XXXVI Ibero-Latin American Congress on Computational Methods in Engineering
CILAMCE
Rio de Janeiro, 22-25 Nov 2015

COMPARATIVE STUDY OF CONSTITUTIVE LAWS IN VIBRATIONS OF RECTANGULAR HYPERELASTIC MEMBRANES

Renato de Sousa e Silva

Renata M. Soares

Zenon J. G. N. del Prado

renatoses@gmail.com

msrenata@gmail.com

zenon@ufg.br

Escola de Engenharia Civil, Universidade Federal de Goiás, UFG

Av. Universitária, s/n, Setor Universitário, 74605-200, Goiânia, GO, Brazil

Paulo B. Gonçalves

paulo@puc-rio.br

Department of Civil Engineering, Pontifical Catholic University, PUC-Rio

Rua Marquês de São Vicente, 225, Gávea, 22451-900 Rio de Janeiro, RJ, Brazil

Abstract. *The aim of the present work is to make a comparative study of the influence of constitutive laws on the vibration response of a pre-stretched rectangular hyperelastic membrane. The membrane is composed of an isotropic, homogeneous, hyperelastic and incompressible material, which is modeled with different constitutive laws, specifically: Yeoh, Mooney-Rivlin and neo-Hookean. The exact solution of the membrane under a biaxial stretch is obtained and the equations of motion of the pre-stretched membrane are derived. From the linearized equations, the natural frequencies and mode shapes of the membrane are obtained analytically and the natural modes are used to approximate the nonlinear deformation field using the Galerkin method. Finally, results with different constitutive laws are compared.*

Keywords: *Rectangular hyperelastic membrane, constitutive law, Large amplitude vibrations.*

1 INTRODUCTION

The study of membranes is an important research topic in nonlinear continuum mechanics and the analysis of elastomeric membranes under finite deformations is a rather challenging subject. The advances in new membrane materials and the development of appropriate constitutive laws also contribute to the importance of research in this area. Most membrane materials are incompressible or almost incompressible and undergo large strains when subjected to loads. These materials usually exhibit very large strains, with a strongly nonlinear stress–strain relation.

The choice of an appropriate constitutive law is a key step in the mathematical modelling of hyperelastic materials. The neo-Hookean, Mooney-Rivlin, Yeoh, Arruda-Boyce, Ogden, Hadamard, Blatz-Ko, and Ogden-Storakers models are among the most frequently used forms of strain-energy density functions in the literature (Boyce and Arruda, 2000; Pascalis, 2010; Bower, 2010). The simplest constitutive model is the neo-Hookean model, which can be viewed as a simplification of the Mooney–Rivlin or Yeoh constitutive laws. These three constitutive laws are widely used in the literature. The reader may refer to the works by Selvadurai (2006) and Saccomandi and Ogden (2004) for critical reviews on constitutive models for hyperelastic materials.

So, hyperelastic membrane analysis finds applications ranging from bio-engineering devices, space structures, actuators, sensors, robotics, and large civil engineering structures. A detailed description of practical applications and the relevant literature can be found in Gonçalves et.al (2009) and Soares and Gonçalves (2012). The first developments in the field of nonlinear finite elasticity plus recent advances and useful historical reviews can be found in the works by Green and Adkins (1970), Treloar (1975), Ogden (1984), Libai and Simmonds (1998), Fu and Ogden (2001), and Selvadurai (2006), among others.

Generally, the strain-energy density is assumed to be dependent on the three principal invariants of the deformation tensor or the related principal stretches. Uniaxial or biaxial experiments are usually employed to determine the material parameters and the most accurate constitutive model applicable to the range of deformations of interest in a particular practical application (Przybylo and Arruda, 1998; Sasso et al, 2008; Selvadurai and Shi, 2012).

Thereby the aim of this paper is to present a parametric analysis clarifying the influence of the membrane initial stretches and constitutive laws on the nonlinear vibrations and instabilities of the rectangular hyperelastic membrane. The membrane material is assumed to be isotropic and incompressible, and its behavior is described by three constitutive laws: neo-Hookean, Mooney-Rivlin and Yeoh. These hypotheses have been widely used to describe the behavior of elastomers.

