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UNCERTAINTY BASED TOPOLOGY OPTIMIZATION USING THE TOPOLOGICAL DERIVATIVE CONCEPT

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Abstract. *Optimization of structural topology has been subject of intense research for many decades. In most cases, the parameters of the problem are taken as deterministic values. However, some parameters, such as loads and material properties, are frequently uncertain in practice. For this reason, several authors proposed approaches to address optimization problems subject to uncertainties, establishing the field of uncertainty based optimization. In this paper we present an approach for uncertainty based topology optimization. The topology optimization is made using an efficient algorithm based on topological derivatives and uncertainties on load magnitudes are considered.*

Keywords: *Topology Design; Structural Optimization; Robust Optimization; Uncertainties; Topological Derivative*

1 INTRODUCTION

Optimization of structural topology has been subject of intense research for many decades. On the other hand, in most cases, the parameters of the problem are taken as deterministic values. However, some parameters, such as loads and material properties, are frequently uncertain in practice. For this reason, several authors proposed approaches to address optimization problems subject to uncertainties, establishing the field of Uncertainty Based Optimization, see for example the reviews presented by Beyer and Sendhoff (2007); Schuëller and Jensen (2008); Lopez and Beck (2012) and references therein for details.

It has been observed that uncertainties consideration may lead to solutions conceptually different from deterministic topology optimization. This fact supports the application of optimization under uncertainties in several cases of practical interest. However, in general, uncertainty based topology optimization requires much more computation effort than its deterministic counterpart. This fact limits the range of practical applications of uncertainty based optimization. In particular, most works on the subject can be grouped into three main classes: Reliability Based Design Optimization (RBDO), Robust Optimization and Risk Based Optimization. In the case of RBDO the constraints of the optimization problem are stated as to impose a maximum failure probability of the system (Maute and Frangopol, 2003; Luo et al., 2014). In the case of Robust Optimization, the goal is to obtain an optimum design that is least sensitive to variations and uncertainties of the variables (Dunning and Kim, 2013; Zhao and Wang, 2014). In the case of Risk Based Optimization, the failure probability of the system is used to evaluate the total cost of failures, that together with other costs compose the objective function of the problem (Beck and Gomes, 2012).

In this context, the goal of this paper is to present an approach for robust compliance topology optimization that is both general and computationally efficient. The compliance is evaluated considering a point-wise worst case scenario, found within an event set of possible outcomes of the random parameters. Analogously to Sequential Optimization and Reliability Assessment (SORA) (Du and Chen, 2004), the resulting robust optimization problem can be decoupled into a deterministic topology optimization step and a reliability analysis step. Since the proposed approach decouples the nested robust optimization problem, it is computationally efficient and easy to implement provided that it requires only existing algorithms. In the reliability analysis step the point-wise worst case scenarios are found as in the Performance Measure Approach (PMA). The topology optimization algorithm proposed by Amstutz and Andrä (2006) is used to solve the associated deterministic problem, which is based on the topological derivative concept and a level-set domain representation method. In addition, from the mathematical point of view, the topological derivative concept has been proved to be robust with respect to uncertainties on the data (Hlaváček et al., 2009). In this paper therefore, the theory developed by Hlaváček et al. (2009) is also confirmed from the numerical point of view. In particular, the numerical examples presented at the end of this paper address the case of topology optimization with load magnitudes as random variables. The general statement of the problem is also presented and can be extended to other random variables as well, which however may require some additional computational effort.

This paper is organized as follows. The robust design optimization problem is stated in Section 2. In particular, we describe how the SORA concept can be used to decouple the nested robust optimization problem into a deterministic optimization step and a reliability analysis step.

The resulting deterministic topology optimization problem is presented in Section 3, which is solved using the topological derivative concept together with a level-set domain representation method. Some numerical examples are presented in Section 4, showing the efficiency and simplicity of the proposed approach as well as the importance of considering uncertainties in the topology optimization problem. Finally, some concluding remarks are presented in Section 5.

2 PROPOSED APPROACH

Let us introduce a hold-all structural domain $\mathcal{D} \subset \mathbb{R}^2$ and a subdomain $\Omega \subset \mathcal{D}$. We consider the minimization of the structural compliance under volume constraint, which can be written as:

$$\text{Find}\, \Omega^d \subset \mathcal{D}\, \text{, such that:}$$
$$\begin{cases} \underset{\Omega \subset \mathcal{D}}{\textbf{Minimize}} & \mathcal{K}_\Omega \;=\; \displaystyle\int_{\mathcal{D}} \sigma_\Omega \cdot \varepsilon_\Omega \;, \\ \textbf{Subject to} & |\Omega| \;\leq\; M \;, \end{cases} \tag{1}$$

where $|\Omega|$ is the Lebesgue measure of Ω (i.e. volume of the structure) and M is a prescribed amount of material. The quantity $\sigma_\Omega \cdot \varepsilon_\Omega$ is twice the strain energy density, with σ_Ω and ε_Ω used to denote the stress and the strain tensors, respectively.

