NEW VARIATIONAL PRINCIPLES FOR THE ASYMPTOTIC ELASTOPLASTIC RESPONSE UNDER CYCLIC LOADS

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Abstract. *The subject of the paper is the direct computation of the asymptotic steady state produced in an elastoplastic body submitted to cyclic loadings exceeding the shakedown limits. We present two new mixed variational principles for the continuum model of the body, whose stationary conditions properly represent known equations characterizing the asymptotic response of stress and plastic strain rate fields. We discuss different time-space discretizations of the mentioned principles giving optimization problems amenable to practical application. The numerical procedure so obtained is demonstrated in one example of application.*

Keywords: *Cyclic plasticity, Variational formulations, Ratcheting*

1 INTRODUCTION

The subject of the paper is the direct computation of the asymptotic steady state produced in an elastoplastic body submitted to cyclic loadings exceeding the shakedown limits (Frederick and Armstrong, 1966; Polizzotto, 2003; Ponter and Chen, 2001; Zouain, 2004).

The fact that many structures attain a stabilized response has been experimentally observed, and also theoretically proven for some general models of inelastic behavior, when submitted to cyclic loadings in excess of elastic shakedown. It was so recognized that the stresses in such structures variate cyclically and with the same period as the applied loading. Further, the plastic strain rates are also periodic and either: (a) exhibit zero net increment per cycle, in cases named alternating plasticity (plastic shakedown) or (b) undergo constant net increment, per cycle, of the plastic strains, in the case of a ratcheting process. A remarkable characteristic of the stabilized solution is that it is independent of initial conditions.

We focus attention on direct methods of computing the asymptotic response under cyclic loads (Spiliopoulos and Panagiotou, 2012; Tereshin and Cherniavsky, 2015), as opposed to incremental integration used to reproduce the steady state solution beyond the transient deformation process.

In summary, we present two new mixed variational principles for the continuum model of the body, whose stationary conditions properly represent known equations characterizing the asymptotic response of stress and plastic strain rate fields. We discuss different time-space discretizations of the mentioned principles giving optimization problems amenable to practical application. The numerical procedure so obtained is demonstrated in one example of application.

The basic tool for the attainment of the present variational formulations for the continuum is a mixed minimization principle, in terms of stress and plastic strain rates, for the flow law of the elastic-ideally plastic material.

2 Elastoplastic analysis

Let us first define some basic notation and definitions.

A body is associated with an open bounded region $\mathcal{B}$ with regular boundary Γ. The space $\mathcal{V}$ is the set of all admissible velocity fields $\boldsymbol{v}$ complying with homogeneous boundary conditions prescribed on a part Γ_u of Γ. The strain rate tensor field $\boldsymbol{d}$ is an element of the space $\mathcal{W}$. The tangent deformation operator $\boldsymbol{\mathcal{D}}$ (the symmetric gradient in 3D models) maps $\mathcal{V}$ into $\mathcal{W}$; we write this relationship as $\boldsymbol{d} = \boldsymbol{\mathcal{D}}\boldsymbol{v}$. Let $\mathcal{W}'$ be the space of stress fields $\boldsymbol{\sigma}$ and $\mathcal{V}'$ the space of load systems $\boldsymbol{F}$. A load $\boldsymbol{F}$ is in equilibrium with a stress field $\boldsymbol{\sigma}$ iff the principle of virtual power is satisfied, that is when

$$\int_{\mathcal{B}} \boldsymbol{\sigma} \cdot \boldsymbol{\mathcal{D}}\boldsymbol{v} d\mathcal{B} = \int_{\mathcal{B}} \boldsymbol{b} \cdot \boldsymbol{v}\, d\mathcal{B} + \int_{\Gamma_\tau} \boldsymbol{\tau} \cdot \boldsymbol{v}\, d\Gamma \qquad \forall\, \boldsymbol{v} \in \mathcal{V} \tag{1}$$

where $\boldsymbol{b}$ and $\boldsymbol{\tau}$ are volume and surface load densities, defining $\boldsymbol{F}$, and $\Gamma_\tau = \Gamma \setminus \Gamma_u$ is part of the boundary Γ where traction is prescribed.

