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# THE TIME-DEPENDENT BOUNDARY ELEMENT METHOD FORMULATION APPLIED TO DYNAMIC ANALYSIS OF EULER-BERNOULLI BEAMS: THE LINEAR $\theta$ METHOD

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***Abstract.*** *In this paper, the dynamic analysis of Euler-Bernoulli beams is performed with the Time-Dependent Boundary Element Method Formulation (TD-BEM). In the standard formulation, the variables related to the essential boundary conditions (displacement and rotation) are assumed to vary linearly in time, i.e., within each time interval, whereas the variables related to the natural boundary conditions (shear force and bending moment) are assumed to have a constant time variation. Different hypothesis concerning the time behaviour of these quantities lead to unstable and inaccurate results. In the linear $\theta$ method, on the other hand, all the variables are assumed to have a linear time variation and reliable results are achieved. These results can be seen in the examples presented in the article, which contains the four usual types of beams under the two more frequently found types of loading.*

***Keywords:*** *Time-dependent BEM, Dynamic analysis, Euler-Bernoulli beams.*

# 1 INTRODUCTION

The Euler-Bernoulli beam theory, also called classic beam theory, is very simple. Its basic hypothesis, according to which plane sections remain plane and perpendicular to the neutral axis after bending, if brought some limitations to the use of the theory, on one hand, on the other gave the designer a quite useful tool for usual projects. Simplicity does not mean that a theory must be neglected. The proof to this assertion is its presence, until today, in several textbooks concerning the discipline of Strength/Mechanics of Materials [see Beer (2015) and Hibbeler (2011)]. When the time appears in the analysis, that is, when the researcher is concerned with dynamic analysis of beams, the matter has also deserved attention for quite some time. For a more complete discussion concerning this matter, the reader is referred to the textbooks by Graff (1991) and Rao (2011), for instance. The solution of the dynamic problem with the Time-Domain Boundary Element Method (TD-BEM) is a matter that brings the researcher some difficulties. They begin with the choice of the fundamental solution that is, according to the authors' opinion, a not trivial one. This fundamental solution can be find, for instance, in Campbell and Foster (1961) and Kythe (1996). A very interesting article, that must be cited, is that due to Sheehan and Debnath (1972). Beside this difficulty, another one, that appeared after the TD-BEM equations were obtained, deserved attention; it can be formulated as: what is the best assumption for the time-behaviour of the variables (displacement, rotation, bending moment and shear force) involved in the analysis?

When began their work, the authors were faced with another challenge: to prove that the TD-BEM formulation was capable of providing accurate results and, at a certain point, to demonstrate that the conclusion found at the work by Langre, Axisa and Guilbaud (1990) is no longer valid. In Langre, Axisa and Guilbaud (1990), the reader can find the following statement: "It is nevertheless clear that the Laplace transform domain method is the most adequate for the solution of linear problems of flexural vibrations of beams". Scuciato (2014) developed a TD-BEM formulation which led to another and completely different conclusion. Such a TD-BEM formulation was developed under the assumption that, at each time interval $\Delta t$, the displacement and the rotation have linear time variation whereas the bending moment and the shear force have constant time variation, and was capable of providing accurate results related not only to the displacements but also to the rotation, bending moment and shear force. It is important to note that the displacement and the rotation are associated with the essential boundary conditions; the bending moment and the shear force, with the natural boundary conditions. Consequently, such a choice concerning the behaviour of the variables follows the recommendation presented by Mansur (1983). In his thesis, Mansur (1983) recommended the use of linear time interpolation functions for the potential (associate to the essential boundary conditions) and constant time interpolation functions for the potential normal derivative (associated to the natural boundary conditions) in the TD-BEM formulation for the scalar wave equation. The reasoning, in Mansur (1983), was that the continuity brought by the linear functions was not desirable for problems with discontinuous potential normal derivative.

The present work is concerned with the presentation of a different version of the TD-BEM formulation. This version is based on the application of a linear $\theta$ method called, from now on, $\theta$-Yu method, which assumes a linear time variation to the displacement, rotation, bending moment and shear force in the time interval $\theta\Delta t$. Although there is no proof that the $\theta$-Yu method is unconditionally stable, stable and accurate results were obtained. The reader is referred to Yu et al. (1998) for additional details concerning the method and its application to the solution of the scalar wave propagation problems. Naturally, care should be taken in the choice of the $\theta$ parameter. In the examples studied, constant cross-section is assumed for the usual kinds of beams, that is: pinned-pinned (PP), clamped-pinned (CP), clamped-clamped (CC) and clamped-free (CF), named, respectively, according to their boundary conditions. For each kind of beam, two types of loading are taken into account: the first one is assumed to be linearly distributed along the domain and to act continuously in time (DD); the second type consists of a concentrated load acting continuously in time (CD). All the numerical results are compared with the corresponding analytical solutions. The good agreement observed between them demonstrates the applicability of the present TD-BEM formulation. The effect of varying $\theta$ in the results is also studied and presented.

# 2 EULER-BERNOULLI BEAM THEORY

## 2.1 Governing Equation

Dynamic bending problems of uniform slender beams are governed by the classical Euler-Bernoulli equation

$$EI\frac{\partial^4 u}{\partial x^4} + \rho A\frac{\partial^2 u}{\partial t^2} = f(x,t), \tag{1}$$

where $u(x,t)$ is the vertical deflection of the beam, $f(x,t)$ is the applied external loading, $EI$ is the flexural rigidity of the beam, $\rho$ is the density of the material, $A$ is the cross-sectional area, $L$ is the length of the beam, $x$ is the spatial variable, and $t$ denotes time. Figure 1 shows an schematic layout of the beam under consideration.

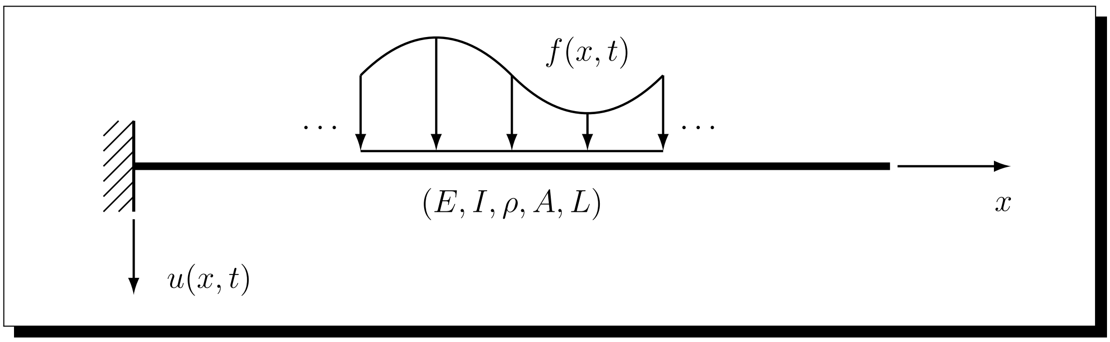


**Figure 1: Schematic layout of the beam under consideration**

Dividing Eq. (1) by $\rho A$ and defining $c = \sqrt{(EI)/(\rho A)}$ one gets the expression

$$c^2\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = \frac{f(x,t)}{\rho A}, \tag{2}$$

which is the governing equation adopted in this work.