When some parameters of the problem (e.g. load magnitudes) are random variables, the compliance becomes a random variable itself. In this case the compliance is not known in the deterministic sense and the deterministic problem from Eq. (1) can result in inefficient designs in practice. In order to take into account such uncertainties it is necessary to apply some Robust Optimization strategy.

In this paper the Robust Optimization strategy adopted is a common non-probabilistic approach called Worst Case Design Optimization (WCDO) (Guo et al., 2009). In this approach, the deterministic optimization problem is solved considering a Worst Case Scenario (WCS) among the possible outcomes of the unknown parameters. The WCS is given by the combination of parameters that lead to the worst performance of the system being optimized.

An important aspect of WCDO is how the set of possible outcomes is defined. Taking the set too large may result in too conservative designs. Taking the set too small, on the other hand, can lead to non robust solutions. In this work we follow the WCDO approach, since it leads to a computationally efficient decoupling of the Robust Optimization problem. However, we use statistical information of the unknown parameters in order to build the bounded set of possible outcomes.

2.1 The bounded set of possible outcomes

Here we assume that the unknown parameters are represented by a random vector $\mathbf{x}$ with known statistical information. In order to write the bounded set of possible outcomes in a concise manner we assume that the random vector $\mathbf{x}$ is composed by independent Gaussian components x_i. Then we can obtain the normalized random vector $\mathbf{u}$ by using the transformation $T : \mathbf{x} \to \mathbf{u}$ given by

$$\mathrm{u}_i = \frac{\mathrm{x}_i - \mu_i}{s_i}, \tag{2}$$

where u_i are independent Normal random variables with mean equal to zero and unitary standard deviation and μ_i and s_i are the mean and the standard deviation of the random variable x_i.

Using the normalized random vector $\mathbf{u} = T(\mathbf{x})$ we can define the event set as

$$\mathcal{E} = \{\mathbf{x} \in \mathbb{R}^m : \|T(\mathbf{x})\| \leq \beta_t\}, \tag{3}$$

where β_t is some prescribed parameter. Note that the event set is composed only by outcomes as distant as β_t from the mean value $T(\mathbf{x}) = \mathbf{0}$. In other words, only outcomes obtained within β_t standard deviations from the mean value are considered.

From the definition of the cumulative distribution function (CDF)

$$P(u \leq \beta_t) = \Phi(\beta_t) \tag{4}$$

where Φ is the cumulative distribution function (CDF) of a Normal random variable. Consequently, by imposing $\|T(\mathbf{x})\| \leq \beta_t$ we are actually imposing a target probability of occurrence

$$P_t = P(\|T(\mathbf{x})\| \leq \beta_t) = \Phi(\beta_t). \tag{5}$$

From Eq. (5) we note that by prescribing a given value β_t we are actually prescribing that the set $\mathcal{E}$ is comprised only by outcomes which CDF correspond to a target probability P_t. Consequently, we can obtain the value of β_t for a given P_t from

$$\beta_t = \Phi^{-1}(P_t). \tag{6}$$

By setting $P_t = 0.95$, for example, the event set will comprise all outcomes that correspond to 95% of the occurrences. The value of P_t can then be adjusted to consider a larger or smaller portion of the possible outcomes as needed. Finally, we note that β_t is related to the Hasofer-Lind reliability index (Haldar and Mahadevan, 2000; Melchers, 1999), in the sense that it defines a distance from the expected value in the normalized space.

2.2 The robust optimization approach

In order to build a Robust Optimization approach, we search for the WCS point-wisely inside the structural domain. At each point of the domain, the strain energy density WCS $\mathbf{x}^*$ for events in the set $\mathcal{E}$ can be found by solving the following problem:

$$\text{Find } \mathbf{x}^* \in \mathbb{R}^m \text{, such that:}$$
$$\begin{cases} \underset{\mathbf{x}\in\mathcal{E}}{\textbf{Maximize}} & \sigma_\Omega(\mathbf{x}) \cdot \varepsilon_\Omega(\mathbf{x}) \,, \\ \textbf{Subject to} & \|T(\mathbf{x})\| \leq \beta \,, \end{cases} \tag{7}$$

where $\sigma_\Omega(\mathbf{x}) \cdot \varepsilon_\Omega(\mathbf{x})$ represents the fact that the strain energy depends on unknown parameters $\mathbf{x}$ and the constraint $\|T(\mathbf{x})\| \leq \beta$ states that we search for a solution in the set $\mathcal{E}$ as defined in Eq. (3).