Under small deformation assumptions, the kinematical and equilibrium relations are then written, in terms of mappings on field spaces, as

$$\varepsilon = \boldsymbol{\mathcal{D}}\boldsymbol{u} \qquad \boldsymbol{\mathcal{D}}'\boldsymbol{\sigma} = \boldsymbol{F} \tag{2}$$

We also assume additive decomposition of strains, then

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p \qquad \boldsymbol{\varepsilon}^e = \mathbb{E}^{-1}\boldsymbol{\sigma} \tag{3}$$

with $\mathbb{E}$ denoting the linear elastic operator.

The material is ideally-plastic and characterized by the set P of plastically admissible stress tensors, for each point x of the body. The set P is convex and the null stress tensor belongs to its strict interior. The plastic flow law is associated, so that the plastic strain rate $\boldsymbol{d}^p(\mathrm{x})$ associated to a given stress $\boldsymbol{\sigma}(\mathrm{x})$ must belong to the cone of normals to the boundary of P at $\boldsymbol{\sigma}(\mathrm{x})$. This is written in compact form as

$$\boldsymbol{d}^p(\mathrm{x}) \in N_P(\boldsymbol{\sigma}(\mathrm{x})) \tag{4}$$

More precisely, by definition this is equivalent to the following set of relations:

$$\boldsymbol{\sigma}(\mathrm{x}) \in P \tag{5}$$

$$[\boldsymbol{\sigma}(\mathrm{x}) - \boldsymbol{\sigma}^*] \cdot \boldsymbol{d}^p(\mathrm{x}) \ \geqslant 0 \quad \forall\, \boldsymbol{\sigma}^* \in P \tag{6}$$

This implies that $\boldsymbol{d}^p(\mathrm{x}) = \mathbf{0}$ if $\boldsymbol{\sigma}(\mathrm{x})$ is strictly interior to P.

The dissipation (per unit volume) function is defined as

$$D(\boldsymbol{d}^p(\mathrm{x})) := \sup_{\boldsymbol{\sigma}^*} \{\boldsymbol{\sigma}^* \cdot \boldsymbol{d}^p(\mathrm{x}) \ \mid \ \boldsymbol{\sigma}^* \in P\} \tag{7}$$

Thus

$$D(\boldsymbol{d}^p(\mathrm{x})) - \boldsymbol{\sigma}(\mathrm{x}) \cdot \boldsymbol{d}^p(\mathrm{x}) \geqslant 0 \qquad \forall \quad (\boldsymbol{\sigma}(\mathrm{x}), \boldsymbol{d}^p(\mathrm{x})) \ \mid \ \boldsymbol{\sigma}(\mathrm{x}) \in P \tag{8}$$

Moreover, the evolution law admits the following equivalent formulations

$$D(\boldsymbol{d}^p(\mathrm{x})) - \boldsymbol{\sigma}(\mathrm{x}) \cdot \boldsymbol{d}^p(\mathrm{x}) = 0 \quad \text{and} \quad \boldsymbol{\sigma}(\mathrm{x}) \in P \qquad \Leftrightarrow \qquad \boldsymbol{d}^p(\mathrm{x}) \in N_P(\boldsymbol{\sigma}(\mathrm{x})) \tag{9}$$

In summary, the relations determining the incremental elastoplastic analysis are: the kinematical and equilibrium equations (2), the state equations (3) and the plastic evolution equation (9), together with suitable prescribed initial conditions.

