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CONVERGENCE PROOF OF A MIXED FINITE ELEMENT FORMULATION FOR TRANSIENT HEAT TRANSFER

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Abstract. *A mixed finite element is developed to discretize one-dimensional transient heat conduction problems in both space and time. The formulation thus involves interpolation of both temperature and heat flow. The convergence is proved analytically by means of eigenvalue bounds, and results of simple numerical models are in close agreement with the exact solution.*

Keywords: *Convergence, Eigenvalue bound, Finite element, Heat conduction*

1 INTRODUCTION

Diffusion is a dissipative process that occurs in nature as a series of different phenomena, like heat conduction (Bergman et al., 2011), soil consolidation (Lambe & Whitman, 1969), chemical species diffusion (Barry, 2002), and so on. The physics of diffusion is usually described with the help of two equations. One equation describes the diffusive flow, as the Fourier law (in the case of heat transfer), which in one-dimensional problems reads

$$\frac{\partial T}{\partial x} + \frac{q''}{k} = 0. \tag{1}$$

The other equation is a balance of energy, which may be described as

$$\rho c \frac{\partial T}{\partial t} + \frac{\partial q''}{\partial x} - \dot{q} = 0 \tag{2}$$

for the case of one-dimensional heat transfer. In those equations, T is the temperature, q'' is the heat flow, t is time, x is the position, and $\dot{q}$ is the rate of energy generation per unit volume. The material properties are provided by the thermal conductivity k, the mass density ρ and the specific heat c.

Isolation of q'' in Eq. (1) and substitution into Eq. (2) leads to the diffusion equation

$$\rho c \frac{\partial T}{\partial t} - \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) - \dot{q} = 0. \tag{3}$$

The temperature-based finite element models are related to the solution of Eq. (3) rather than to the solution of Eq. (1) and Eq.(2). They are traditionally the most currently used in heat conduction analysis because of their simplicity and general effectiveness. However, there are certain analyses in which these models are not sufficiently effective. As an example, the heat flow field provided by them is likely to display greater error than the temperature field. The reason for this behavior is that the heat flow field is obtained by differentiation of the temperature field, and differentiation discloses differences. Computed heat flow is usually most accurate at locations within an element rather than on its boundaries. This is unfortunate because the heat flow of greatest interest usually appears at boundaries. Therefore, heat flow may be calculated at certain points within an element, extrapolated to element boundaries and, if desired, the discontinuity arisen from this process may be treated by some smoothing scheme (Hinton & Campbell, 1974; Diersch, 2014). The lower quality of the predicted heat flow field becomes yet worse in the presence of sharp discontinuities at interface of materials or phase changes, where the heat flow is the most relevant thermal quantity.

The mixed finite element models, which are related to the solution of Eq. (1) and Eq. (2) rather than to the solution of Eq. (3), have been developed to offer some advantages compared to the standard temperature-based models. In a mixed model, the heat flow and temperature are simultaneously and independently approximated. As the temperature is not favored over the heat flow or vice versa, a better heat flow description permits the improvement of the heat flow approximation (Kardestuncer, 1987).

A mixed finite element is developed in this paper, with the introduction of both space and time discretization into the element formulation. In other words, no additional technique will be required for time-domain integration. The discrete problem has its convergence proved analytically by invoking a theorem due to Lax & Richtmyer (1956) and another due to Irons & Treharne (1971).

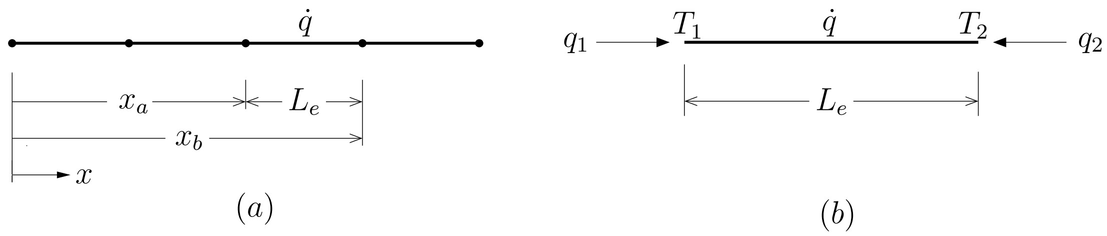

Figure 1: (a) Finite element mesh; (b) a typical two-node element

2 FINITE ELEMENT FORMULATION

Figure 1 shows the problem space domain divided into a number of finite elements, and a typical element of length L_e with endpoints at $x = x_a$ and $x = x_b$ isolated from the mesh. In order to develop the finite element for discretization in both space and time, Eq. (1) and Eq. (2) are restated in the integral form

$$\int_{x_a}^{x_b}\int_{t_n}^{t_{n+1}}\left(\frac{\partial T}{\partial x}+\frac{q''}{k}\right)\delta q''\,dx\,dt = 0$$

$$\int_{x_a}^{x_b}\int_{t_n}^{t_{n+1}}\left(\rho c\frac{\partial T}{\partial t}+\frac{\partial q''}{\partial x}-\dot{q}\right)\delta T\,dx\,dt = 0 \tag{4}$$

where δT and $\delta q''$ are the first variations of the temperature and heat flow. Just as for the length $L_e = x_b - x_a$ in the x direction shown in the figure, the element has also a length of $\Delta t = t_{n+1} - t_n$ in the t direction.