Minimization of the strain energy density considering a point-wise WCS can be achieved by substitution of $\mathbf{x}^*$ into (1). The resulting optimization problem is:

$$\text{Find } \Omega^p \subset \mathcal{D}\text{, such that:}$$
$$\begin{cases} \underset{\Omega \subset \mathcal{D}}{\textbf{Minimize}} & \mathcal{K}_\Omega \;=\; \displaystyle\int_{\mathcal{D}} \sigma_\Omega(\mathbf{x}^*) \cdot \varepsilon_\Omega(\mathbf{x}^*) \,, \\ \textbf{Subject to} & |\Omega| \;\leq\; M \,, \end{cases} \tag{8}$$

where $\sigma_\Omega(\mathbf{x}^*) \cdot \varepsilon_\Omega(\mathbf{x}^*)$ indicates that the strain energy density is evaluated considering the point-wise WCS $\mathbf{x}^*$ as defined in (7).

In the context of this work it is important to note the difference between Ω^d, solution to (1), and Ω^p solution to (8). Note that Ω^p is the optimum solution of the Robust Optimization problem while Ω^d is the solution of the deterministic optimization problem.

2.3 Sequential Optimization and Reliability Assessment

Unfortunately, it is not efficient to address the problem from (8) directly, since $\mathbf{x}^*$ is defined implicitly by means of (7). This leads to a nested optimization problem, where it is necessary to solve the maximization problem (7) at each point of the domain before evaluating the objective function from (8).

However, it is possible to decouple these two optimization problems using concepts from Sequential Optimization and Reliability Assessment (SORA). We first define the solution of the problem from (8) as the pair $\{\Omega^p, \mathbf{x}^p\}$. We note that, $\mathbf{x}^*$ is defined as the WCS obtained for an arbitrary topology, while $\mathbf{x}^p$ is defined as the special WCS obtained with the optimum topology Ω^p.

If, for some reason, $\mathbf{x}^p$ was known beforehand, it would not be necessary to solve the problem from (7) before solving the problem from (8). In this case, the optimum topology Ω^p can be found directly by taking the WCS $\mathbf{x}^p$ in the problem from (8). This puts in evidence that, once the WCS $\mathbf{x}^p$ is known, the problem (8) becomes a deterministic topology optimization problem. On the other hand, if the optimum topology Ω^p is known, the WCS $\mathbf{x}^p$ can be found directly by solving (7) point-wisely with the optimum topology Ω^p.

The problem from (7) is stated in the same form frequently encountered in RBDO using the Performance Measure Approach (PMA) (Lopez and Beck, 2012; Tu et al., 1999). Consequently, it can be solved by efficient schemes already developed in the literature. Here this problem is solved using the Advanced Mean Value (AMV) algorithm, originally proposed by (Wu et al., 1990).

Obviously, neither Ω^p nor $\mathbf{x}^p$ are known beforehand, unless the problem is already solved. However, it is possible to start from initial approximations and iterate until accurate solutions are found. In fact, let us indicate some iteration number with (k), then an iterative algorithm for solving the Robust Optimization problem can be summarized as presented in Algorithm 1. The procedure is started with an arbitrary topology $\Omega^{(0)}$. With this topology we find the WCS $\mathbf{x}^{(0)}$

at each point of interest by solving (7) with $\Omega^{(0)}$. With the WCS $\mathbf{x}^{(0)}$ we find a new topology $\Omega^{(1)}$ and so on, until convergence is achieved.

Convergence of the solution can be checked on changes of Ω or $\mathbf{x}$ between two successive iterations. Besides, the point-wise WCS $\mathbf{x}$ is evaluated only at interest points of the domain. In the case the problem is solved using the Finite Element Method, for example, the WCS can be evaluated only at integration points.

Algorithm 1: Robust Optimization algorithm proposed

input : Initial topology $\Omega^{(0)}$
output: $\{\Omega^p, \mathbf{x}^p\}$

Set $k \leftarrow 0$;
Compute $\mathbf{x}^{(0)}$ by solving (7) with $\Omega = \Omega^{(0)}$;
while *convergence is not achieved* **do**
    Make $k \leftarrow k+1$;
    Compute $\Omega^{(k)}$ by solving (8) with $\mathbf{x}^* = \mathbf{x}^{(k-1)}$;
    Compute $\mathbf{x}^{(k)}$ by solving (7) with $\Omega = \Omega^{(k)}$;
end while
Return $\{\Omega^\star, \mathbf{x}^\star\}$ as solution.