Combining (2) and (3) we obtain

$$\boldsymbol{u} = \boldsymbol{\mathcal{K}}^{-1} \left(\boldsymbol{F} + \boldsymbol{\mathcal{D}}'\mathbb{E}\boldsymbol{\varepsilon}^p\right) \qquad \boldsymbol{\sigma} = \mathbb{E}\boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{K}}^{-1}\boldsymbol{F} + \boldsymbol{\mathcal{Z}}\boldsymbol{\varepsilon}^p \tag{10}$$

where

$$\boldsymbol{\mathcal{K}} := \boldsymbol{\mathcal{D}}'\mathbb{E}\boldsymbol{\mathcal{D}} \qquad \boldsymbol{\mathcal{Z}} := \mathbb{E}\boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{K}}^{-1}\boldsymbol{\mathcal{D}}'\mathbb{E} - \mathbb{E} \tag{11}$$

are the stiffness mapping (from $\mathcal{V}$ to $\mathcal{V}'$) and the mapping (from $\mathcal{W}$ to $\mathcal{W}'$) giving the residual stress field associated with certain plastic strain field, respectively.

The operator $\boldsymbol{\mathcal{Z}}$ has the following properties: (i) it is symmetric and negative semi-definite, (ii) its range coincides with the linear space of residual stresses (the kernel of $\boldsymbol{\mathcal{D}}'$) and (iii) its null-space coincides with the linear space of compatible strain fields (the range of $\boldsymbol{\mathcal{D}}$).

We also write (10) as

$$\boldsymbol{u} = \boldsymbol{u}^E + \boldsymbol{\mathcal{K}}^{-1}\boldsymbol{\mathcal{D}}'\mathbb{E}\boldsymbol{\varepsilon}^p \qquad \boldsymbol{\sigma} = \boldsymbol{\sigma}^E + \boldsymbol{\sigma}^r \qquad \boldsymbol{\sigma}^r = \boldsymbol{\mathcal{Z}}\boldsymbol{\varepsilon}^p \tag{12}$$

where

$$\boldsymbol{u}^E = \boldsymbol{\mathcal{K}}^{-1}\boldsymbol{F} \qquad \boldsymbol{\sigma}^E = \mathbb{E}\boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{K}}^{-1}\boldsymbol{F} \tag{13}$$

are the solutions of the ideally elastic analogous structure.

2.1 The equations of the steady state

Loads are periodic with period T. It is always possible in inviscid plasticity to use a time parameter such that $T = 1$; thus periodic loads verify $\boldsymbol{F}(\hat{t}) = \boldsymbol{F}(\hat{t} + 1)$ for all $\hat{t}$. Further, time in each period m is denoted $t := \hat{t} - (m - 1) \in (0, 1)$.

The cyclic loading is represented by the fictitious cyclic elastic field $\boldsymbol{\sigma}^E(t)$.

We state in the following the equations governing the asymptotic cyclic response. This system of equations is the same system, described above, applicable to a general loading program, except that the initial conditions are substituted by a constraint enforcing that the asymptotic stress is cyclic (with same period as the loading). However, in formulating the system of equations we prefer to use residual stresses rather than total stresses. Accordingly, we obtain the formulation given below.

The equations of the asymptotic response - Given $\boldsymbol{\sigma}^E(t)$, find $\boldsymbol{\sigma}^r(t)$, $\boldsymbol{\varepsilon}^p(t)$, $\boldsymbol{d}^p(t)$, for $t \in [0, 1]$, and $\Delta\boldsymbol{u}$ such that

$$\boldsymbol{d}^p = \dot{\boldsymbol{\varepsilon}}^p \tag{14}$$

$$\boldsymbol{d}^p \in N_P(\boldsymbol{\sigma}^E + \boldsymbol{\sigma}^r) \tag{15}$$

$$\int_0^1 \boldsymbol{d}^p \, dt = \boldsymbol{\mathcal{D}} \Delta\boldsymbol{u} \tag{16}$$

$$\boldsymbol{\sigma}^r = \boldsymbol{\mathcal{Z}} \boldsymbol{\varepsilon}^p \tag{17}$$

In the above formulation, the self-equilibrium condition $\boldsymbol{\mathcal{D}}'\boldsymbol{\sigma}^r = \boldsymbol{0}$ is a consequence of (17), because range$\boldsymbol{\mathcal{Z}}$ = null$\boldsymbol{\mathcal{D}}'$, and the plastic admissibility of $\boldsymbol{\sigma}^E + \boldsymbol{\sigma}^r$ is implicitly enforced by (15).