In view of the integration by parts

$$\int_{x_a}^{x_b}\frac{\partial q''}{\partial x}\delta T\,dx = q''\delta T|_{x_a}^{x_b} - \int_{x_a}^{x_b} q''\frac{\partial\,\delta T}{\partial x}dx, \tag{5}$$

the second of Eq. (4) becomes

$$\int_{x_a}^{x_b}\int_{t_n}^{t_{n+1}}\left(\rho c\frac{\partial T}{\partial t}\delta T - q''\frac{\partial\,\delta T}{\partial x} - \dot{q}\,\delta T\right)dx\,dt - \int_{t_n}^{t_{n+1}}\left(q_1\delta T(x_a) + q_2\delta T(x_b)\right)dt = 0 \tag{6}$$

where $q_1 = q''(x_a)$ and $q_2 = -q''(x_b)$ are the heat flow into the element at the nodes. The first of Eq. (4) and Eq. (6) will be used for the finite element discretization in space and time.

The temperature will be sought in the form

$$T(x,t) = T_1(t)\phi_1(x) + T_2(t)\phi_2(x), \tag{7}$$

with the variables x and t separated. The usual linear interpolation functions

$$\phi_1(x) = \frac{x_b - x}{L_e} \qquad \phi_2(x) = \frac{x - x_a}{L_e} \tag{8}$$

will be adopted in space, and the nodal values T_i will also be linearly interpolated in time:

$$T_1(t) = \frac{t_{n+1}-t}{\Delta t}T_1^0 + \frac{t-t_n}{\Delta t}T_1^{\Delta t} \qquad T_2(t) = \frac{t_{n+1}-t}{\Delta t}T_2^0 + \frac{t-t_n}{\Delta t}T_2^{\Delta t}. \tag{9}$$

The temperature will be continuous between elements in space and time, with nodal values T_1^0, T_2^0 at time t_n and $T_1^{\Delta t}$, $T_2^{\Delta t}$ at time t_{n+1}.

Just as for the temperature, the heat flow will also be sought in the form

$$q''(x,t) = q_1''(t)\phi_1(x) + q_2''(t)\phi_2(x) \tag{10}$$

varying linearly in both space and time:

$$q_1''(t) = \frac{t_{n+1}-t}{\Delta t} q_1''^0 + \frac{t-t_n}{\Delta t} q_1''^{\Delta t} \qquad q_2''(t) = \frac{t_{n+1}-t}{\Delta t} q_2''^0 + \frac{t-t_n}{\Delta t} q_2''^{\Delta t}. \tag{11}$$

The heat flow will be continuous between elements in space and time, with nodal values $q_1''^0$, $q_2''^0$ at time t_n and $q_1''^{\Delta t}$, $q_2''^{\Delta t}$ at time t_{n+1}.

In order to develop a step-by-step approach for the solution in the t direction, the temperature and heat flow are supposed to be known at time t_n ($\delta T_1^0 = \delta T_2^0 = \delta q_1''^0 = \delta q_2''^0 = 0$) and to be determined at time t_{n+1} ($\delta T_1^{\Delta t}$, $\delta T_2^{\Delta t}, \delta q_1''^{\Delta t}, \delta q_2''^{\Delta t}$ are arbitrary). Substitution of Eq. (7) and Eq. (10) into the first of Eq. (4) and Eq. (6) yields the element equations

$$\begin{bmatrix} \frac{\rho c L_e}{12}\mathbf{A} & \frac{\Delta t}{6}\mathbf{B} \\ \frac{\Delta t}{6}\mathbf{B}^T & -\frac{L_e \Delta t}{18k}\mathbf{A} \end{bmatrix} \begin{Bmatrix} T_1^{\Delta t} \\ T_2^{\Delta t} \\ q_1''^{\Delta t} \\ q_2''^{\Delta t} \end{Bmatrix} = \begin{bmatrix} \frac{\rho c L_e}{12}\mathbf{A} & -\frac{\Delta t}{12}\mathbf{B} \\ -\frac{\Delta t}{12}\mathbf{B}^T & \frac{L_e \Delta t}{36k}\mathbf{A} \end{bmatrix} \begin{Bmatrix} T_1^0 \\ T_2^0 \\ q_1''^0 \\ q_2''^0 \end{Bmatrix} + q \begin{Bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \\ 0 \\ 0 \end{Bmatrix} \tag{12}$$

where

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} (t-t_n)\, dt \qquad q = \frac{L_e}{2\,\Delta t}\int_{t_n}^{t_{n+1}} \dot{q}\,(t-t_n)\, dt. \tag{13}$$

For simplicity, $\dot{q}$ has been assumed uniformly distributed along the space length of the element.

