origin, O, from whose solution the ψ_0 warping function is obtained, and the position of the torsion center ($\bar{x}_s$, $\bar{y}_s$) can be determined by solving the system of equations (Athanasiadis (1989), Hillesheim (2013))

$$\begin{cases} I_{\bar{x}\bar{x}}\bar{x}_s - I_{\bar{x}\bar{y}}\bar{y}_s + M_{\bar{x}}c &= -\int_\Omega \bar{y}\psi_0 d\Omega \\ \\ I_{\bar{x}\bar{y}}\bar{x}_s - I_{\bar{y}\bar{y}}\bar{y}_s + M_{\bar{y}}c &= -\int_\Omega \bar{x}\psi_0 d\Omega \quad , \\ \\ M_{\bar{x}}\bar{x}_s - M_{\bar{y}}\bar{y}_s + \Omega c &= -\int_\Omega \psi_0 d\Omega \end{cases} \tag{6}$$

where $I_{\bar{x}_i\bar{x}_j} = \int_\Omega \bar{x}_i\bar{x}_j d\Omega$, $M_{\bar{x}_i} = \int_\Omega \bar{x}_j d\Omega$, and Ω is the area of the section. With ψ_0 determined, the corresponding warping function referred to the x, y system centered at the torsion center (see Fig. 3) is calculated by Athanasiadis (1989)

$$\psi(x,y) = \psi_0(\bar{x},\bar{y}) - y_s\bar{x} + x_s\bar{y} + c \quad . \tag{7}$$

Notice that the corresponding prescribed fluxes referred to the $\bar{x}$, $\bar{y}$ system are given by (c.f. 2)

$$p_0(\mathbf{x}) = \bar{\mathbf{x}}\cdot\mathbf{t}(\mathbf{x}), \quad \bar{\mathbf{x}} \in \Gamma. \tag{8}$$

2.2 Calculation of other geometrical section properties

Simpler mathematical formulas for calculating geometrical section properties as areas, moments of inertia, and moments of areas can be easily obtained by applying Green's integral theorems to convert domain into boundary integrals. One obtains:

$$\begin{cases} \Omega = \oint_\Gamma x n_x d\Gamma \\ \\ I_{xx} = \oint_\Gamma x y^2 n_x d\Gamma, \quad I_{yy} = \oint_\Gamma x^2 y n_y d\Gamma, \quad I_{xy} = \oint_\Gamma x^2 y n_x d\Gamma \\ \\ x_c = \frac{1}{\Omega} \oint_\Gamma x y n_y d\Gamma, \quad y_c = \frac{1}{\Omega} \oint_\Gamma x y n_x d\Gamma \end{cases} , \tag{9}$$

Additionally, the domain integrals at right-hand side of equation 6 may also be expressed in terms of the following boundary integrals:

$$\begin{cases} \int_\Omega \bar{y}\psi_0 d\Omega = \oint_\Gamma [(\frac{1}{2}\bar{y}^2 n_{\bar{y}})\psi_0(\bar{\mathbf{x}}) - (\frac{1}{6}\bar{y}^3)p(\bar{\mathbf{x}})]d\Gamma \\ \\ \int_\Omega \bar{x}\psi_0 d\Omega = \oint_\Gamma [(\frac{1}{2}\bar{x}^2 n_x)\psi_0(\bar{\mathbf{x}}) - (\frac{1}{6}\bar{x}^3)p(\bar{\mathbf{x}})]d\Gamma \\ \\ \int_\Omega \psi_0 d\Omega = \oint_\Gamma [(\bar{x} n_x)\psi_0(\bar{\mathbf{x}}) - (\frac{1}{2}\bar{x}^2)p(\bar{\mathbf{x}})]d\Gamma \end{cases} . \tag{10}$$

Thus, with the expressions above, all geometrical properties of the 3D frame element, including all the other coefficients of system of equations 6, can be exclusively calculated by means of boundary integrals.

2.3 Shear-stress shape factor

Shear-stress effects at a certain x_i direction can be included on the analysis by replacing the real area by the corresponding shear effective area at that direction, Ω_{eff,x_i}, expressed by

$$\Omega_{eff,x_i} = \frac{\Omega}{\chi_{x_i}} \quad , \tag{11}$$

where χ_{x_i} is the section shear shape factor along x_i. To calculate the element stiffness matrices, we will need to know the shear shape factor for the section principal directions, which particularly for the y_p principal axis is given by (see Fig. 5)

$$\chi_{y_p} = \frac{\Omega}{I^2_{x_p x_p}} \int_{y_b}^{y_t} \frac{M^2_{x_p}(s)}{b_{x_p}(s)} ds \quad , \tag{12}$$

where M_{x_p} is the 1st moment of area related to x_p, and $b_{x_p}(s)$ is the width of the section at the s ordinate, along the x_p axis. We see then that, with the aid of relations given above in this section, all the terms needed for calculating the shear shape factor in eq. 12 can also be converted into boundary integrals. Analogously, the shear shape factor for the other principal direction, χ_{x_p}, can be determined.