The cyclic constraint $\boldsymbol{\sigma}(1) = \boldsymbol{\sigma}(0)$ was substituted in the system above by (16); this requires further explanations. Indeed, from (17) and (14) we obtain $\dot{\boldsymbol{\sigma}}^r = \boldsymbol{\mathcal{Z}} \boldsymbol{d}^p$ and by integration and (16) that $\boldsymbol{\sigma}(1) - \boldsymbol{\sigma}(0) = \boldsymbol{\mathcal{Z}}\boldsymbol{\mathcal{D}}\Delta\boldsymbol{u}$. But this is zero due to the properties of $\boldsymbol{\mathcal{Z}}$.

It is convenient to eliminate from the unknowns of the system above, (14) to (17), the plastic strain field $\boldsymbol{\varepsilon}^p$, which is not cyclic in general. To this end, we adopt instead the following equations, which equivalently ensure $\boldsymbol{\sigma}^r = \boldsymbol{\mathcal{Z}}\boldsymbol{\varepsilon}^p$.

Given $\boldsymbol{\sigma}^E(t)$, find $\boldsymbol{\sigma}^r(t)$, $\boldsymbol{d}^p(t)$ and $\Delta\boldsymbol{u}$ such that

$$\boldsymbol{d}^p \in N_P(\boldsymbol{\sigma}^E + \boldsymbol{\sigma}^r) \tag{18}$$

$$\int_0^1 \boldsymbol{d}^p \, dt = \boldsymbol{\mathcal{D}} \Delta\boldsymbol{u} \tag{19}$$

$$\dot{\boldsymbol{\sigma}}^r = \boldsymbol{\mathcal{Z}} \boldsymbol{d}^p \tag{20}$$

$$\boldsymbol{\mathcal{D}}'\boldsymbol{\sigma}^r(1) = \boldsymbol{0} \tag{21}$$

The system of equations (18) to (21) is used in the following as the characterization of the asymptotic response to cyclic loadings.

Adopting (19) in the definition of the steady state problem is an important option. In fact, we could substitute (19) by the equivalent relation $\boldsymbol{\mathcal{Z}} \int_0^1 \boldsymbol{d}^p \, dt = \boldsymbol{0}$, with the result that the displacement unknown $\Delta \boldsymbol{u}$ would be eliminated from the system. However, it is simpler and computationally more convenient to impose the compatibility condition (19) rather than the mentioned equivalent relation.

3 Principles for the asymptotic cyclic response

Due to (8) and (9), the set of fields $\boldsymbol{\sigma}^r$ and $\boldsymbol{d}^p$ merely satisfying (at all times and all points) the constitutive equation (18) coincides with the set of solutions of the following minimization problem

$$\inf_{\boldsymbol{\sigma}^r, \boldsymbol{d}^p} \left\{ \int_0^1 \int_{\mathcal{B}} \left[D(\boldsymbol{d}^p) - (\boldsymbol{\sigma}^E + \boldsymbol{\sigma}^r) \cdot \boldsymbol{d}^p \right] d\mathbf{x}\, dt \quad \Big| \quad \forall t \quad \boldsymbol{\sigma}^E + \boldsymbol{\sigma}^r \in P \right\} \tag{22}$$

The asymptotic response can be selected now by joining (19-21) as additional constraints. This leads to our first variational principle for the continuum.