2 FORMULATION

This work considers a homogeneous, isotropic, rectangular hyperelastic membrane of undeformed lengths in the x and y directions of, respectively, L_{xo} and L_{yo}, thickness h, and mass density Γ, which under goes finite elastic deformations, as illustrated in Fig. 1, where the deformed and undeformed configurations, basic geometric parameters, and associated coordinate systems are depicted.

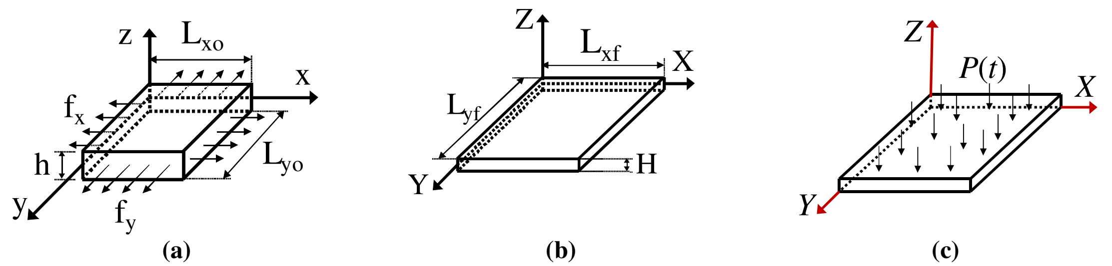


Figure 1 - Geometry of membrane. (a) Undeformed geometry; (b) Streched geometry; (c) Dynamic load.

It is assumed that h/L_{xo} , h/L_{yo} << 1 so that the deformed membrane can be described by the theory of hyperelastic membranes under finite deformations (Green and Adkins, 1960). A key point in the analysis of hyperelastic membranes using the theory of finite elasticity is the choice of an appropriate strain energy density function. Strain-invariant constitutive models are usually used to describe the behavior of hyperelastic materials. If the material is incompressible the third strain invariant, $I_3 = 1$.

For the representation of elastomers, there are in the literature (Boyce and Arruda, 2000) different constitutive laws with the strain energy density function, represented by a polynomial function that, in this work, are approached by the neo-Hookean, Mooney-Rivlin and Yeoh, which is the strain energy density function shown below.

neo-Hookean $$W = C_1(I_1 - 3) \tag{1}$$

Mooney-Rivlin $$W = C_1(I_1 - 3) + C_2(I_2 - 3) \tag{2}$$

Yeoh $$W = \sum_{i=1}^{N} C_i (I_1 - 3)^i \tag{3}$$

Where C_1, C_2, and C_i are the material parameters. The neo-Hookean strain density function can be considered as a special case of Eq. (2) with $C_2 = 0$ or of Eq.(3) with N=1. For the Yeoh strain density function will be considered N = 2. The three strain invariants of the deformation field can be written in terms of the principal stretches λ_i (i=1, 2, 3) as:

$$\begin{aligned} I_1 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\ I_2 &= (\lambda_1\lambda_2)^2 + (\lambda_1\lambda_3)^2 + (\lambda_2\lambda_3)^2 \\ I_3 &= (\lambda_1\,\lambda_2\,\lambda_3)^2 \end{aligned} \tag{4}$$

First, the membrane is subjected to an uniform finite stretch ratio in the x direction, δ_x, and an uniform stretch ratio in the y direction, δ_y, attaining the final dimensions L_{xf} and L_{yf} and a rectangular deformed configuration, as illustrated in Fig 1(b). It is considered that the stretched membrane is fixed along the four edges. Thus, the in-plane stretches are given by:

$$\lambda_1 = \frac{dX}{dx} = X_{,x} \qquad \lambda_2 = \frac{dY}{dy} = Y_{,y} \qquad \lambda_3 = \frac{1}{\lambda_1\,\lambda_2} = \frac{1}{X_{,x}Y_{,y}} \tag{5}$$