3 AMPLIFICATION MATRIX

According to (Zienkiewicz et al., 2005), the concept of amplification matrix is applied to Eq. (12) by doing $q = Q_i = 0$. In this case, the equations reduce to

$$\begin{Bmatrix} T_1^{\Delta t} \\ T_2^{\Delta t} \\ q_1''^{\Delta t} \\ q_2''^{\Delta t} \end{Bmatrix} = \mathbf{k} \begin{Bmatrix} T_1^0 \\ T_2^0 \\ q_1''^0 \\ q_2''^0 \end{Bmatrix} \tag{14}$$

where

$$\mathbf{k} = \begin{bmatrix} \dfrac{\rho c L_e}{12}\mathbf{A} & \dfrac{\Delta t}{6}\mathbf{B} \\ \dfrac{\Delta t}{6}\mathbf{B}^T & -\dfrac{L_e \Delta t}{18k}\mathbf{A} \end{bmatrix}^{-1} \begin{bmatrix} \dfrac{\rho c L_e}{12}\mathbf{A} & -\dfrac{\Delta t}{12}\mathbf{B} \\ -\dfrac{\Delta t}{12}\mathbf{B}^T & \dfrac{L_e \Delta t}{36k}\mathbf{A} \end{bmatrix}. \tag{15}$$

The same concept can also be extended to the global equations

$$\left\{T^{\Delta t}\right\} = \mathbf{K}\left\{T^0\right\}. \tag{16}$$

The usual discussion on stability of any numerical formulation takes for granted that the numerical scheme will be stable if all eigenvalues λ_i of the global matrix $\mathbf{K}$ are such that $|\lambda_i| \leq 1$. In fact, the rigorous discussion of stability has small subtleties, involving negative and complex eigenvalues. As time increments go on (Ramis, 2015), eigenvectors associated with:

- $|\lambda_i| < 1$ converge to zero;
- $|\lambda_i| > 1$ diverge;
- $\lambda_i = 1$ do not change;
- $|\lambda_i| = 1$ (not including $\lambda_i = 1$) oscilate *ad æternum* without diverging.

Since our formulation deals with real symmetric matrices, for which all the eigenvalues are real and the eigenvectors are mutually orthogonal (Strang, 2009), the fourth condition stated above becomes $\lambda_i = -1$.

The eigenvectors $\{u_i\}$ of the global matrix are clearly a basis for the nodal value vector space. Hence, the vector $\{T^0\}$ of nodal values (temperature and heat flow) at the initial condition can be decomposed into the linear combination

$$\left\{T^0\right\} = \sum a_i \, \{u_i\} \tag{17}$$

where a_i are the weights. With reference to Eq. (16) and the step-by-step approach, the nodal value profile $\left\{T^{n\,\Delta t}\right\}$ at any instant $t = n\Delta t$ can be evaluated by means of

$$\begin{aligned} \left\{T^{n\Delta t}\right\} &= \mathbf{K}^n \left\{T^0\right\} \\ &= \sum_i a_i \, \mathbf{K}^n \{u_i\} \\ &= \sum_i a_i \, \lambda_i^n \{u_i\} \end{aligned} \tag{18}$$

where $\mathbf{K}^n = \mathbf{K}\,\mathbf{K}\cdots\mathbf{K}$. Hence, the multiplier of the each initial eigenvector component $a_i \, \{u_i\}$ of $\left\{T^{n\,\Delta t}\right\}$ is simply λ_i^n and the previous discussion on the convergence becomes obvious, as the evolution of each eigenvector component $a_i \, \lambda_i^n \{u_i\}$ in time may be treated as a geometric progression with ratio λ_i.

The special case of $\lambda_i = 1$ is important because it allows the existence of non-homogeneous (constant in time) boundary conditions. For this reason, this case will always be present in most finite element global matrices before the elimination of variables caused by Dirichlet boundary conditions.

According to the second law of thermodynamics, all systems without heat source should converge to thermal equilibrium. Hence, as time increments go on, homogeneous heat conduction should converge to finite values, and divergence or oscillation *ad æternum* should not take place. In other words, all eigenvalues of the global matrices in heat conduction should be such that $|\lambda_i| < 1$, except for the special case of $\lambda_i = 1$.

4 THE EIGENVALUE BOUND THEOREM

According to Irons & Treharne (1971), this theorem is due to Tong (1971) and uses the concept of the Rayleigh quotient

$$\rho\left(\{T\}\right) = \frac{\{T\}^T \mathbf{K} \{T\}}{\{T\}^T \{T\}} \tag{19}$$

for symmetric matrices. As a vector $\{T\}$ of nodal values may be described as a linear combination of the eigenvectors $\{u_i\}$, from Eq. (17) one writes

$$\rho\left(\{T\}\right) = \frac{\sum a_i^2 \lambda_i \{u_i\}^T \{u_i\}}{\sum a_i^2 \{u_i\}^T \{u_i\}} \tag{20}$$

taking into account that $\mathbf{K}\{u_i\} = \lambda_i \{u_i\}$ and the eigenvectors are mutually orthogonal. Since $\{u_i\}^T \{u_i\} > 0$, one can see that $\rho\left(\{T\}\right)$ is a linear combination of the eigenvalues with non-negative weights whose sum is unity.