3 STRUCTURAL TOPOLOGY OPTIMIZATION PROBLEM

The robust topology optimization we are dealing with consists in solving problem (8) evaluated at the worst case scenario, solution to the problem (7). Without loss of generality, we restrict ourselves to uncertainties on the intensity of the applied loads (note that the problem stated through (7) and (8) is general and can be applied when other parameters are random variables). In this case, we can use linear superposition of the effects in order to save computational effort. Let us consider that the applied load can be written as a linear combination of q_i independent loads, with $i = 1, ..., m$. Then, the stress σ_Ω and strain ε_Ω tensors at each point can be written as

$$\sigma_\Omega := \sum_{i=1}^{m} \mathrm{x}_i^* \sigma(u_i) \quad \text{and} \quad \varepsilon_\Omega := \sum_{i=1}^{m} \mathrm{x}_i^* \varepsilon(u_i) \ , \tag{9}$$

where, in this context, x_i^* can be interpreted as load scale factors. The *canonical* strain $\varepsilon(u_i)$ and stress $\sigma(u_i)$ tensors are obtained from each individual load q_i. For this purpose, we must to solve the following set of canonical variational problems related to the structural response when each load q_i is applied:

$$\text{Find } u_i \in \mathcal{V}\text{, such that}$$
$$\left\{ \begin{array}{l} \displaystyle\int_{\mathcal{D}} \sigma(u_i) \cdot \varepsilon(\eta) = \int_{\Gamma_N} q_i \cdot \eta \quad \forall \eta \in \mathcal{V} \ , \\ \qquad\qquad\qquad\quad \text{with } \sigma(u_i) := \rho \mathbb{C} \varepsilon(u_i), \end{array} \right. \tag{10}$$

where

$$\varepsilon(\varphi) = \frac{1}{2}(\nabla\varphi + (\nabla\varphi)^\top) \tag{11}$$

is the linearized Green tensor,

$$\mathbb{C} = \frac{E}{1-\nu^2}((1-\nu)\mathbb{I} + \nu \mathrm{I} \otimes \mathrm{I}) \tag{12}$$

is the elasticity tensor, I and $\mathbb{I}$ are the second and fourth identity tensors, respectively, E is the Young modulus and ν the Poisson ratio. The piecewise constant function ρ

$$\rho(x) = \begin{cases} 1, & \text{if} \quad x \in \Omega \ , \\ \rho_0, & \text{if} \quad x \in \mathcal{D} \setminus \overline{\Omega} \ , \end{cases} \tag{13}$$

with $0 < \rho_0 \ll 1$, is used to mimic voids. That is, the original structural problem, where the structure itself consists of the domain Ω of given elastic properties and the remaining part $\mathcal{D} \setminus \overline{\Omega}$ of the hold-all is empty (has no material), is approximated by means of the two-phase material distribution given by (13) over $\mathcal{D}$ where the empty region $\mathcal{D} \setminus \overline{\Omega}$ is occupied by a material (the soft phase) with Young's modulus, $\rho_0 E$, much lower than the given Young's modulus, E, of the structure material (the hard phase). The space $\mathcal{V}$ is defined as

$$\mathcal{V} := \{\varphi \in H^1(\Omega; \mathbb{R}^2) : \varphi_{|\Gamma_D} = 0\} \ . \tag{14}$$

Here, Γ_D and Γ_N are Dirichlet and Neumann boundaries, respectively, such that $\partial\mathcal{D} = \overline{\Gamma_D} \cup \overline{\Gamma_N}$ with $\Gamma_D \cap \Gamma_N = \emptyset$, and q_i is the prescribed traction on Γ_N. See details in Fig. 1.

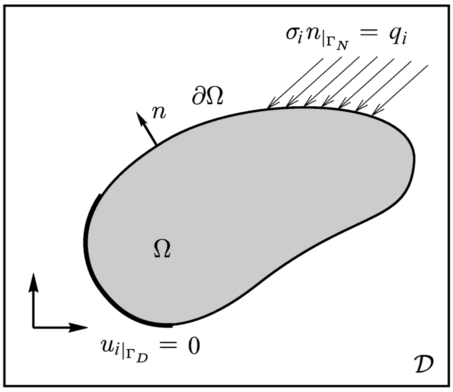


Figure 1: Sketch of the elasticity problem.