From $u(x,t)$ the following quantities are defined:

$$\phi(x,t) = \frac{\partial u}{\partial x}, \quad M(x,t) = -EI\frac{\partial^2 u}{\partial x^2}, \quad Q(x,t) = -EI\frac{\partial^3 u}{\partial x^3}, \quad v(x,t) = \frac{\partial u}{\partial t}, \tag{3}$$

called, respectively, rotation, bending moment, shear force, and velocity.

## 2.2 Analytical Solution

The analytical solution of Eq. (2) can be written as

$$u(x,t) = \sum_{n=1}^{\infty} W_n(x) q_n(t), \tag{4}$$

where $W_n(x)$ is the $n$-th normal mode and $q_n(t)$ is the $n$-th generalized coordinate.

The generalized coordinates, taking into account null initial conditions, are given by

$$q_n(t) = \frac{1}{\rho A b_n \omega_n} \int_0^t F_n(\tau) \sin \omega_n (t-\tau)\, \mathrm{d}\tau, \tag{5}$$

where

$$b_n = \int_0^L W_n^2(x)\, \mathrm{d}x, \qquad \omega_n = \beta_n^2 c, \qquad \text{and} \qquad F_n(\tau) = \int_0^L W_n(x) f(x,\tau)\, \mathrm{d}x. \tag{6}$$

In Eq. (5) and (6), $\omega_n$ represents the $n$-th natural frequency.

Expressions for the normal modes $W_n(x)$ and for the coefficients $\beta_n$, for the beams analysed in this work, are given bellow:

a) PP-beam:

$$W_n(x) = \sin \beta_n x; \qquad \beta_n = \frac{n\pi}{L}; \tag{7}$$

b) CP-beam:

$$\begin{aligned} &W_n(x) = (\sin \beta_n x - \sinh \beta_n x) - \alpha_n (\cos \beta_n x - \cosh \beta_n x); \\ &\alpha_n = \frac{\sin \beta_n L - \sinh \beta_n L}{\cos \beta_n L - \cosh \beta_n L}; \qquad \tan \beta_n L = \tanh \beta_n L; \end{aligned} \tag{8}$$

c) CC-beam:

$$\begin{aligned} &W_n(x) = (\sinh \beta_n x - \sin \beta_n x) - \alpha_n (\cosh \beta_n x - \cos \beta_n x); \\ &\alpha_n = \frac{\sinh \beta_n L - \sin \beta_n L}{\cosh \beta_n L - \cos \beta_n L}; \qquad \cos \beta_n L \cosh \beta_n L = +1; \end{aligned} \tag{9}$$

d) CF-beam:

$$\begin{aligned} &W_n(x) = (\sin \beta_n x - \sinh \beta_n x) - \alpha_n (\cos \beta_n x - \cosh \beta_n x); \\ &\alpha_n = \frac{\sin \beta_n L + \sinh \beta_n L}{\cos \beta_n L + \cosh \beta_n L}; \qquad \cos \beta_n L \cosh \beta_n L = -1. \end{aligned} \tag{10}$$

More details can be found, for instance, in Rao (2011).

# 3 MATHEMATICAL FORMULATION

## 3.1 TD-BEM Formulation

The basic TD-BEM equation can be obtained from a weighted residuals statement; it reads [see Scuciato (2014) for a complete discussion concerning this matter]:

$$\begin{aligned} u(\xi,t) = &-\frac{1}{\rho A}\int_0^t \Big[+u^*Q-\phi^*M+M^*\phi-Q^*u\Big]_{x=0}\mathrm{d}\tau \\ &+\frac{1}{\rho A}\int_0^t \Big[+u^*Q-\phi^*M+M^*\phi-Q^*u\Big]_{x=L}\mathrm{d}\tau \\ &+\frac{1}{\rho A}\int_0^t\int_0^L u^*f\,\mathrm{d}x\,\mathrm{d}\tau. \end{aligned} \tag{11}$$

The fundamental solution, that is, the solution of the equation

$$c^2\frac{\partial^4 u^*}{\partial x^4}+\frac{\partial^2 u^*}{\partial t^2}=\delta(x-\xi)\delta(t-\tau), \tag{12}$$

has the following expression [see Campbell and Foster (1961) and Kythe (1996)]:

$$u^*(x,\xi,t,\tau)=\frac{1}{c}\left\{\frac{r}{2}\left[\mathrm{S}\left(\frac{r}{\sqrt{2\pi a}}\right)-\mathrm{C}\left(\frac{r}{\sqrt{2\pi a}}\right)\right]+\frac{\sqrt{a}}{\sqrt{2\pi}}\left[\sin\left(\frac{r^2}{4a}\right)+\cos\left(\frac{r^2}{4a}\right)\right]\right\}. \tag{13}$$

In Eq. (13), $r$ is the distance between the field, $x$, and the source, $\xi$, points and $a=c(t-\tau)$. The Fresnel integrals S and C are defined according to

$$\mathrm{S}(z)=\int_0^z \sin\left(\frac{\pi}{2}\zeta^2\right)\mathrm{d}\zeta, \qquad \text{and} \qquad \mathrm{C}(z)=\int_0^z \cos\left(\frac{\pi}{2}\zeta^2\right)\mathrm{d}\zeta. \tag{14}$$

The quantities $\phi^*$, $M^*$, and $Q^*$ are obtained from Eq. (13) and Eq. (3). One has:

$$\phi^*(x,\xi,t,\tau)=+\frac{1}{c}\left\{\frac{1}{2}\left[\mathrm{S}\left(\frac{r}{\sqrt{2\pi a}}\right)-\mathrm{C}\left(\frac{r}{\sqrt{2\pi a}}\right)\right]\right\}\left(\frac{\partial r}{\partial x}\right), \tag{15}$$

$$M^*(x,\xi,t,\tau)=-\frac{EI}{c}\left\{\frac{1}{2\sqrt{2\pi a}}\left[\sin\left(\frac{r^2}{4a}\right)-\cos\left(\frac{r^2}{4a}\right)\right]\right\}\left(\frac{\partial r}{\partial x}\right)^2, \tag{16}$$

$$Q^*(x,\xi,t,\tau)=-\frac{EI}{c}\left\{\frac{r}{4\sqrt{2\pi a^3}}\left[\sin\left(\frac{r^2}{4a}\right)+\cos\left(\frac{r^2}{4a}\right)\right]\right\}\left(\frac{\partial r}{\partial x}\right)^3. \tag{17}$$