Minimum principle I

For given loadings, represented by the ideally elastic stress fields $\boldsymbol{\sigma}^E(t)$, the asymptotic response, comprising $\{\boldsymbol{\sigma}^r(t), \boldsymbol{d}^p(t);\ t \in [0,1]\}$ and $\Delta \boldsymbol{u} = \boldsymbol{u}(1) - \boldsymbol{u}(0)$, solves the minimization problem

$$\hat{\Upsilon} := \inf_{\Delta \boldsymbol{u}, \boldsymbol{\sigma}^r, \boldsymbol{d}^p} \left\{ \int_0^1 \int_{\mathcal{B}} \left[D(\boldsymbol{d}^p) - (\boldsymbol{\sigma}^E + \boldsymbol{\sigma}^r) \cdot \boldsymbol{d}^p \right] d\mathbf{x}\, dt \ \Big| \right.$$
$$\left. \int_0^1 \boldsymbol{d}^p\, dt = \boldsymbol{\mathcal{D}} \Delta \boldsymbol{u};\ \boldsymbol{\mathcal{D}}' \boldsymbol{\sigma}^r(1) = \boldsymbol{0};\ \forall t \quad \boldsymbol{\sigma}^E + \boldsymbol{\sigma}^r \in P,\ \dot{\boldsymbol{\sigma}}^r = \boldsymbol{\mathcal{Z}} \boldsymbol{d}^p \right\} \tag{23}$$

according to the following conditions:

Proposition A - It holds that $\hat{\Upsilon}$ is finite and non-negative and either: (i) $\hat{\Upsilon} = 0$ and the minimizers $\boldsymbol{\sigma}^r$ and $\boldsymbol{d}^p$ of (23) are also asymptotic responses of the steady state problem given by equations (18) to (21), or (ii) $\hat{\Upsilon} > 0$ and there are no possible asymptotic responses, solving (18-21), for the given cyclic load program.

Consequently, if the sufficient conditions for existence of asymptotic response are guaranteed, then the mixed minimum principle (23) is an equivalent formulation of the steady state problem.

Proof. - In view of (8), the integrand in the objective function is non-negative at all times and all points because the stress field $\boldsymbol{\sigma}^E + \boldsymbol{\sigma}^r$ is constrained to be plastically admissible, at all points, for all times. This proves $\hat{\Upsilon} \geqslant 0$.

In case (i), $\hat{\Upsilon} = 0$ leads to conclude that the integrand in the objective function is zero at any time and any point; thus, the flow law (18) is obeyed by the minimizers of (23). The other equations in the system (18-21) are also satisfied as constraints of the minimization problem. This proves that the minimizers are steady state solutions.

On the other hand, if there exists a solution set to (18-21), it is feasible to the minimization principle and annihilates the integrand in the objective, thus proving $\hat{\Upsilon} = 0$. As a consequence, whenever it is found $\hat{\Upsilon} > 0$ in (23) (case (ii)) there cannot exist any steady state solution. This completes the proof of the proposition. $\square$

The objective functional in (23) can be simplified as explained in the following. Firstly, we claim that

$$\int_0^1 \int_{\mathcal{B}} \boldsymbol{\sigma}^r \cdot \boldsymbol{d}^p \, d\mathbf{x} \, dt = 0 \tag{24}$$

under the set of conditions, implied by the constraints in (23):

$$\boldsymbol{\mathcal{D}}' \boldsymbol{\sigma}^r(1) = \mathbf{0} \tag{25}$$

$$\dot{\boldsymbol{\sigma}}^r = \boldsymbol{\mathcal{Z}} \boldsymbol{d}^p \qquad \forall t \tag{26}$$

$$\boldsymbol{\mathcal{Z}} \Delta \boldsymbol{\varepsilon}^p = \mathbf{0} \tag{27}$$

where

$$\Delta \boldsymbol{\varepsilon}^p := \boldsymbol{\varepsilon}^p(1) - \boldsymbol{\varepsilon}^p(0) = \int_0^1 \boldsymbol{d}^p \, dt \tag{28}$$