Defining the element matrices in Eq. (12) by

$$\mathbf{k^{\Delta t}} = \begin{bmatrix} \dfrac{\rho c L_e}{12}\mathbf{A} & \dfrac{\Delta t}{6}\mathbf{B} \\ \dfrac{\Delta t}{6}\mathbf{B}^T & -\dfrac{L_e \Delta t}{18k}\mathbf{A} \end{bmatrix} \qquad \mathbf{k^0} = \begin{bmatrix} \dfrac{\rho c L_e}{12}\mathbf{A} & -\dfrac{\Delta t}{12}\mathbf{B} \\ -\dfrac{\Delta t}{12}\mathbf{B}^T & \dfrac{L_e \Delta t}{36k}\mathbf{A} \end{bmatrix} \tag{21}$$

and their global counterparts by $\mathbf{K^{\Delta t}}$ and $\mathbf{K^0}$, the Rayleigh coefficient for a generalised eigenvalue problem writes

$$\rho\left(\{T\}\right) = \frac{\{T\}^T \mathbf{K^0} \{T\}}{\{T\}^T \mathbf{K^{\Delta t}} \{T\}}. \tag{22}$$

If $\mathbf{K^{\Delta t}}$ is positive definite (has only positive eigenvalues), it is assured that all the previous reasoning still hold and the eigenvalues of the amplification matrix form a convex hull for $\rho\left(\{T\}\right)$.

This closes the proof that $\rho\left(\{T\}\right)$ always lies within the convex hull of the global matrix eigenvalues, that is, the Rayleigh quotient is always bounded by the eigenvalues of $\mathbf{K}$.

Locally, all the previous reasoning is also valid for each element: every vector $\{T\}$ may be described as a linear combination of the element eigenvectors $\{u_k^e\}$, in a similar way to Eq.(17):

$$\left\{T^0\right\} = \sum c_i \{u_i^e\}. \tag{23}$$

The decoupling of the global matrix as a sum of the element matrices makes clear that the Rayleigh quotient is also bounded by the eigenvalues λ_k^e of all elements:

$$\rho(\{T\}) = \frac{\sum c_i^2 \lambda_i^e \{u_i^e\}^T \{u_i^e\}}{\sum c_i^2 \{u_i^e\}^T \{u_i^e\}}. \tag{24}$$

The requirement of temperature continuity between elements imposes additional restrictions to the multipliers c_k so that the convex hull of the global eigenvalues is a restricted subset of the convex hull of the eigenvalues of any element. Hence, any eigenvalue of the global matrix is in the interval defined by the minimum and maximum eigenvalues of any element:

$$\min \lambda_k^e \leq \lambda_k \leq \max \lambda_k^e. \tag{25}$$

Irons & Treharne (1971) say that this theorem is "familiar but underevalued" and, in fact, it is seldom used in the proof of convergence of finite element formulations. For some unknown reason, Zienkiewicz et al. (2005) state the theorem as if it were valid to the squared eigenvalues:

$$\min \lambda_k^{e2} \leq \lambda_k^2 \leq \max \lambda_k^{e2}. \tag{26}$$

The subsequent edition of this book (Zienkiewicz et al., 2013) simply omits the theorem and focuses on the time domain with usual techniques for second-order time derivatives, following many other major textbooks on finite element method.

5 APPLICATION TO THE MIXED ELEMENT

The element developed in Section 2 is now taken as a didactic example of the convergence analysis proposed in the previous section. To do so, the dimensionless parameter

$$\Theta = \frac{k\,t}{L_e^2 \rho c} \tag{27}$$

is defined. It can be verified that the amplification matrix $\mathbf{k}$ in Eq. (15) has eigenvalues

$$\lambda_1^e = 1 \qquad \lambda_2^e = \frac{1-4\Theta}{1+8\Theta} \qquad \lambda_3^e = \lambda_4^e = -\frac{1}{2}. \tag{28}$$

One can see that λ_1^e is related to the steady state, and has eigenvector $\{1,1,0,0\}^T$. One can also see that λ_2^e will always be in the interval $]-1/2, 1]$ for $t > 0$. The matrix $\mathbf{k}^{\Delta \mathbf{t}}$ has only one eigenvalue negative $L_e \Delta t / 18k$, which is related to the eigenvector $\{0,0,1,-1\}^T$, which has no physical meaning as initial condition. Hence, it should be not present in practical problems and the convex hull of all eigenvalues may be warranted.

As all eigenvalues are real, they constitute bounds for the eigenvalues of any global matrix obtained from a mesh of this element. Even in the cases of heterogeneous media or meshes with different element sizes, the minimum and maximum eigenvalues of the global amplification matrix will always be within the interval $]-1/2, 1]$ and the modelling of a heat conduction problem with this element will always be stable and convergent.