It is important to mention that the solution of the problem requires an additional equation, as the problem presents, at each boundary node (as matter of fact, the two boundary nodes at $x=0$ and $x=L$ constitute the boundary of the problem), four variables of which only two are known from the boundary conditions. The chosen additional equation is that related to the rotation, obtained after deriving Eq. (11) with respect to the source point. It reads:

$$\begin{aligned} \phi(\xi,t) = &-\frac{1}{\rho A}\frac{\partial}{\partial\xi}\left\{\int_0^t \Big[+u^*Q-\phi^*M+M^*\phi-Q^*u\Big]_{x=0}\mathrm{d}\tau\right\} \\ &+\frac{1}{\rho A}\frac{\partial}{\partial\xi}\left\{\int_0^t \Big[+u^*Q-\phi^*M+M^*\phi-Q^*u\Big]_{x=L}\mathrm{d}\tau\right\} \\ &+\frac{1}{\rho A}\frac{\partial}{\partial\xi}\left\{\int_0^t\int_0^L u^*f\,\mathrm{d}x\,\mathrm{d}\tau\right\}, \end{aligned} \tag{18}$$

It seems reasonable that such equation could be that related to the bending moment or that related to the shear force. In this work, however, these equations were employed only in the post-processing stage. They are written as follows:

$$\begin{aligned} M(\xi,t) = &+ c^2 \frac{\partial^2}{\partial \xi^2}\left\{ \int_0^t \Big[ + u^* Q - \phi^* M + M^* \phi - Q^* u \Big]_{x=0} \mathrm{d}\tau \right\} \\ &- c^2 \frac{\partial^2}{\partial \xi^2}\left\{ \int_0^t \Big[ + u^* Q - \phi^* M + M^* \phi - Q^* u \Big]_{x=L} \mathrm{d}\tau \right\} \\ &- c^2 \frac{\partial^2}{\partial \xi^2}\left\{ \int_0^t \int_0^L u^* f \,\mathrm{d}x \,\mathrm{d}\tau \right\}, \end{aligned} \tag{19}$$

and

$$\begin{aligned} Q(\xi,t) = &+ c^2 \frac{\partial^3}{\partial \xi^3}\left\{ \int_0^t \Big[ + u^* Q - \phi^* M + M^* \phi - Q^* u \Big]_{x=0} \mathrm{d}\tau \right\} \\ &- c^2 \frac{\partial^3}{\partial \xi^3}\left\{ \int_0^t \Big[ + u^* Q - \phi^* M + M^* \phi - Q^* u \Big]_{x=L} \mathrm{d}\tau \right\} \\ &- c^2 \frac{\partial^3}{\partial \xi^3}\left\{ \int_0^t \int_0^L u^* f \,\mathrm{d}x \,\mathrm{d}\tau \right\}, \end{aligned} \tag{20}$$

## 3.2 The $\theta$-Yu Method

Equations (11) and (18)–(20) constitute what could be called the standard TD-BEM formulation, under the assumption that, inside each time interval $\Delta t$, the displacement and the rotation are assumed to vary linearly, whereas the bending moment and the shear force present constant time variation. The standard TD-BEM formulation was capable of providing accurate results turning the conclusion of Langre, Axisa and Guilbaud (1990), mentioned earlier, no longer valid. At the Introduction it was mentioned that the assumption of linear time variation for a variable implies that this variable presents continuity along the time of analysis; the continuity, however, can be undesirable to the variables related to the natural boundary conditions (bending moment and shear force in the present study) and its imposition leads to very inaccurate results [see Scuciato (2014) and Mansur (1983)]. In spite of that, linear time variation, for all the variables, can be assumed with the help of the $\theta$-Yu method [see Yu et al. (1998)]. The method is very simple and its application to the standard TD-BEM formulation is quite straightforward: basically, a different last time step, say $\theta\Delta t = t_{n+\theta} - t_n$, $\theta > 1$, instead of $\Delta t = t_{n+1} - t_n$, must be used. The responses at $t_{n+1}$ and $t_{n+\theta}$ are related according to:

$$u^{n+1} = \left(\frac{1}{\theta}\right) u^{n+\theta} + \left(\frac{\theta-1}{\theta}\right) u^n, \tag{21}$$

$$\phi^{n+1} = \left(\frac{1}{\theta}\right) \phi^{n+\theta} + \left(\frac{\theta-1}{\theta}\right) \phi^n, \tag{22}$$

$$M^{n+1} = \left(\frac{1}{\theta}\right) M^{n+\theta} + \left(\frac{\theta-1}{\theta}\right) M^n, \tag{23}$$

$$Q^{n+1} = \left(\frac{1}{\theta}\right) Q^{n+\theta} + \left(\frac{\theta-1}{\theta}\right) Q^n. \tag{24}$$

The above expressions can easily be implemented into the existing standard TD-BEM equations. The $\theta$-Yu method proved to be useful in the dynamic analysis of Euler-Bernoulli beams. Some results are presented in the next section.

# 4 NUMERICAL EXAMPLES

In the examples of this section, the $\theta$-Yu TD-BEM results are always compared with the available analytical solutions presented earlier at section 2.2. The CP beam has $L = 2$m; for the other three types of beams one has $L = 4$m. The geometrical and material properties are presented in Tab. 1; the load values for each type of loading are listed in Tab. 2. The prescribed boundary conditions and initial conditions are null in all examples. The results presented here were obtained after performing several analysis, ranging the time-step value from $100\mu$s to $5\mu$s and the $\theta$ parameter from $1.0$ to $2.0$. By choosing $\Delta t = 5\mu$s and $\theta = 1.4$, good results were obtained and these values were adopted for all the graphics included here.

**Table 1: Geometrical and material properties**

| Quantity | Value | Unit |
|---|---|---|
| $E$ | 2.0000E+11 | N/m$^2$ |
| $I$ | 1.7393E−04 | m$^4$ |
| $\rho$ | 7.8500E+03 | kg/m$^3$ |
| $A$ | 6.2000E−03 | m$^2$ |

**Table 2: Load value for each type of loading**

| Loading | Value | Unit | Position |
|---|---|---|---|
| DD | 10 | kN/m | — |
| CD | $10 \times L$ | kN | $3L/4$ |

For each type of loading, the results for each type of beam are presented: first, those related to the boundary unknowns; then, those related to $u$, $\phi$, $M$, and $Q$ always at $x = L/4$. In this way, the boundary results are: for the PP beam, $\phi$ and $Q$; for the CP beam, $M$ and $Q$ at $x = 0$, and $\phi$ and $Q$ at $x = L$; for the CC beam, $M$ and $Q$; and for CF beam, $M$ and $Q$ at $x = 0$, and $u$ and $\phi$ at $x = L$. Fig. 2–5 depicts results for the DD loading; Fig. 6–9, the results for CD loading.