Indeed, we integrate (26) to obtain $\boldsymbol{\sigma}^r(t) = \bar{\boldsymbol{\sigma}}^r + \boldsymbol{\mathcal{Z}} \boldsymbol{\varepsilon}^p(t)$, with the constant of integration satisfying $\boldsymbol{\mathcal{D}}' \bar{\boldsymbol{\sigma}}^r = \mathbf{0}$ because $\boldsymbol{\sigma}^r(1)$ and $\boldsymbol{\mathcal{Z}} \boldsymbol{\varepsilon}^p(1)$ are self-equilibrated. Then

$$\int_0^1 \int_{\mathcal{B}} \boldsymbol{\sigma}^r \cdot \boldsymbol{d}^p \, d\mathbf{x} \, dt = \int_{\mathcal{B}} \bar{\boldsymbol{\sigma}}^r \cdot \Delta \boldsymbol{\varepsilon}^p \, d\mathbf{x} + \int_0^1 \int_{\mathcal{B}} \boldsymbol{\mathcal{Z}} \boldsymbol{\varepsilon}^p \cdot \boldsymbol{d}^p \, d\mathbf{x} \, dt = \tfrac{1}{2} \left[\int_{\mathcal{B}} \boldsymbol{\mathcal{Z}} \boldsymbol{\varepsilon}^p \cdot \boldsymbol{\varepsilon}^p \, d\mathbf{x} \right]_0^1 \tag{29}$$

where we used (28) and the fact that $\Delta \boldsymbol{\varepsilon}^p$ is compatible because of (27). Moreover, since $\boldsymbol{\mathcal{Z}}$ is symmetric, we have

$$\boldsymbol{\mathcal{Z}} \boldsymbol{\varepsilon}^p(1) \cdot \boldsymbol{\varepsilon}^p(1) = \boldsymbol{\mathcal{Z}} [\boldsymbol{\varepsilon}^p(0) + \Delta \boldsymbol{\varepsilon}^p] \cdot [\boldsymbol{\varepsilon}^p(0) + \Delta \boldsymbol{\varepsilon}^p] = \boldsymbol{\mathcal{Z}} \boldsymbol{\varepsilon}^p(0) \cdot \boldsymbol{\varepsilon}^p(0) \tag{30}$$

Finally, (29) and (30) prove our claim (24).

The above results imply, under the constraints of (23), that

$$\int_0^1 \int_{\mathcal{B}} \left(\boldsymbol{\sigma}^E + \boldsymbol{\sigma}^r \right) \cdot \boldsymbol{d}^p \, d\mathbf{x} \, dt = \int_0^1 \int_{\mathcal{B}} \boldsymbol{\sigma}^E \cdot \boldsymbol{d}^p \, d\mathbf{x} \, dt \tag{31}$$

With respect to the objective function in (23), we remark that $D(\boldsymbol{d}^p) - (\boldsymbol{\sigma}^E + \boldsymbol{\sigma}^r) \cdot \boldsymbol{d}^p$ is nonnegative at all times, due to constraints, while $D(\boldsymbol{d}^p) - \boldsymbol{\sigma}^E \cdot \boldsymbol{d}^p$ changes sign along the cycle, although its integral is nonnegative under the assumed constraints.

Finally, we have:

Minimum principle II

For given loadings, represented by $\boldsymbol{\sigma}^E(t)$, the minimization problem

$$\hat{\Upsilon} := \inf_{\Delta \boldsymbol{u}, \boldsymbol{\sigma}^r, \boldsymbol{d}^p} \left\{ \int_0^1 \int_{\mathcal{B}} \left[D(\boldsymbol{d}^p) - \boldsymbol{\sigma}^E \cdot \boldsymbol{d}^p \right] d\mathbf{x} \, dt \;\middle|\; \int_0^1 \boldsymbol{d}^p \, dt = \boldsymbol{\mathcal{D}} \Delta \boldsymbol{u};\ \boldsymbol{\mathcal{D}}' \boldsymbol{\sigma}^r(1) = \mathbf{0};\ \forall t \quad \boldsymbol{\sigma}^E + \boldsymbol{\sigma}^r \in P,\ \dot{\boldsymbol{\sigma}}^r = \boldsymbol{\mathcal{Z}} \boldsymbol{d}^p \right\} \tag{32}$$

has optimum value $\hat{\Upsilon} = 0$ and the minimizers constitute the asymptotic cyclic response, solving (18-21), in the conditions described in Proposition A.