The nonlinear equilibrium equations of the stretched membrane, considering the three constitutive laws, are given by:

neo-Hookean
$$C_1\left[X_{,xx}+\frac{3X_{,xx}}{X_{,x}{}^4 Y_{,y}{}^2}\right]=0 \quad C_1\left[Y_{,yy}-\frac{3Y_{,yy}}{X_{,x}{}^2 Y_{,y}{}^4}\right]=0 \tag{6}$$

Mooney-Rivlin
$$\begin{gathered} C_1\left[X_{,xx}+\frac{3X_{,xx}}{X_{,x}{}^4 Y_{,y}{}^2}\right]+C_2\left[X_{,xx}Y_{,y}{}^2+\frac{3X_{,xx}}{X_{,x}{}^4}\right]=0 \\ C_1\left[Y_{,yy}-\frac{3Y_{,yy}}{X_{,x}{}^2 Y_{,y}{}^4}\right]+C_2\left[X_{,x}{}^2\, Y_{,yy}+\frac{3\,Y_{,yy}}{Y_{,y}{}^4}\right]=0 \end{gathered} \tag{7}$$

Yeoh
$$\begin{gathered} X_{,xx}\left[C_1\left(1+\frac{3}{X_{,x}^4Y_{,y}^2}\right)+2C_2\left(3X_{,x}^2+Y_{,y}^2+\frac{3}{X_{,x}^4}+\frac{5}{X_{,x}^6Y_{,y}^4}-\frac{9}{X_{,x}^4Y_{,y}^2}-3\right)\right]=0 \\ Y_{,yy}\left[C_1\left(1+\frac{3}{X_{,x}^2Y_{,y}^4}\right)+2C_2\left(X_{,x}^2+3Y_{,y}^2+\frac{3}{Y_{,y}^4}+\frac{5}{X_{,x}^4Y_{,y}^6}-\frac{9}{X_{,x}^2Y_{,y}^4}-3\right)\right]=0 \end{gathered} \tag{8}$$

The exact solution of the nonlinear equilibrium equations is given by:

$$\begin{aligned} X_o &= \delta_x\, x; \qquad \delta_x = L_{xf}\,/\,L_{xo} \\ Y_o &= \delta_y\, y; \qquad \delta_y = L_{yf}\,/\,L_{yo} \end{aligned} \tag{9}$$

A dynamic displacement field is superimposed to the static solution and the nonlinear equations of motion of the stretched membrane are obtained:

$$\begin{aligned} X &= X_o\,(x)+u\,(x,\,y,\,t) \\ Y &= Y_o\,(y)+v\,(x,\,y,\,t) \\ Z &= w\,(x,\,y,\,t) \end{aligned} \tag{10}$$

Due to the incompressibility condition, the mass density remains constant during the deformation process. The non-linear equations of motion are then obtained by applying Hamilton’s principle. They are:

$$\begin{aligned} &-\frac{\partial}{\partial x}\left(h\frac{\partial W}{\partial X_{,x}}\right)-\frac{\partial}{\partial y}\left(h\frac{\partial W}{\partial X_{,y}}\right)+h\Gamma\frac{\partial^2 u}{\partial t^2}-\zeta\frac{\partial u}{\partial t}=0 \\ &-\frac{\partial}{\partial x}\left(h\frac{\partial W}{\partial Y_{,x}}\right)-\frac{\partial}{\partial y}\left(h\frac{\partial W}{\partial Y_{,y}}\right)+h\Gamma\frac{\partial^2 v}{\partial t^2}-\zeta\frac{\partial v}{\partial t}=0 \\ &-\frac{\partial}{\partial x}\left(h\frac{\partial W}{\partial Z_{,x}}\right)-\frac{\partial}{\partial y}\left(h\frac{\partial W}{\partial Z_{,y}}\right)+h\Gamma\frac{\partial^2 w}{\partial t^2}-2\Gamma\omega\zeta\frac{\partial w}{\partial t}=P_0\cos(\Omega t)\delta_x\delta_y \end{aligned} \tag{11}$$

Where P_0 is the excitation magnitude and Ω is the excitation frequency.