3.1 Deterministic Topology Optimization Method

The volume constraint in problem (8) is trivially imposed via the Augmented Lagrangian Method (Campeão et al., 2014). In particular, it can be rewritten as an unconstrained optimization problem as follows:

$$\underset{\Omega \subset \mathcal{D}}{\textbf{Minimize}} \ \mathcal{J}_\Omega = \mathcal{K}_\Omega + \lambda_1 g_\Omega^+ + \frac{\lambda_2}{2}(g_\Omega^+)^2 \ , \tag{15}$$

where $g_\Omega^+ := \max\{g_\Omega, -\lambda_1/\lambda_2\}$, with $g_\Omega = (|\Omega| - M)/M$, $\lambda_2 > 0$ is a fixed multiplier and λ_1 is updated according to the following recursive formula:

$$\begin{aligned} \lambda_1^{(0)} &= 0 \\ \lambda_1^{(n+1)} &= \max\{0, \lambda_1^{(n)} + \lambda_2 g_\Omega\} \ . \end{aligned} \tag{16}$$

The deterministic structural compliance optimization problem (8) is solved by using the topological derivative concept, which has been shown to be robust with respect to uncertainties on the data (Hlaváček et al., 2009). The topological derivative measures the sensitivity of a given shape functional with respect to an infinitesimal singular domain perturbation, such as the insertion of holes, inclusions or source-terms (Novotny and Sokołowski, 2013). In particular, the topological derivative of the shape functional $\mathcal{J}_\Omega$ with respect to the nucleation of a small circular inclusion with different material property from the background, represented by a contrast γ, is given by the sum

$$D_T\mathcal{J}_\Omega = D_T\mathcal{K}_\Omega \mp \max\{0, \lambda_1 + \lambda_2 g_\Omega\} \;, \tag{17}$$

where $\mp$ means that if we remove material the volume becomes smaller, on the other hand, if we insert material the volume increases. The topological derivative for the compliance denoted by $D_T\mathcal{K}_\Omega$ is know and can be found in the book (Novotny and Sokołowski, 2013, ch.5 pp.158), for instance. It is given by:

$$D_T\mathcal{K}_\Omega = \mathbb{P}\sigma_\Omega \cdot \varepsilon_\Omega \;, \tag{18}$$

where the Pólya-Szegö polarization tensor $\mathbb{P}$ is

$$\mathbb{P} = \frac{1-\gamma}{1+\gamma a_2}\left((1+a_2)\mathbb{I} + \frac{1}{2}(a_1 - a_2)\frac{1-\gamma}{1+\gamma a_1}\mathrm{I}\otimes\mathrm{I}\right) \;, \tag{19}$$

with

$$a_1 = \frac{1+\nu}{1-\nu} \quad \text{and} \quad a_2 = \frac{3-\nu}{1+\nu} \;. \tag{20}$$

In addition, the contrast γ in the material property, is defined as follows

$$\gamma(x) = \begin{cases} \rho_0, & \text{if} \quad x \in \Omega \;, \\ \frac{1}{\rho_0}, & \text{if} \quad x \in \mathcal{D}\setminus\overline{\Omega} \;. \end{cases} \tag{21}$$

In fact, the above result together with a level-set domain representation method is used for solving the deterministic compliance topology optimization problem (8) with $\mathbf{x}^* = \mathbf{x}^{(k)}$ fixed, necessary in step (3) of the algorithm proposed in Section 2.3.

3.2 Deterministic Topology Design Algorithm

The resulting deterministic topology optimization algorithm is now explained in details for the reader convenience. It has been proposed by Amstutz and Andrä (2006) and consists basically in looking for a local optimality condition for the minimization problem (15), written in terms of the topological derivative and a level-set function. Therefore, the elastic part Ω as well as the complacent material $\mathcal{D}\setminus\overline{\Omega}$ are characterized by a level-set function $\Psi \in L^2(\mathcal{D})$ of the form:

$$\Omega = \{\Psi(x) < 0 \text{ a.e. in } \mathcal{D}\} \;, \tag{22}$$

$$\mathcal{D}\setminus\overline{\Omega} = \{\Psi(x) > 0 \text{ a.e. in } \mathcal{D}\} \;, \tag{23}$$

where Ψ vanishes on the interface $\partial\Omega$. A local sufficient optimality condition for Problem (15), under the considered class of domain perturbation given by circular inclusions, can be stated as Amstutz (2011)

$$D_T \mathcal{J}_\Omega(x) > 0 \ \ \forall x \in \mathcal{D} \ . \tag{24}$$

Therefore, let us define the quantity

$$g(x) := \begin{cases} -D_T \mathcal{J}_\Omega(x), & \text{if } \Psi(x) < 0, \\ \ \ D_T \mathcal{J}_\Omega(x), & \text{if } \Psi(x) > 0, \end{cases} \tag{25}$$

allowing for rewrite the condition (24) in the following equivalent form

$$\begin{cases} g(x) < 0, & \text{if } \Psi(x) < 0, \\ g(x) > 0, & \text{if } \Psi(x) > 0. \end{cases} \tag{26}$$