4 On the discretization of the minimum principles

Discrete counterparts of the proposed continuum principles are generated based on a time grid for the interval $[0,1]$ of the cycle and a mixed finite element interpolation for the stress and displacement fields. This was already performed by the authors, using all mixed finite elements presented by Zouain, Borges and Silveira (2014). We do not describe this here (it will be published elsewhere) but we address in the following the most important issues concerning this discretization.

The main option in our discretization, that deserves justification, is that we use the Minimum Principle I, given by (23), instead of Principle II, (32), which is strictly equivalent in the continuum framework. The reason to this choice is that with Principle I we can generate, using standard approximations, a finite dimensional minimization problem in which the crucial condition of reaching a minimum exactly equal to zero is attainable. Consequently, the exact evolution equation (the associated flow law) is induced for the set of discrete points in the finite element approximation.

Alternatively, one could use truncated Fourier series for the discretization in time. In doing so, the constraint $\dot{\boldsymbol{\sigma}}^r = \boldsymbol{\mathcal{Z}} \boldsymbol{d}^p$ can be exactly fulfilled and the strict equivalence of principles is recovered. This was tried by the authors but the results were much poorer than with our first choice, described before.

Our discretization adopts the backward Euler in the approximation of the plastic flow law, so that the results of our direct approach exactly match the converged incremental solution.

5 A block under cyclic thermo-mechanical loadings

The minimization enunciated in Principle I, and given by (23), was applied to the block under thermo-mechanical loads shown in Figure 1 (Zouain, 2004). The mechanical load p remains constant while the thermal load q, representing imposed temperature, fluctuates between zero and a prescribed value $q_{\max}$, as shown in the first graph of Figure 1. Peak load values was chosen to produce the combined mechanism of incremental collapse (ratcheting) also shown in the exact solutions of the same figure.

Figure 2 displays the approximate solution, shown by small circles representing discrete values obtained dividing the cycle in 32 intervals. The exact solution is plotted by solid lines. Not surprisingly, exact and approximate plots are coincident.

Although very simple, this structure undergoes the most complex (non-shakedown) failure mechanism. The solution presented here was obtained by a code incorporating all the features required to process complex geometries by finite element interpolations, presently under development.

6 Conclusions

Two variational principles are proposed in this paper. They are strictly equivalent in the context of the continuum formulation where the corresponding constraints are supposed to be