6 NUMERICAL EXAMPLE

Consider a bar of length $L = 1$ m composed of two materials. In the region $0 < x < 1/2$, there is a material with $k_1 = 1$ kW m^{-1} °C $^{-1}$ and $\rho_1 c_1 = 1$ kJ m^{-3} °C^{-1}, while in the region $1/2 < x < 1$ there is another material with $k_2 = 0.25$ kW m^{-1} °C^{-1}and $\rho_2 c_2 = 1$ kJ m^{-3} °C^{-1}. The boundary conditions are

$$T(0,t) = 0 \qquad q(1,t) = 0 \qquad\qquad t > 0 \tag{29}$$

and the initial condition is

$$T(x,0) = 1\text{°C} \qquad\qquad 0 < x < 1. \tag{30}$$

The exact solution of Eq. (3) subjected to Eq. (29) and Eq. (29) is found in Queiroz & Vidal (2008) as being

$$\begin{aligned} T(x,t) \;&=\; \sum_{i=1}^{\infty} \left[\frac{2}{\lambda_i''} \sin(\lambda_i'' x) e^{-\lambda_i''^2 t} + \frac{1}{\lambda_i'} \sin(\lambda_i' x) e^{-\lambda_i'^2 t} + \frac{1}{\lambda_i} \sin(\lambda_i x) e^{-\lambda_i^2 t} \right] \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \le x \le \frac{1}{2} \\ T(x,t) \;&=\; \sum_{i=1}^{\infty} (-1)^i \left\{ \frac{2}{\lambda_i''} \cos\left[2\lambda_i''(1-x)\right] e^{-\lambda_i''^2 t} - \frac{\sqrt{3}}{\lambda_i'} \cos\left[2\lambda_i'(1-x)\right] e^{-\lambda_i'^2 t} \right. \\ &\qquad\qquad \left. - \frac{\sqrt{3}}{\lambda_i} \cos\left[2\lambda_i(1-x)\right] e^{-\lambda_i^2 t} \right\} \qquad\qquad \frac{1}{2} \le x \le 1 \end{aligned} \tag{31}$$

for $t \ge 0$, where

$$\lambda_i = 2\left[\pi(i-1) + \tan^{-1}\frac{\sqrt{2}}{2}\right] \qquad \lambda_i' = 2\left(\pi i - \tan^{-1}\frac{\sqrt{2}}{2}\right) \qquad \lambda_i'' = \pi(2i-1) \tag{32}$$

In order to verify the element performance, a small mesh with twelve elements in space ($L_e = 1/12$) and $\Delta t = 0.02$ is used to solve the problem for ten increments in time direction.

Figure 2 shows the results for Δt, $4\Delta t$ and $10\Delta t$. For each time instant, the error

$$e = \sqrt{\frac{\displaystyle\int_0^L [\tilde{T}(x,t_i) - T(x,t_i)]^2 \mathrm{d}x}{\displaystyle\int_0^L T^2(x,t_i)\mathrm{d}x}} \tag{33}$$

based on a normalized Lebesgue norm is presented. The parameter $L = 1$ stands for the length of the bar, $\tilde{T}$ the finite element approximation and T the exact solution. It can be seen that, despite of the coarse mesh, few time increments are enough to provide a moderately accurate results for $t = 0.08$ and $t = 0.2$.

Figure 3 shows the results of heat flow, for the same time steps of Fig. 2. The results for $t = 0.02$ and $x = 0$ are evidently less accurate due to the proximity of the singularity in $t = x = 0$, that is, the sudden change of temperature from the initial condition ($T = 1°C$) to the boundary condition ($T = 0°C$).