An overview of Fig. 2–5 enables one to conclude that, for the loading DD, the BEM results are always in good agreement with the analytical solutions, both at the boundary nodes of the problem and at the selected internal point. Even the results for shear forces can be considered very good, in spite of the oscillations that appear in the respective analytical solution. For the CD loading, the results in Fig. 6–9 allow the same conclusion.

It is important to mention that nor the standard neither the $\theta$-Yu TD-BEM formulations present the restrictions found in the TD-BEM formulation for the scalar wave propagation problem [see Mansur (1983)]. On the other hand, it is also important to bear in mind that the use of high values of the $\theta$ parameter produces more numerical damping. The value $\theta = 1.4$ seems to be the most adequate one.

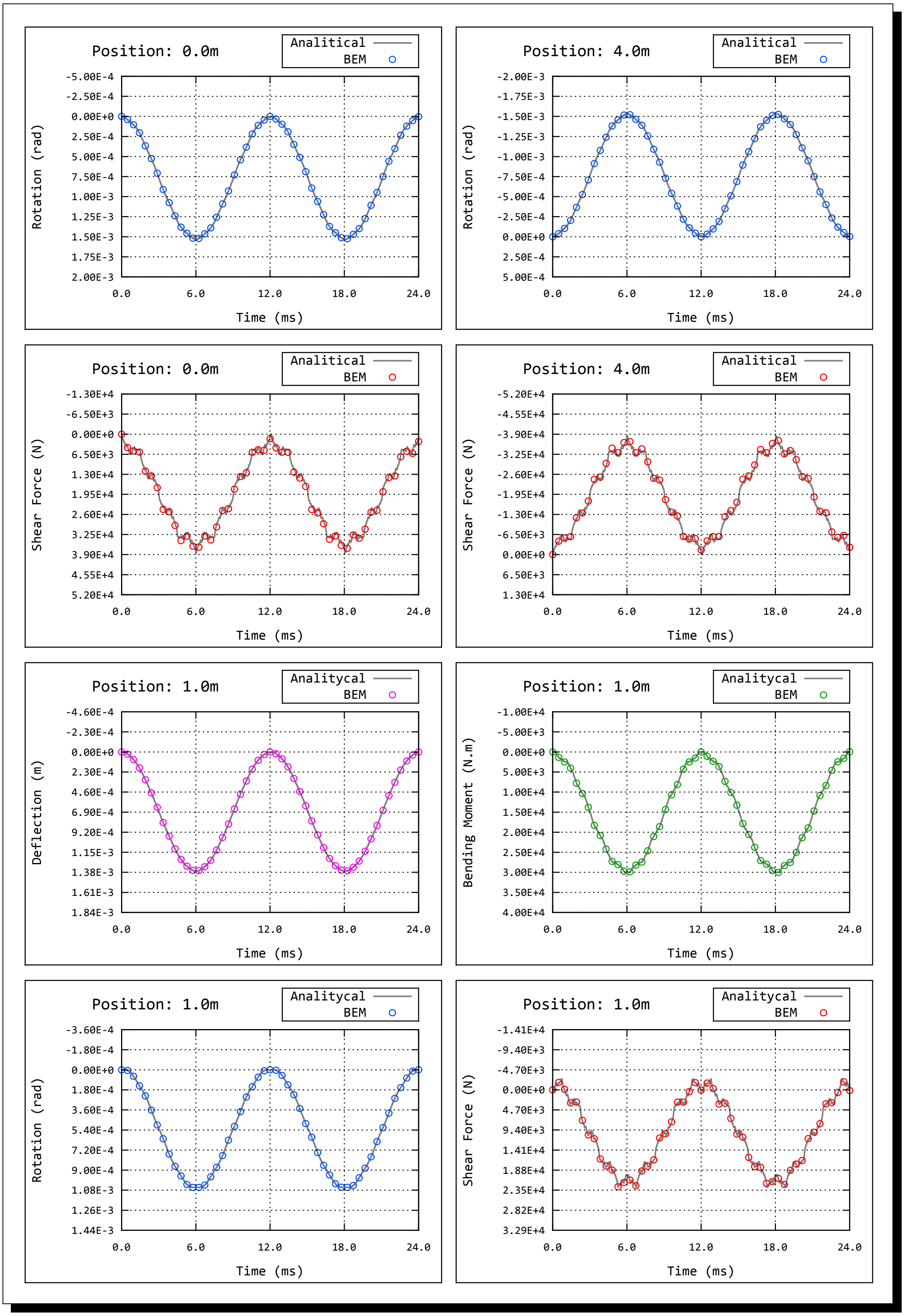


**Figure 2: Load Type: DD / Beam Type: PP**

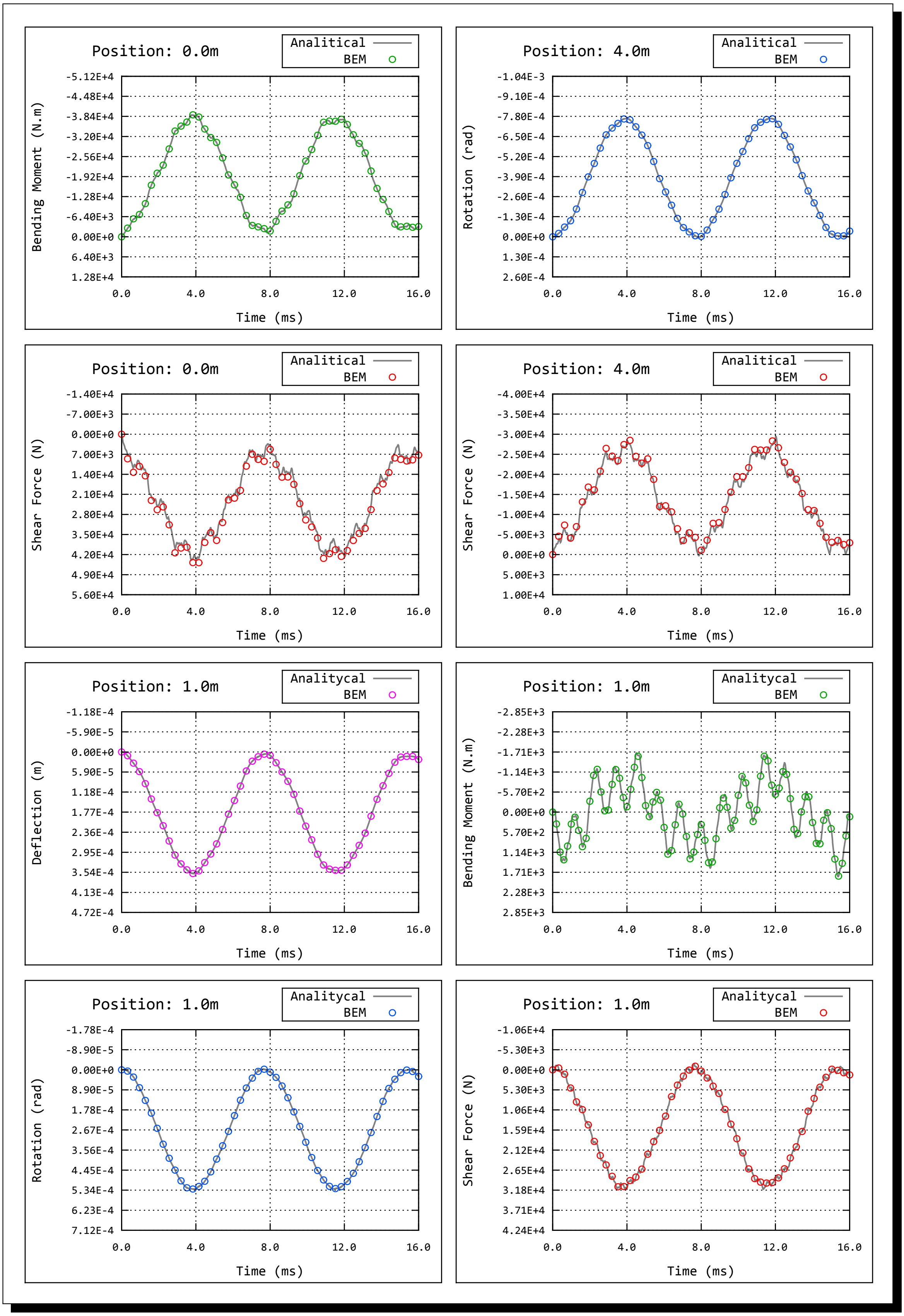