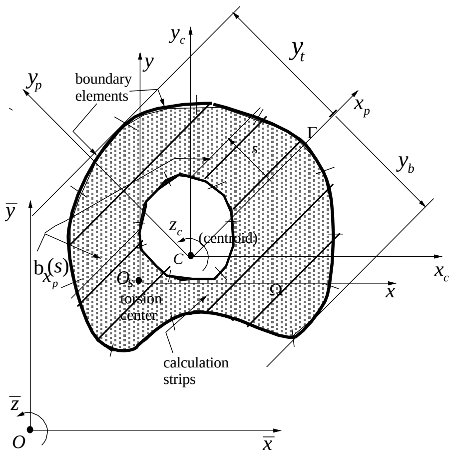


Figure 4: Section calculation domain

2.4 BE-based numerical procedures

To solve the BVP given in equations 1 and 2, the BEM is applied, and for that isoparametric linear (2-node) and quadratic (3-node) boundary elements are considered to discretize the boundary Γ (see BE domain in Fig. 5). Here, a standard BEM formulation based on fundamental solutions for 2D potential problems defined in an infinite domain Ω^* is employed. After integrating over the boundary elements, an algebraic system of equations of the form

$$\mathbf{H}\psi_0 = \mathbf{G}\bar{\mathbf{p}}_0 \tag{13}$$

is originated, where ψ_0 is the vector with the unknown warping boundary values referred to the $\bar{x}$, $\bar{y}$ system, and the $\bar{\mathbf{p}}_0$ vector contains the prescribed fluxes given by equation 8. To remove the singularity of this system of equations, it assumed that $\psi_{n0} = 0$, where n is the last degree of freedom of the BE model.

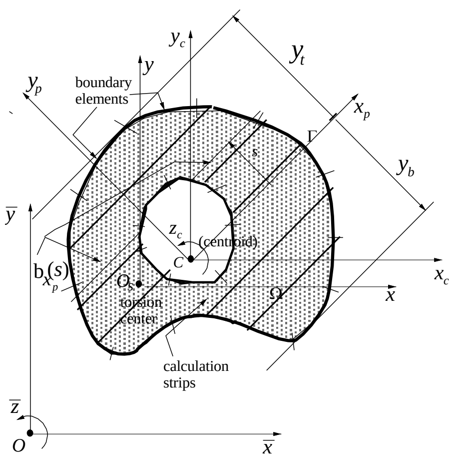


Figure 5: Section calculation domain

A fundamental step in the BE algorithm is the evaluation of singular and, mainly, nearly-singular integrals involved. Here, an analytical integration process is implemented for the linear elements, while a numerical quadrature based on the Telles coordinate transformation, Telles (1987), is considered for both linear and quadratic elements. As the latter quadrature has been largely employed in many works, Araújo and Gray (2008), no additional will be made here. Comments on the analytical scheme are presented below. Essentially, the coordinate transformation indicated in Fig. 6 so as to express the radius modulus, r, by

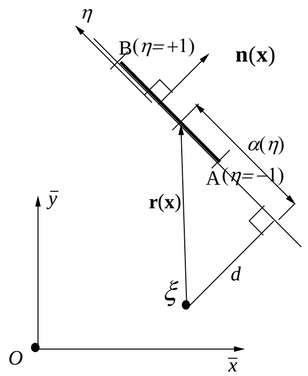


Figure 6: Coordinate transformation for analytical integrations

$$r(\eta) = \sqrt{d^2 + \alpha^2(\eta)} \quad , -1 \leq \eta \leq +1 \quad . \tag{14}$$

This transformation allows to get analytical expressions for the either singular or quasi-singular integrals involved, and so to construct stiffness matrices in a unified general way for either thin-walled and thick-walled sections.

After solving the torsion BVP by using BE models, the existing BE mesh is then employed to calculate all other geometrical properties properties needed, for which the boundary integrals shown previously are considered. Particularly for the determination of the shear shape factor, the section domain is subdivided into an arbitrary number of calculation strips ($nstrip$), over which the Gauss-Legendre quadrature is applied (see Fig. 5 for the calculation of χ_{y_p}). The following expression is available for computational implementation (npg is the number of integration points per strip):

$$\chi_{y_p} = \frac{\Omega}{I^2_{x_p x_p}} \sum_{i=1}^{nstrip} \sum_{j=1}^{npg} \frac{M^2_{x_p}(s_j)}{b_{x_p}(s_j)} \frac{\Delta_{strip}}{2} \omega_j \quad . \tag{15}$$

Applications

To show the performance of the algorithm developed, geometrical properties for the 2 different sections shown in Figs. 7(a) and 8(a) are calculated. Namely, area, moments of inertia, torsion center, torsional constant, and shear shape factor are determined by using the BE models and calculation strips given in Figs. 7(b),(c),(d) and 8(b),(c). Linear (2-node) boundary elements and analytical integration are employed. The results are presented in Tables 1 and 2 and compared to the ones obtained using the SAP2000 software. As seen, excellent agreement between the results calculated with the strategies proposed in this paper and with SAP2000 are observed.