We observe that (26) is satisfied wether the quantity g coincides with the level-set function Ψ up to a strictly positive number, namely $\exists\ \tau > 0 : g = \tau\Psi$, or equivalently

$$\theta := \arccos\left[\frac{\langle g, \Psi\rangle_{L^2(\mathcal{D})}}{\|g\|_{L^2(\mathcal{D})}\|\Psi\|_{L^2(\mathcal{D})}}\right] = 0 \ , \tag{27}$$

which shall be used as optimality condition in the topology design algorithm, where θ is the angle between the functions g and Ψ in $L^2(\mathcal{D})$. Let us now explain the algorithm. We start by choosing an initial level-set function $\Psi_0 \in L^2(\mathcal{D})$. In a generic iteration n, we compute function g_n associated with the level-set function $\Psi_n \in L^2(\mathcal{D})$. Thus, the new level-set function Ψ_{n+1} is updated according to the following linear combination between the functions g_n and Ψ_n

$$\begin{aligned} &\Psi_0 \in L^2(\mathcal{D}), \\ &\Psi_{n+1} = \frac{1}{\sin\theta_n}\left[\sin((1-\kappa)\theta_n)\Psi_n + \sin(\kappa\theta_n)\frac{g_n}{\|g_n\|_{L^2(\mathcal{D})}}\right] \quad \forall n \in \mathbb{N} \ , \end{aligned} \tag{28}$$

where θ_n is the angle between g_n and Ψ_n, and κ is a step size determined by a linear-search performed in order to decrease the value of the objective function $\mathcal{J}_{\Omega_n}$, with Ω_n used to denote the elastic part associated to Ψ_n. The process ends when the condition $\theta_n \leq \epsilon_\theta$ and at the same time the required volume $|1 - |\Omega_n|/M| \leq \epsilon_M$ are satisfied in some iteration, where ϵ_θ and ϵ_M are given small numerical tolerances. In particular, we can choose

$$\Psi_0 \in \mathcal{S} = \{x \in L^2(\mathcal{D}); \|x\|_{L^2(\mathcal{D})} = 1\} \ , \tag{29}$$

and by construction $\Psi_{n+1} \in \mathcal{S},\ \forall n \in \mathbb{N}$. If at some iteration n the linear-search step size κ is found to be smaller then a given numerical tolerance $\epsilon_\kappa > 0$ and the optimality condition is not satisfied, namely $\theta_n > \epsilon_\theta$, then a uniform mesh refinement of the hold all domain $\mathcal{D}$ is carried out and the iterative process is continued.

4 NUMERICAL EXAMPLES

In the numerical examples we assume that the structures are under a plane stress state. The Young's modulus and the Poisson ratio are respectively given by $E = 1.0$ and $\nu = 0.3$, while the contrast $\rho_0 = 10^{-4}$. The angle θ defined in Amstutz and Andrä (2006), representing the optimality condition, has converged to a value smaller than 1^o in all cases. The mechanical problem is discretized into linear triangular finite elements and two steps of uniform mesh refinement were performed during the iterative process. Since only load magnitudes are taken as random variables, linear superposition of effects is used wherever it is possible. Finally, the mean compliance and the compliance standard deviation of the structures are computed using Monte Carlo Simulation with 10^4 samples (Haldar and Mahadevan, 2000).

4.1 Example 1

Let us consider a square panel of size 1×1 clamped on the top and submitted to a pair of loads, as shown in Fig. 2. The loading consists of two forces $q_1 = (2.0, -1.0)$ and $q_2 = (-2.0, -1.0)$ applied on the middle of the bottom edge. The hold-all domain is discretized into a uniform mesh with 6400 elements and 3281 nodes. The required volume fraction is set as $M = 25\%$, while the parameter $\lambda_2 = 1.0$.