The following expressions are obtained for the natural frequencies:

Neo/-Hookean
$$\omega_{mn}^2=\frac{2\pi^2 C_1}{\Gamma L_{xo}^2}\left[m^2\left(1-\frac{k_o^2}{\delta_x^6\,k_f^2}\right)+\frac{n^2}{k_o^2}\left(1-\frac{k_o^4}{\delta_x^6\,k_f^4}\right)\right] \tag{12}$$

Mooney-Rivlin
$$\omega_{mn}^2=\frac{2\pi^2 C_1}{\Gamma L_{xo}^2}\left[m^2\left(1-\frac{k_o^2}{\delta_x^6\,k_f^2}\right)\left(1+\alpha\delta_x^2\frac{k_f^2}{k_o^2}\right)+\frac{n^2}{k_o^2}\left(1-\frac{k_o^4}{\delta_x^6\,k_f^4}\right)\left(1+\alpha\delta_x^2\right)\right] \tag{13}$$

Yeoh
$$\omega_{mn}^2=\frac{2\pi^2 C_1}{\Gamma L_{xo}^2\delta_x^{12}k_f^6}\left[\delta_x^4\frac{k_f^2}{k_o^2}+2\alpha\left(\delta_x^6\frac{k_f^2}{k_o^2}+\delta_x^6\frac{k_f^4}{k_o^4}-3\delta_x^4\frac{k_f^2}{k_o^2}+1\right)\right]$$
$$\left[m^2\left(\delta_x^8 k_f^4k_o^2-\delta_x^2 k_f^2k_o^4\right)+\frac{n^2}{k_o^2}\left(\delta_x^8 k_f^4k_o^2-\delta_x^2 k_o^6\right)\right] \quad (14)$$

In the following sections, using as a reference the undeformed membrane dimensions, the following non dimensional geometrical parameters are introduced:

$$\bar{x}=\frac{x}{L_{xo}};\bar{y}=\frac{y}{L_{yo}};\bar{z}=\frac{z}{h};\tau=\omega_{mn}t;\bar{X}=\frac{X}{L_{xo}};\bar{Y}=\frac{Y}{L_{yo}};\bar{Z}=\frac{Z}{h};$$
$$\beta_x=\frac{h}{L_{xo}};\beta_y=\frac{h}{L_{yo}};\Delta=\frac{L_{xo}}{L_{yo}};\bar{\Omega}=\frac{\Omega}{\omega_{mn}};\bar{P}_o=\frac{P_o}{C_1};\bar{\Gamma}=\frac{\Gamma L_{xo}^2\omega_{mn}^2}{C_1} \quad (15)$$

From the linearized equation of motion (11) in the transversal direction:

neo-Hookeano
$$\frac{\partial^2 w}{\partial\tau^2}=\frac{2}{\bar{\Gamma}}\left[\frac{\left(\delta_x^6k_f^2-k_o^2\right)}{\delta_x^6k_f^2}w_{,\bar{x}\bar{x}}+\frac{\left(\delta_x^6k_f^4-k_o^4\right)}{\delta_x^6k_o^2k_f^4}w_{,\bar{y}\bar{y}}\right] \quad (16)$$

Mooney-Rivlin
$$\frac{\partial^2 w}{\partial\tau^2}=\frac{2}{\bar{\Gamma}}\left\{\frac{\left(\delta_x^6k_f^2-k_o^2\right)}{k_o^2\delta_x^6k_f^2}\left(\alpha\delta_x^2k_f^2+k_o^2\right)w_{,\bar{x}\bar{x}}+\frac{\left(\delta_x^6k_f^4-k_o^4\right)}{\delta_x^6k_o^2k_f^4}\left(\alpha\delta_x^2+1\right)w_{,\bar{y}\bar{y}}\right\} \quad (17)$$