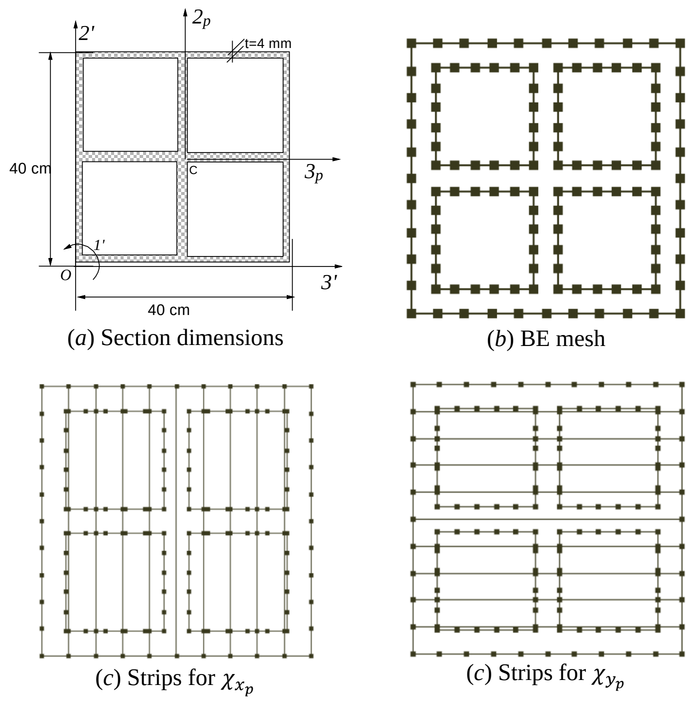
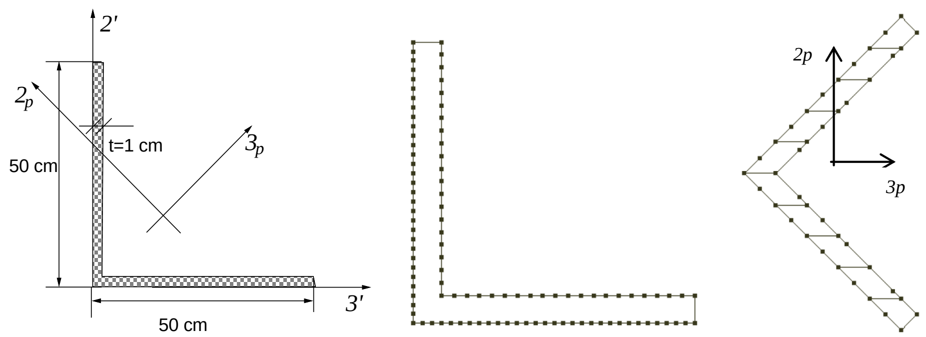


Figure 7: Stiffened box section

Table 1: Geometrical properties for the stiffened box section

	BEM	SAP2000
$\Omega\ (m^2)$	9.456E-03	9.456E-03
$I_{2_p}\ (m^4)$	1.857E-04	1.857E-04
$I_{3_p}\ (m^4)$	1.857E-04	1.857E-04
$J\ (m^4)$	2.519E-04	2.507E-04
χ_{2_p}	2.02*	1.99
χ_{3_p}	2.02*	1.99

*values calculated with nstrip=20, npg/strip=3



(a) Section dimensions (b) BE mesh (c) Strips for $\chi_{x_{2p}}$

Figure 8: L shape

Table 2: Geometrical properties for L shape

	BEM	SAP 2000
$\Omega\ (m^2)$	9.900×10^{-03}	9.900×10^{-03}
$I_{2_P}\ (m^4)$	4.043×10^{-04}	4.043×10^{-04}
$I_{3_P}\ (m^4)$	1.012×10^{-04}	1.012×10^{-04}
$J\ (m^4)$	3.292×10^{-07}	3.292×10^{-07}
χ_{2_p}	1.20*	1.19
χ_{3_p}	1.20*	1.20
$\bar{x}_s(m)$	0.1287	0.1287
$\bar{y}_s(m)$	0.1287	0.1287

*values calculated with nstrip=10, npg/strip=2

Conclusions

In this paper, strategies essentially based on boundary element models are presented either for solving the Neumann's BVP governing the torsion in 3D frame elements or just to calculate the many other geometrical section properties needed to construct 3D element stiffness matrices. As all the calculations involved in the strategies take into account only boundary integrals, the process is very convenient for dealing with any section geometry, and efficiently applies either thin- or thick-walled sections, with or without a number of holes. Besides, it can be easily incorporated into a DSM-based code to analyze spatial frames. In fact, the computational implementation done allows a user-friendlily completely automated process to model 3D frame elements. Although linear elements and analytical integration have been employed the problems presented in this paper, quadratic elements and numerical integration based on the Telles coordinate transformation have been incorporated into the computer code and proven very efficient. Indeed, the CPU time required for calculating the section properties is very small.

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