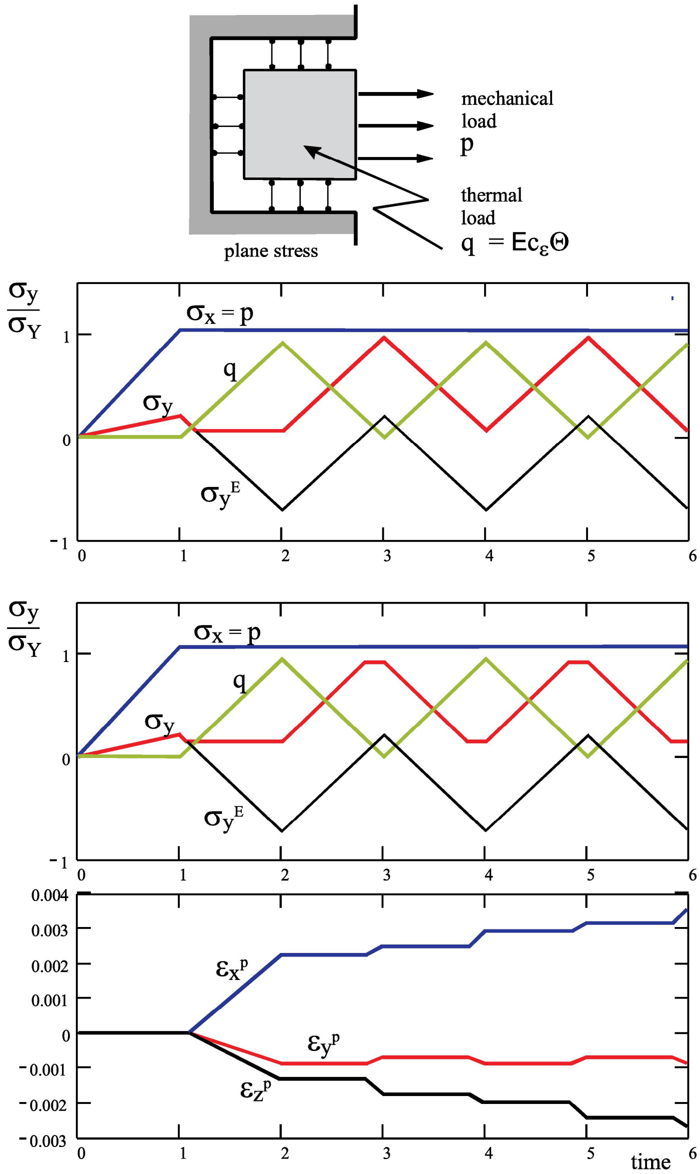

Figure 1: The response of a restrained block under two cyclic thermo-mechanical loadings with $p = p_{\max}$ and $q \in (0, q_{\max})$: (i) elastic shakedown for $(p_{\max}, q_{\max}) = (1.0268\sigma_Y, 0.91496\sigma_Y)$, and (ii) combined mechanism of incremental collapse for $(p_{\max}, q_{\max}) = (1.0627\sigma_Y, 0.94698\sigma_Y)$. Stresses and loads are plotted divided by σ_Y. The elastic constants are $E = 1000\,\sigma_Y$ and $\nu = 0.2$. Reprinted from E. Stein, R. de Borst and T. J. R. Hughes, eds., *Encyclopedia of Computational Mechanics*; Zouain, N.,, Shakedown and Safety Assessment, pp. 291-334. 2004; John Wiley.

valid at any material point as well as at any instant, for exact time rates. However, one of the formulations leads directly to a consistent discrete principle while the other loses crucial properties when discretized under usual assumptions such as backward Euler approximation of evolution equations.

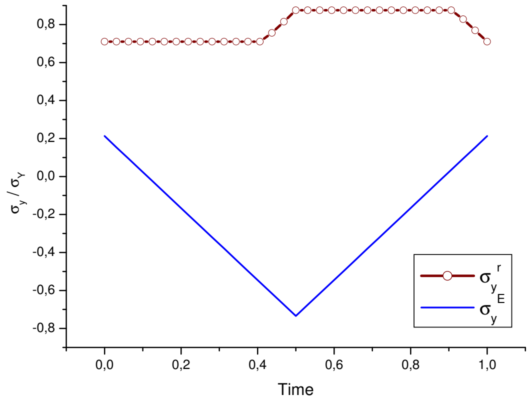

Figure 2: Discrete solution for the residual stress in the block under thermo-mechanical loadings.

A formulation similar to our Minimum Principle II was given by Tereshin and Cherniavsky (2015, p. 86) and then it is used in their discretization (Tereshin and Cherniavsky, 2015, p. 95). Polizzotto (2003, p. 2683) states a minimization problem analogous to our Principle II.

Discrete counterparts of the Minimum Principle I have been developed by the authors (to be published elsewhere); one of them is demonstrated here in a very simple numerical application.

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