Yeoh
$$\frac{\partial^2 w}{\partial\tau^2}=\frac{4}{\bar{\Gamma}}\left[1+2\alpha\left(-3+\delta_x^2+\frac{\delta_x^2k_f^2}{k_o^2}-\frac{k_o^6}{\delta_x^{10}k_f^6}+\frac{3k_o^4}{\delta_x^6k_f^4}-\frac{k_o^4}{\delta_x^4k_f^4}\right)-\frac{k_o^4}{\delta_x^6k_f^4}\right]\frac{\beta_y}{k_o}w_{,\bar{y}\bar{y}}$$
$$\left[1+2\alpha\left(-3+\delta_x^2+\frac{\delta_x^2k_f^2}{k_o^2}-\frac{k_o^4}{\delta_x^{10}k_f^4}+\frac{3k_o^2}{\delta_x^6k_f^2}\right)\right]\beta_x w_{,\bar{x}\bar{x}} \quad (18)$$

Where $k_f=L_{yf}/L_{xf}$, $k_o=L_{yo}/L_{xo}$ and the displacement is given by:

$$w\left(\bar{x},\bar{y},\tau\right)=\sum A_{mn}\sin\left(m\pi\bar{x}\right)\sin\left(n\pi\bar{y}\right)\cos\left(\bar{\omega}_{mn}\tau\right) \quad (19)$$

Where m is the half wave number of the corresponding vibration mode the x direction, n is the half wave number in the y direction and A_{mn} is the amplitude of displacement.

The Galerkin method is used to discretize the continuous system and the resulting nonlinear ordinary differential equations of motion in time domain are solved by numerical integration and continuation techniques. As shown in Soares (2009), the influence of the in-plane dynamic displacements u and v are negligible even for large amplitude vibrations and therefore are neglected in the present analysis.

3 NUMERICAL RESULTS

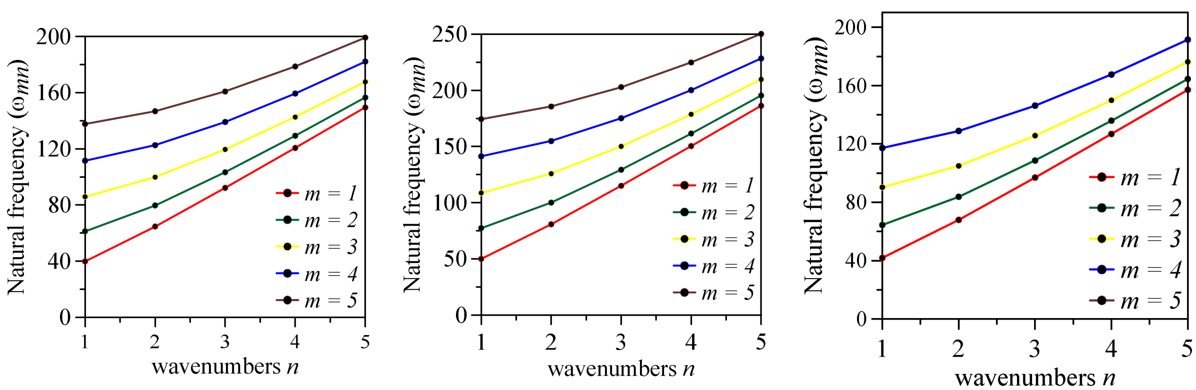
For the numerical analysis, a rectangular membrane with initial lengths $L_{xo} = 1$ *m*, k_o variable, thickness $h = 0.001$ *m*, and mass density $\Gamma = 1200$ Kg/m^3 is considered. The material constants are given in Soares (2014) and are presented in Table 1.

Table 1. Material Constants

neo-Hookean	$C_1 = 0{,}1361$ MPa
Mooney-Rivlin	$C_1 = 0{,}1361$ MPa; $C_2 = 0{,}0806$ MPa
Yeoh	$C_1 = 0{,}15$ MPa; $C_2 = 0{,}003$ Mpa