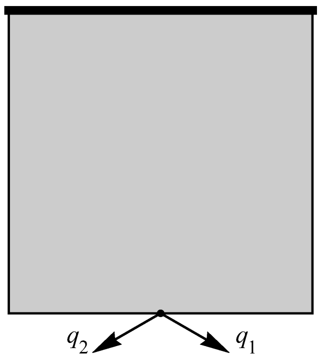


Figure 2: Example 1: initial guess and boundary conditions.

If topology optimization is made considering all parameters as deterministic, the optimal topology obtained is that presented in Fig. 3, which is a benchmark solution to this problem.

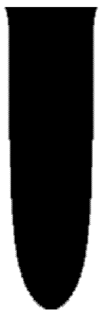


Figure 3: Example 1: result for the deterministic case.

We now assume that the load scale factors are represented by Gaussian random variables. The event set is defined with $\beta = 2.0$ and the load scale factors have unitary means, namely $\mu_1 = \mu_2 = 1.0$. In the first case, the standard deviations are equal and given by $s_1 = s_2 = 0.20$. The optimum topology is presented in Fig. 4. As expected, the optimum solution of the robust optimization problem leads to a V-bracket structure, able to handle the horizontal components of the loads that result for cases where the two forces do not have the same magnitude.

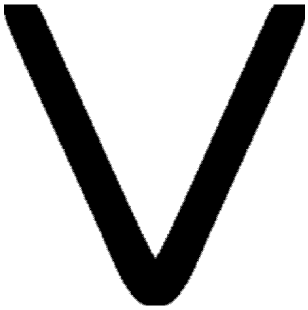
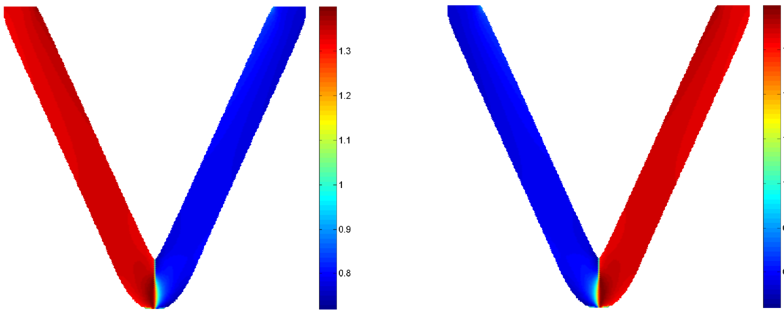


Figure 4: Example 1: result for the robust case with $s_1 = s_2 = 0.20$.

The distributions of the load scale factors in the worst case scenario x_1 and x_2 are shown in Fig. 5.



(a) index x_1 (b) index x_2

Figure 5: Example 1: load scale factors x_1 and x_2 for the robust case with $s_1 = s_2 = 0.20$.

Finally, we repeat the same experiment by setting $s_1 = 0.20$ and $s_2 = 0.02$. As expected, in this case the final topology is not symmetric as can be seen in Fig. 6.

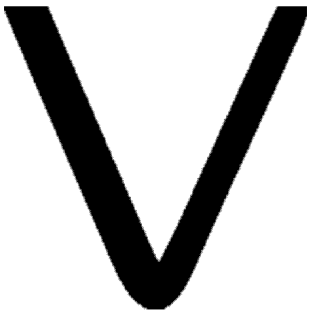
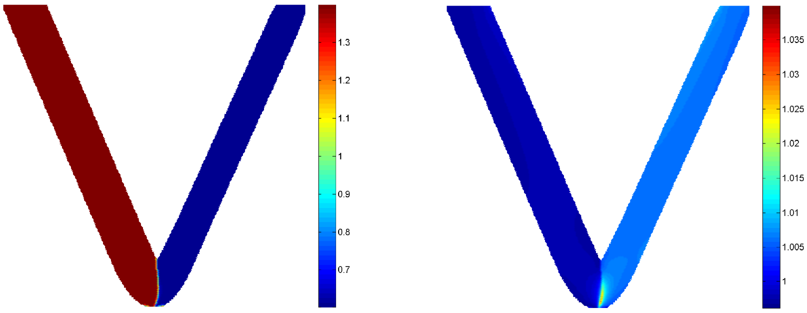


Figure 6: Example 1: result for the robust case with $s_1 = 0.20$ and $s_2 = 0.02$.

The distributions of the load scale factors in the worst case scenario x_1 and x_2 are shown in Fig. 7.



(a) index x_1 (b) index x_2

Figure 7: Example 1: load scale factors x_1 and x_2 for the robust case with $s_1 = 0.20$ and $s_2 = 0.02$.