**Figure 3: Load Type: DD / Beam Type: CP**

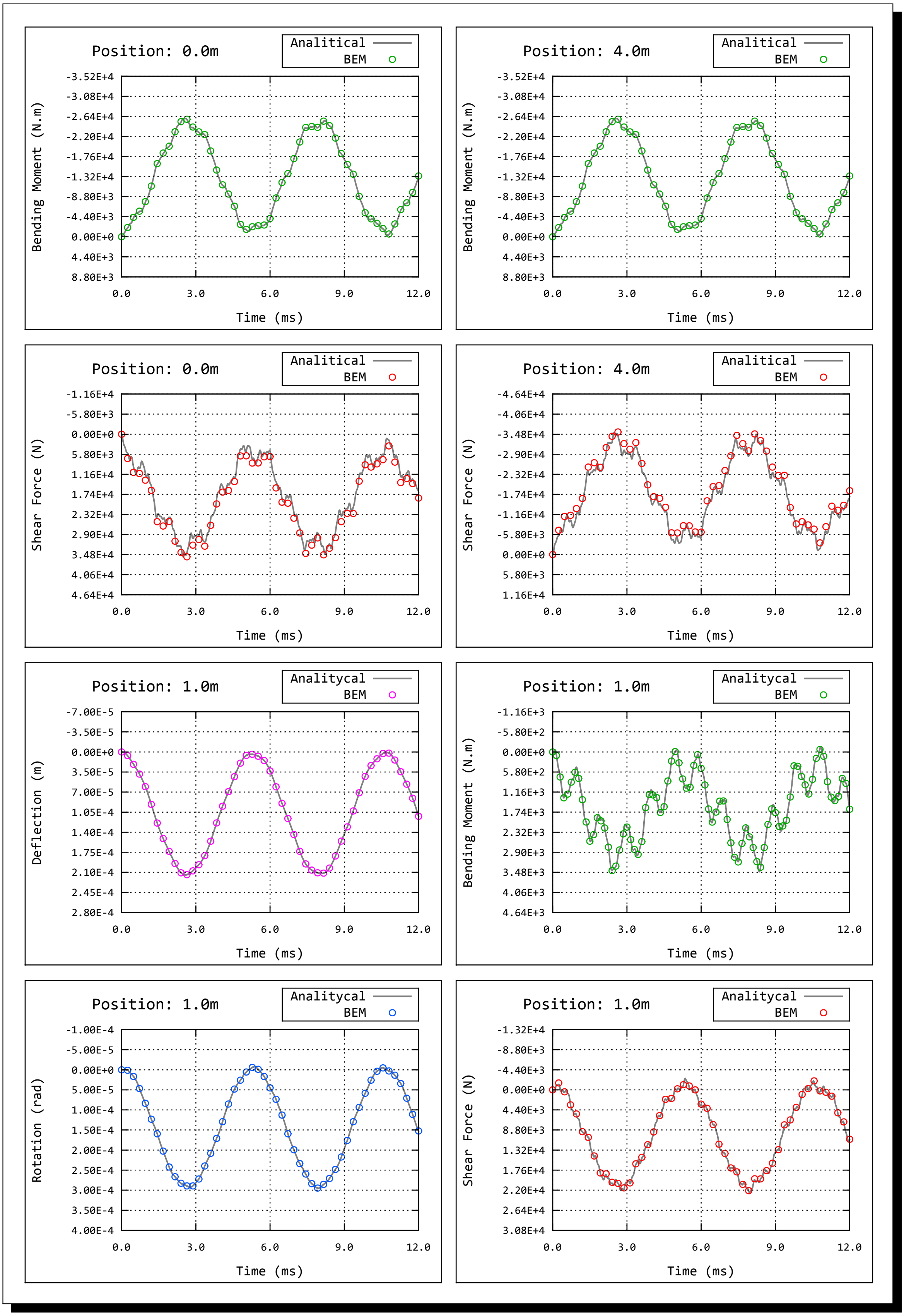


**Figure 4: Load Type: DD / Beam Type: CC**

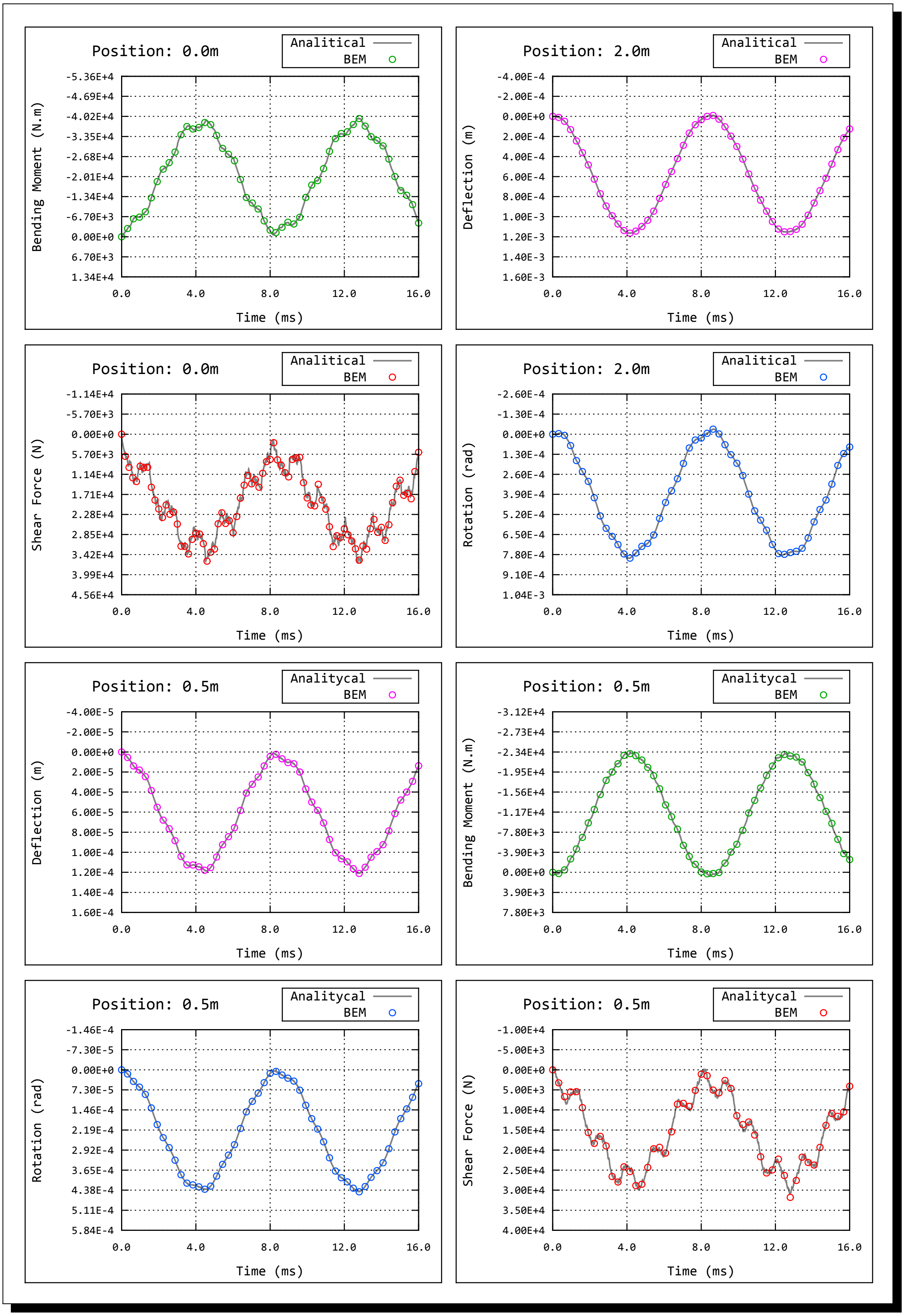


**Figure 5: Load Type: DD / Beam Type: CF**

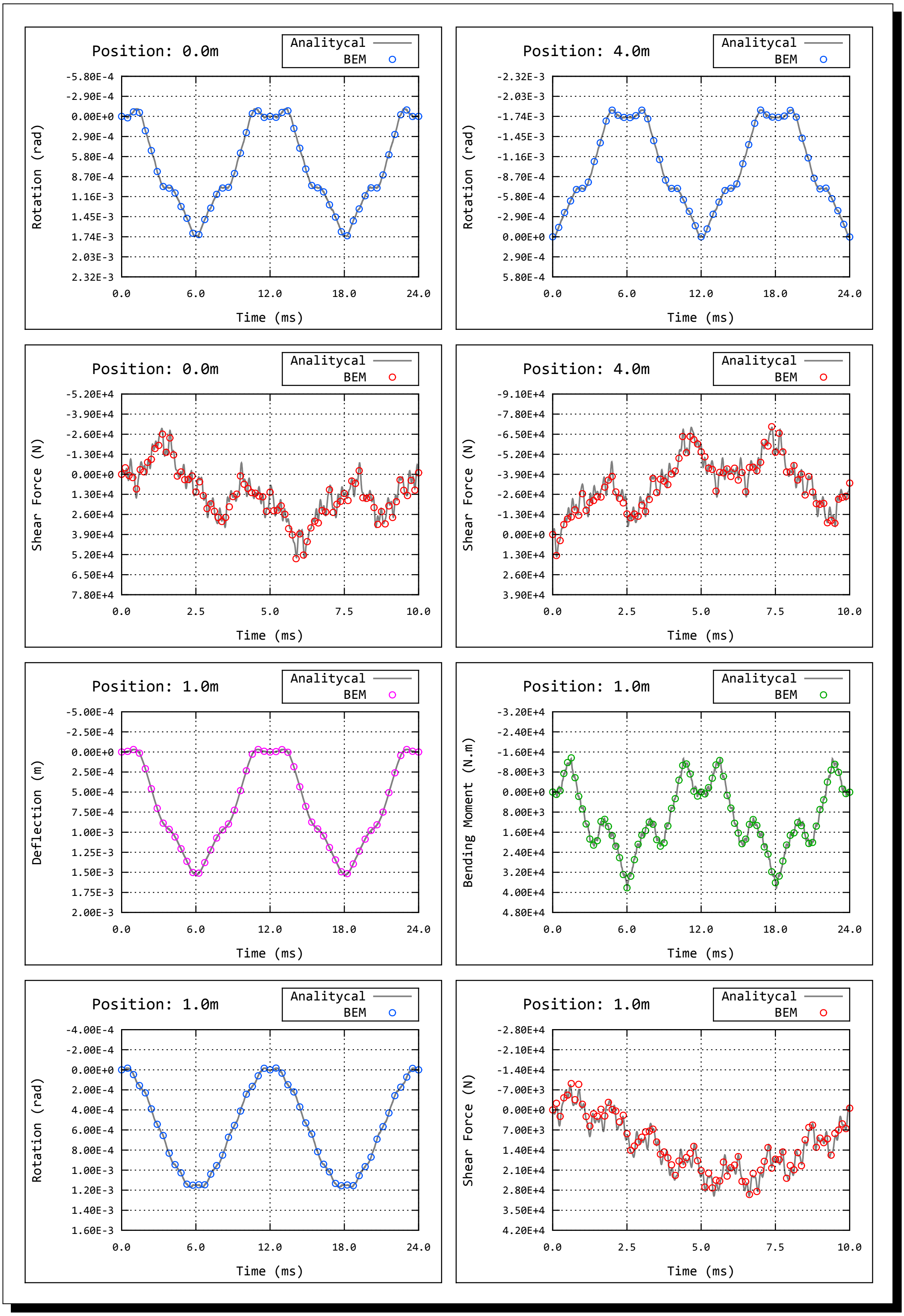


**Figure 6: Load Type: CD / Beam Type: PP**

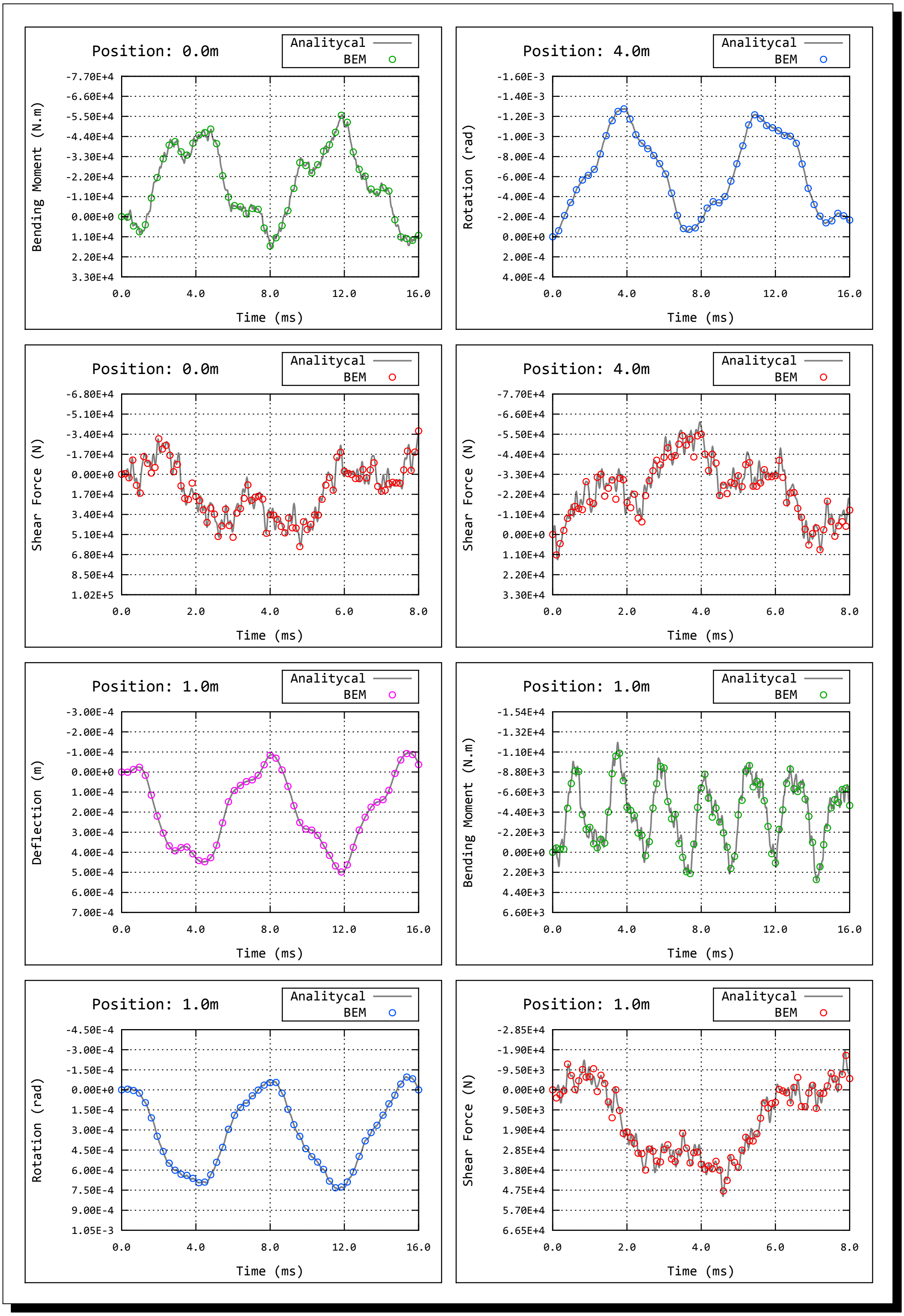


**Figure 7: Load Type: CD / Beam Type: CP**

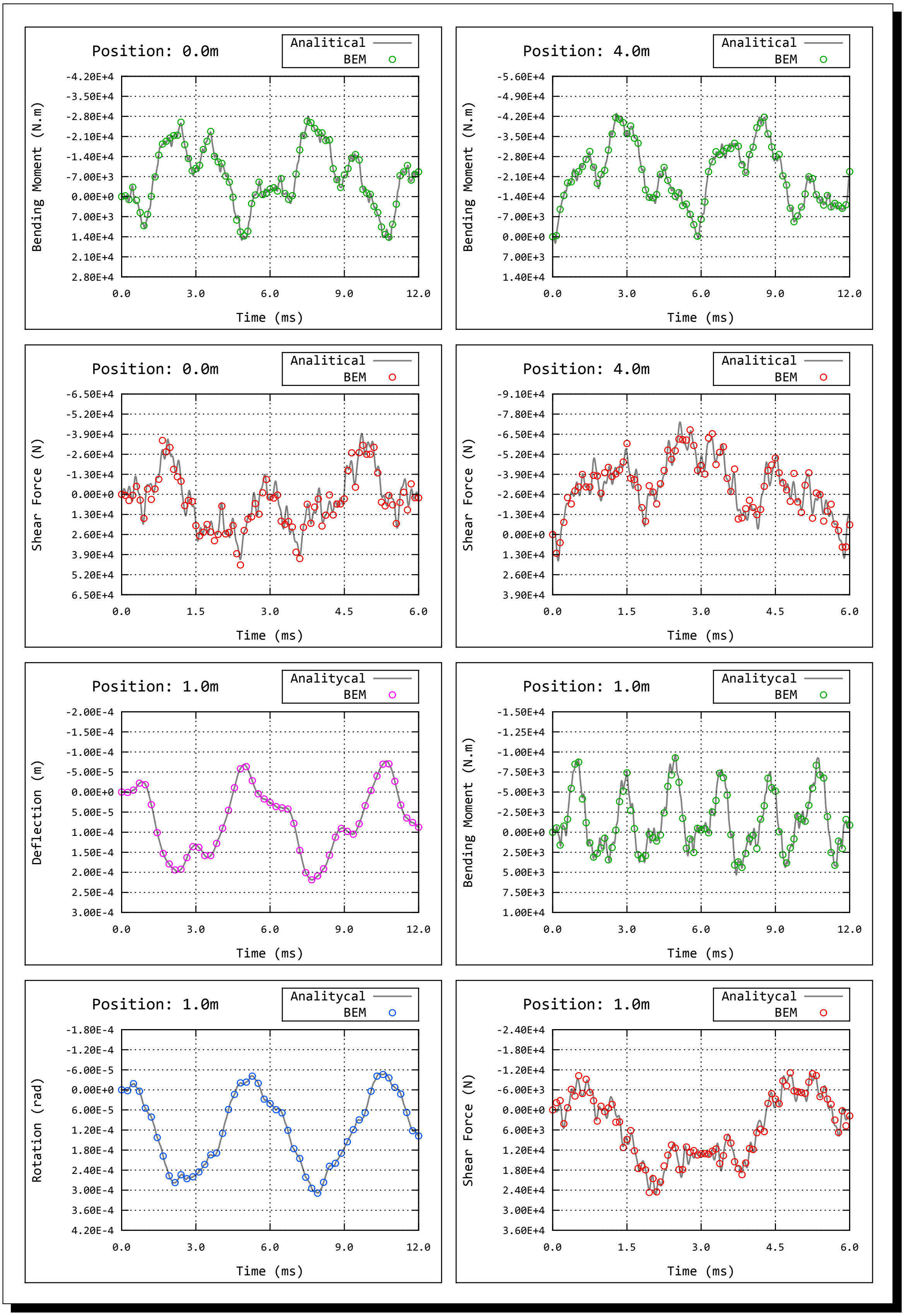


**Figure 8: Load Type: CD / Beam Type: CC**

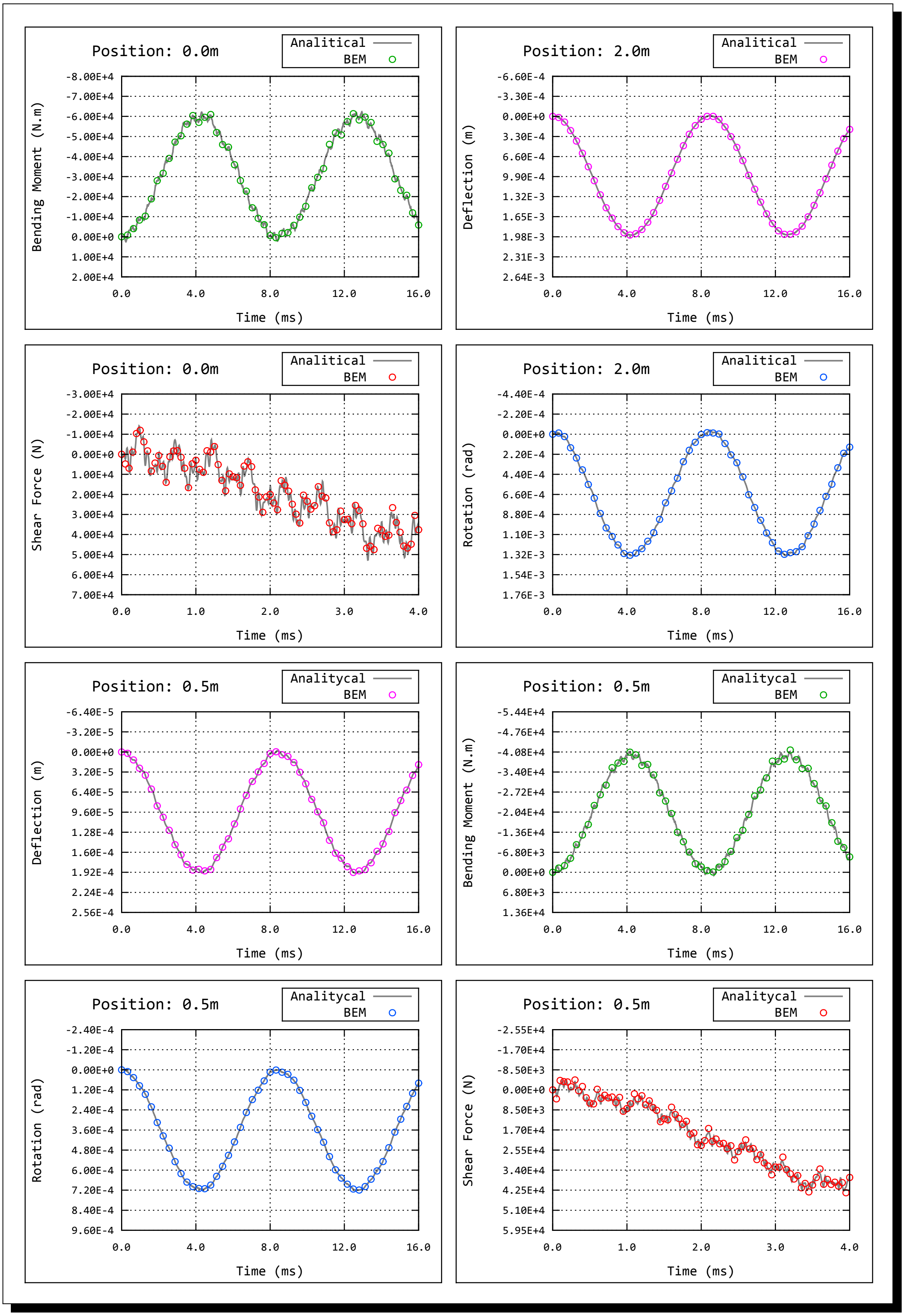


**Figure 9: Load Type: CD / Beam Type: CF**

# 5 CONCLUSIONS

Although the Euler-Bernoulli beam theory is relatively simple, it gives, from the practical point of view, very acceptable results, which justifies the attention it has received for quite some time and the development of the TD-BEM formulation presented in this work. A new version of the standard TD-BEM formulation is presented here. This new version is based on the use of the so-called $\theta$-Yu method, which allows for the adoption of the assumption of linear time behaviour, inside each time interval, for all the variables ($u$, $\phi$, $M$, and $Q$) that appear in the beam analysis. The numerical results are quite good and confirm the potentialities of the formulation based on the $\theta$-Yu method. Besides, another related studies, such as continuous beams, beams with variable cross section, and beams on an elastic foundation, can be carried out as well.

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