3.1 Linear vibration analysis

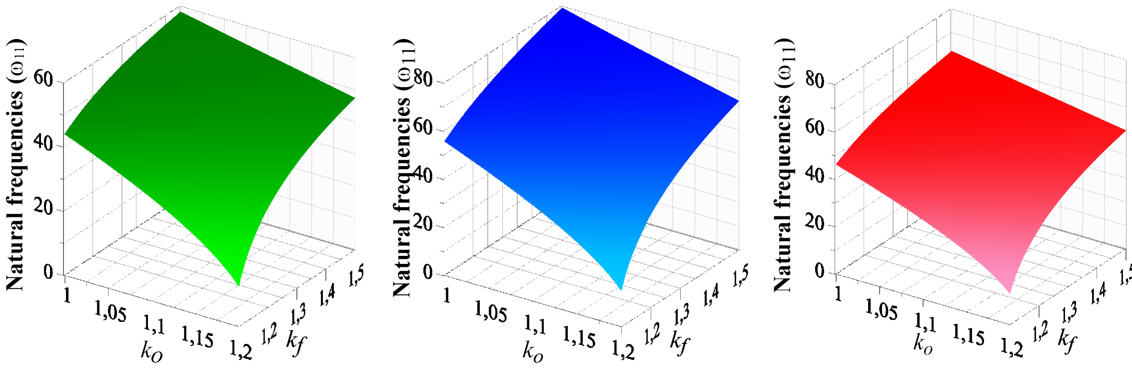
Figure 2 shows the typical variation of the natural frequencies as a function of the wavenumbers m and n for $\delta_x = 1.05$, $k_f = 1.05$ and $k_o = 1.0$. As can be observed, the natural frequencies increase with m and n and the lowest frequency always occurs for $m = 1$ and $n = 0$.



(a) **(b)** **(c)**

Figure 2. Characteristic frequency spectrum of a rectangular membrane. (a) neo-Hookean model; (b) Mooney-Rivlin model; (c) Yeoh model.

The influence of the stretching ratio and undeformed geometry on the lowest natural frequency, ω_{11}, is illustrated in Fig. 3 for a membrane under biaxial stretching with $\delta_x = 1.05$. It is observed that the three materials behave similarly, it is also possible to observe that increasing the constant k_f and decreasing the constant k_o, the natural frequency of the membrane increases meanwhile, increasing the value of k_o and decreasing k_f, the natural frequency also decreases.



(a) (b) (c)

Figure 3. Natural frequency (m = 1, n = 1) for a membrane under biaxial stretching. (a) neo-Hookean model; (b) Mooney-Rivlin model; (c) Yeoh model.

Table 2 displays the natural frequencies for membrane considering the variations in the initial geometry (k_o), the final geometry (k_f) and the stretch ratio (δ_x). It is observed that if traction (δ_x or k_f) is increased the natural frequency will also be increased, thereby the membrane gains stiffness. In the other side, increasing the initial geometry (k_o) the natural frequency of the membrane will be reduced. Furthermore, Mooney-Rivlin model material, provides higher frequency values than other materials for membranes with the same characteristics. These values were used in adimensionalization of equations for nonlinear analysis.

Table 2. Natural frequency (ω_{11}) with variation of membrane parameters (δ_x, k_o, k_f)

Membrane Parameters			Natural frequency ω_{11} (rad/s)		
δ_x	k_o	k_f	neo-Hookean	Mooney-Rivlin	Yeoh
1.01	1.00	1.00	16.109	19.795	16.912
1.05	1.00	1.00	33.709	41.985	35.409
1.10	1.00	1.00	44.159	55.945	46.455
1.10	1.02	1.00	41.965	52.960	44.129
1.10	1.05	1.00	38.508	48.295	40.476
1.10	1.00	1.02	45.775	58.208	48.176
1.10	1.00	1.05	47.871	61.203	50.420

The influence of the stretching ratio on the lowest natural frequency, ω_{11}, is illustrated in Fig. 4 for an initially square membrane (k_o = 1) under uniform equi-biaxial stretching (k_f = 1). As can be observed, for the three material models the frequency increases rapidly from zero as δ increases from 1 (unstretched configuration). For the neo-Hookean material the natural frequency tends to an upper bound by $\delta > 2$ and for the Mooney-Rivlin and Yeoh materials, the frequency exhibits an almost linear variation for $\delta > 2$, with a lower declivity for the Yeoh material model.