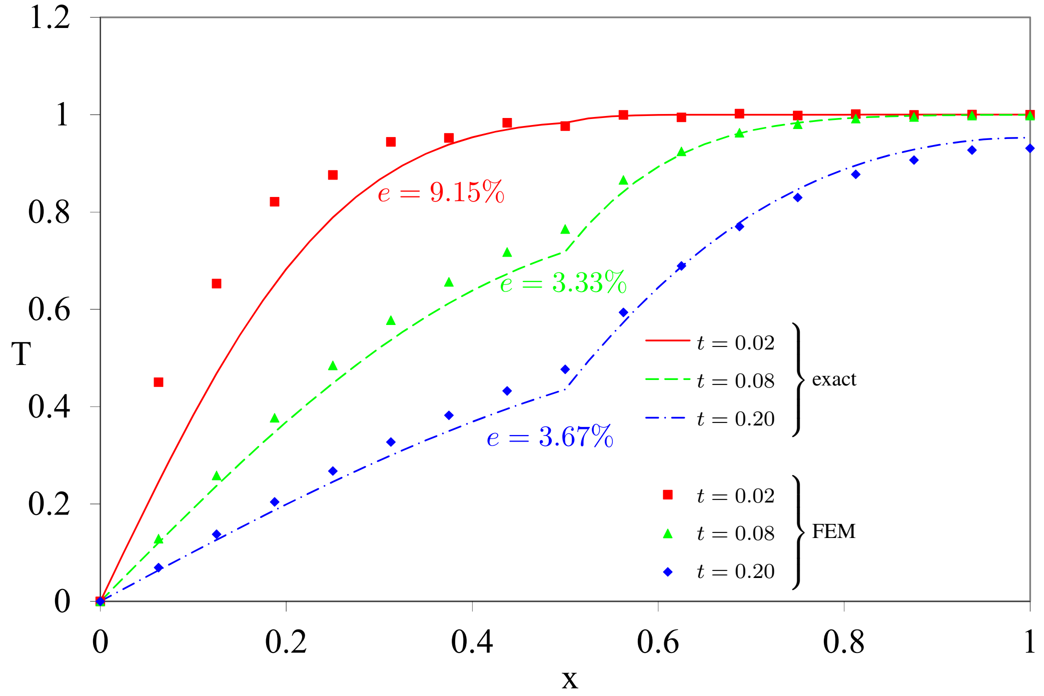

Figure 2: Distribution of temperature obtained with 12 elements in space and $\Delta t = 0.02$

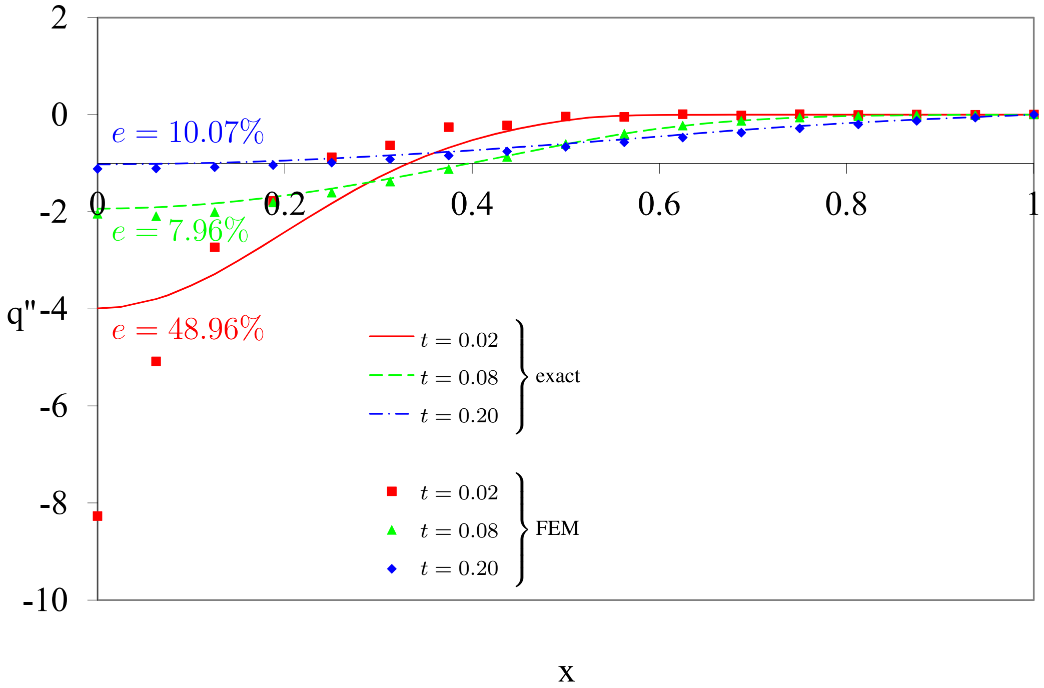

Figure 3: Heat flow profiles obtained with 12 elements in space and $\Delta t = 0.02$