4.2 Example 2

Now, let us consider the design of a supported beam submitted to the loads as shown in Fig. 8. The hold-all domain is rectangular with dimensions 2×1. It is discretized into a uniform mesh with 6400 elements and 3321 nodes. The loading consists of a pair of forces $q_1 = (0.0, -1.0)$ and $q_2 = (0.0, -1.0)$ applied on the bottom edge as shown in Fig. 8. The required volume fraction is set as $M = 30\%$, while the parameter $\lambda_2 = 1.0$. The load scale factors have unitary means $\mu_1 = \mu_2 = 1.0$ and identical standard deviations given by $s_1 = s_2 = 0.20$.

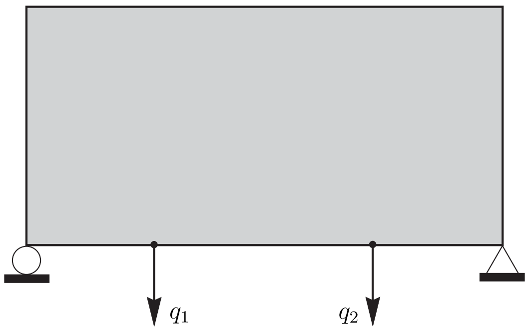


Figure 8: Example 2: initial guess and boundary conditions.

The optimal topology obtained considering all parameters as deterministic is presented in Fig. 9. The mean compliance and the compliance standard deviation of this structure were found to be 59.83 and 17.17, respectively. This results in a coefficient of variation of 0.29.

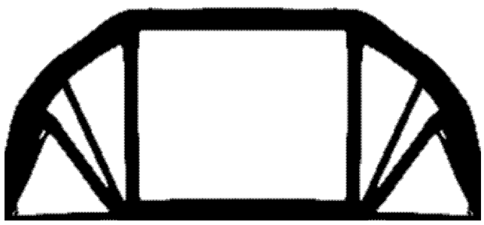


Figure 9: Example 2: result for the deterministic case.

Now we consider the robust case, with the load scale factors represented again by Gaussian random variables. The event set is defined with $\beta = 2.0$. The optimum topology is presented in Fig. 10. The mean compliance and the compliance standard deviation of this structure are given respectively by 56.88 and 15.78, which results in a coefficient of variation of 0.28. This result indicates that the proposed robust approach is able to obtain better solutions than its deterministic counterpart.

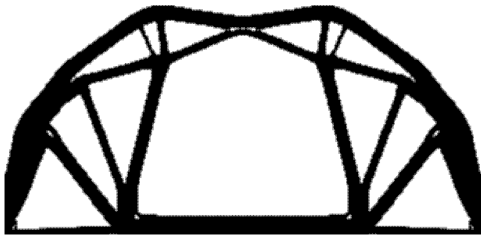
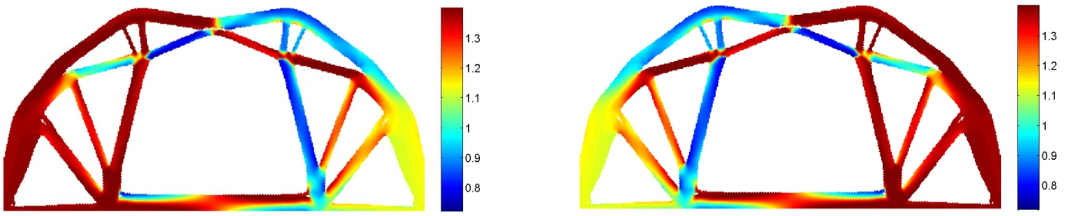


Figure 10: Example 2: result for the robust case.

The distributions of the load scale factors in the worst case scenario x_1 and x_2 are shown in Fig. 11. It is interesting to note that the point-wise worst case scenarios present significant variation over the structural domain.



(a) index x_1 (b) index x_2

Figure 11: Example 2: load scale factors x_1 and x_2 for the robust case.

5 CONCLUDING REMARKS

In this paper an alternative approach for robust topology optimization has been presented. Thanks to the SORA approach, the problem can be rewritten as a deterministic optimization problem with modified parameters, that are obtained with inverse reliability analysis. This allows standard deterministic algorithms already developed to be used in the context of robust optimization. Here, the deterministic topology optimization problem has been solved using an efficient method based on the topological derivative concept together with a level-set domain representation method as proposed in Amstutz and Andrä (2006).

The examples presented here considered only the magnitude of the applied load as random variables. In this case, the stress and strain tensors can be written as a linear combination of the load scale factors and the canonical stress and strain tensors, reducing drastically the computational effort. The examples presented show that the approach is able to obtain optimum solutions that take into account uncertainties of some parameters.

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