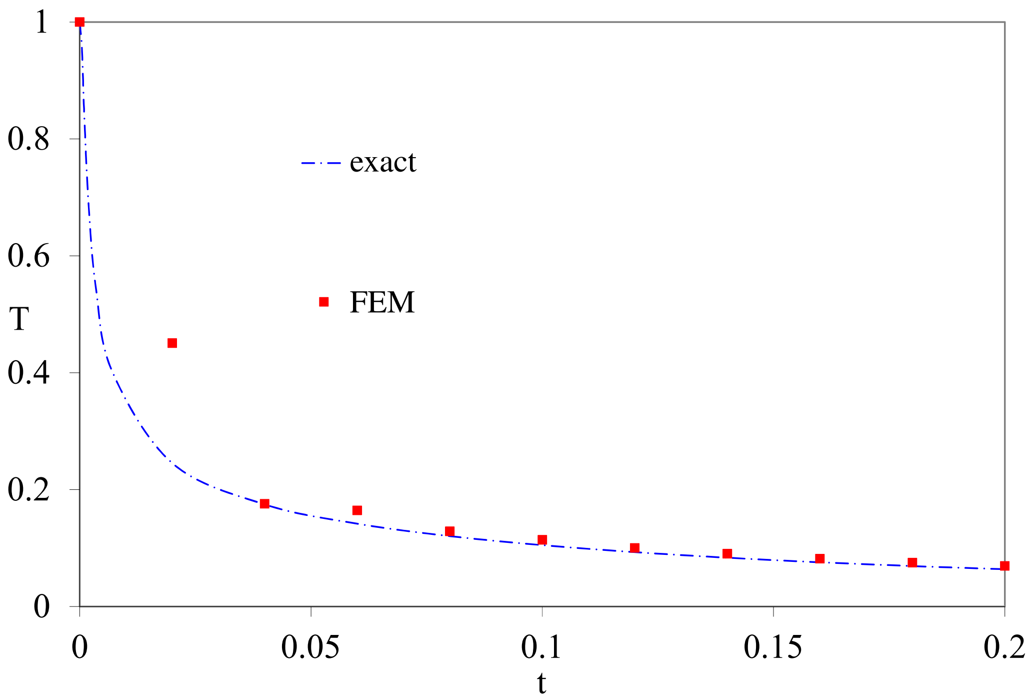

Figure 4: Temperature calculated at the node closest to the non-isolated edge ($x = 0.0625$**) along ten time increments (**$\Delta t, 2\Delta t, \ldots, 10\Delta t$ **with** $\Delta t = 0.02$**)**

Figures 4 and 5 show respectively the temperature and heat flow at the second node of the mesh, near to the non-isolated edge of the bar, which is next to the singularity point $x = t = 0$. It is clear that the diffusion equation has the property of "smoothing" the solution, which becomes very accurate after the third time increment.

Figures 6 and 7 show respectively the temperature and heat flow at $x = 0.5$, the transition between two different materials. In this case, although the temperatures in the model are reasonably accurate, heat flow indicates a necessity of mesh refinement, in order to improve the accuracy.

7 CONCLUDING REMARKS

This paper presented an eigenvalue bound theorem as a tool for the analytical proof of convergence of finite element formulations. A simple 1-D element for heat conduction was assessed and proved to be unconditionally convergent. A practical assessment of the element performance was carried out by comparing its numerical results with an exact solution for transient conduction in heterogeneous media. The results were in reasonable agreement, despite the fact that a coarse mesh and few time increments were used. Future works related to this development include an atempt to extend this theorem for unsymmetric matrices.

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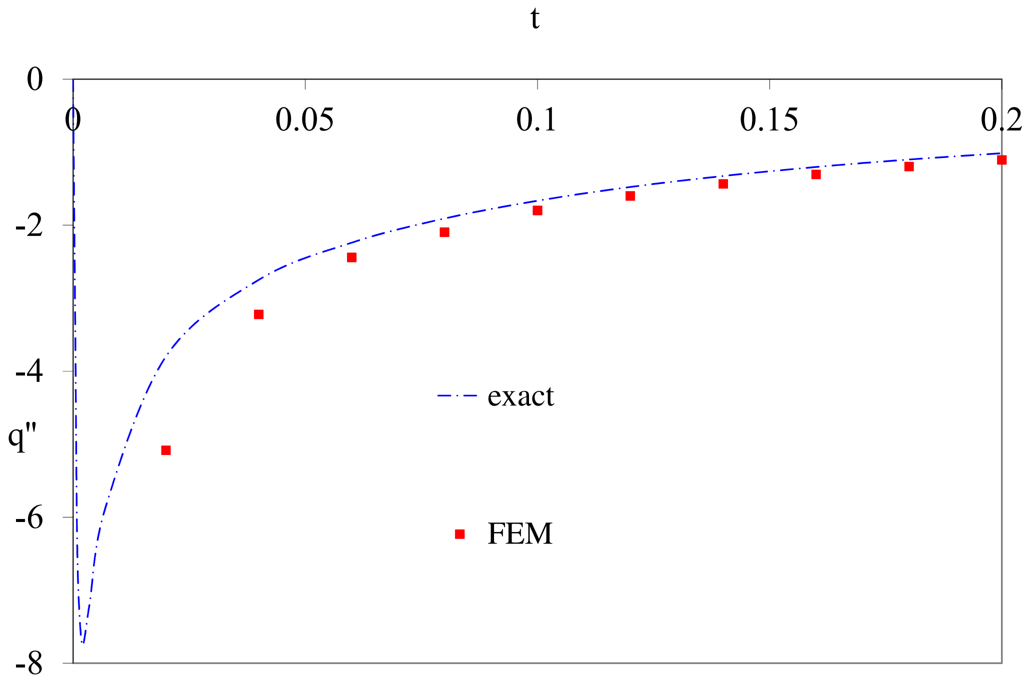

Figure 5: Heat flow calculated at the node closest to the non-isolated edge ($x = 0.0625$) along ten time increments ($\Delta t, 2\Delta t, \ldots, 10\Delta t$ with $\Delta t = 0.02$)

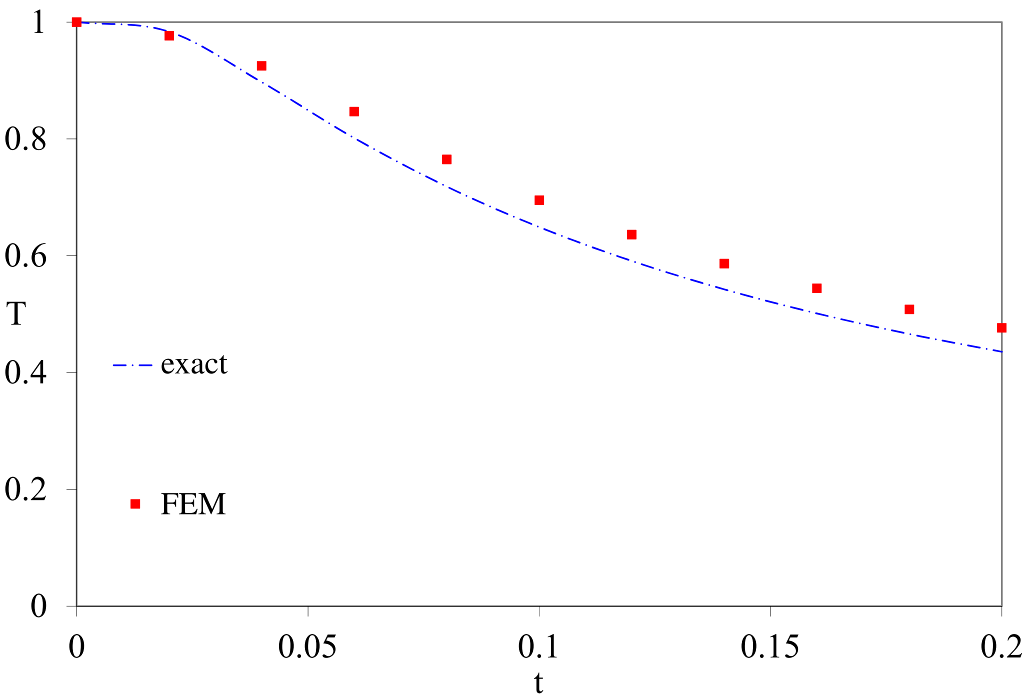

Figure 6: Temperature calculated at the interface between two materials ($x = 0.5$) along ten time increments ($\Delta t, 2\Delta t, \ldots, 10\Delta t$ with $\Delta t = 0.02$)

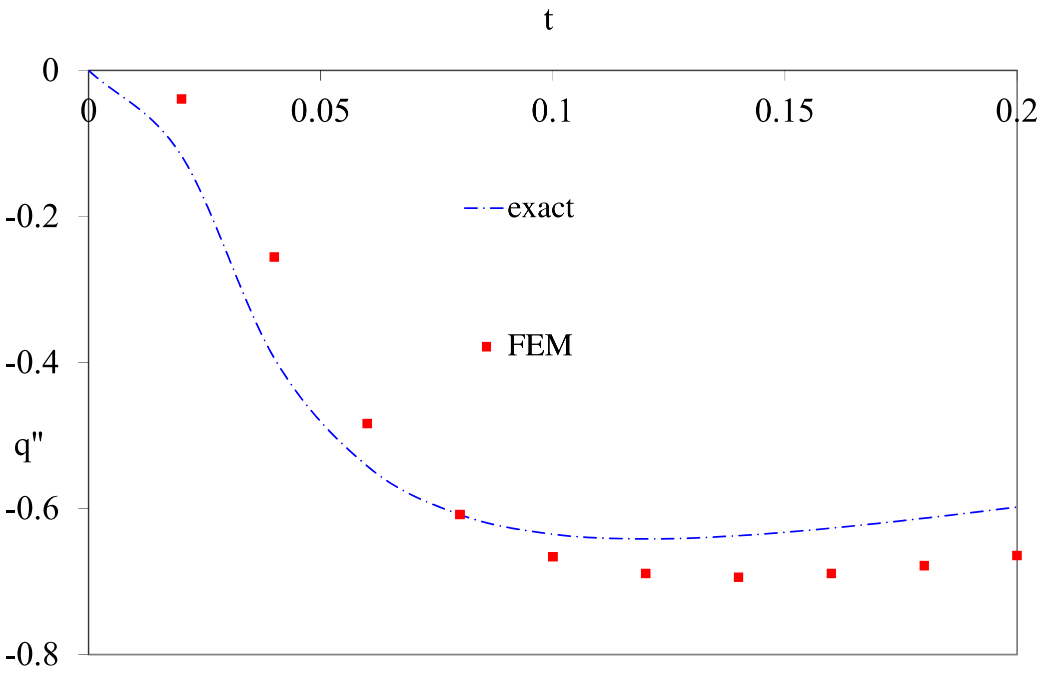

Figure 7: Heat flow calculated at the interface between two materials ($x = 0.5$) along ten time increments ($\Delta t, 2\Delta t, \ldots, 10\Delta t$ with $\Delta t = 0.02$)

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