Figure 5 depicts the influence of the stretching ratio on the lowest natural frequency for an initially square membrane (k_o = 1) with increasing values of biaxial stretching δ_x and constant k_f. In this figure, the membrane depicts similar behavior as described in Fig. 4.

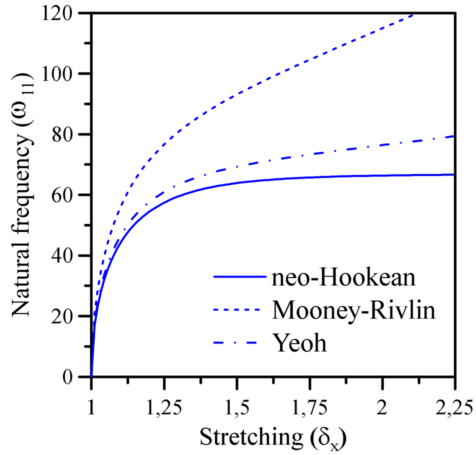
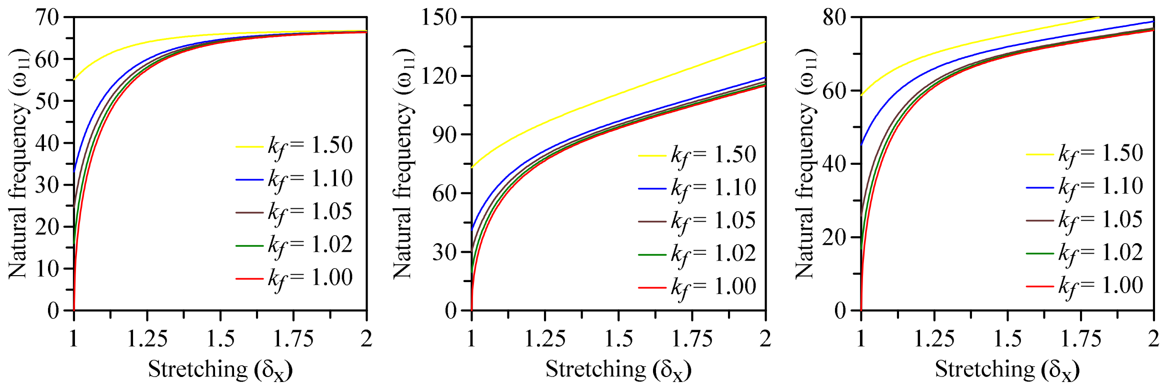


Figure 4. Square membrane (k_o = 1) under uniform equi-biaxial (k_f = 1) stretching. Variation of the lowest natural frequency (rad/s) as a function of the initial stretching ratio δ_x.



(a) (b) (c)

Figure 5. Variation of the lowest natural frequency (rad/s) as a function of the initial stretching ratio δ_x for the square membrane (k_o = 1) and biaxial stretching (k_f = variable). (a) Neo-Hookeano model; (b) Mooney-Rivlin model; (c) Yeoh model.

3.2 Nonlinear vibration analysis

The nonlinear vibration response of the membrane under large and small-amplitude vibrations is obtained using numerical integration and continuation techniques (Prado, 2001). In the following sections all results are nondimensional.

Figure 6 illustrates the influence of the stretching ratio for the membrane under initial uniform equi-biaxial stretching ($k_o = k_f = 1$) on the nonlinear frequency–amplitude relation. In Fig. 7 the backbone curves are depicted considering δ_x = 1.10 and k_f = 1 and increasing values of the undeformed geometry relation (k_o). Figure 8 displays the frequency-amplitude relation of the membrane considering increasing values of the deformed geometry relation (k_f) and fixed values of δ_x = 1.10 and k_o = 1. In these figures, the magnitude of the first harmonic A_{11} is reported. In all cases, for small vibration amplitudes, the response shows a strong increase in the natural frequency. For the neo-Hookean and Mooney-Rivlin materials, the vibration amplitude increases, the hardening effect decreases and the curve veers upward tending to a constant frequency value for large vibration amplitudes. However, for the Yeoh material, the vibration amplitude increases as the frequency value increased as well.

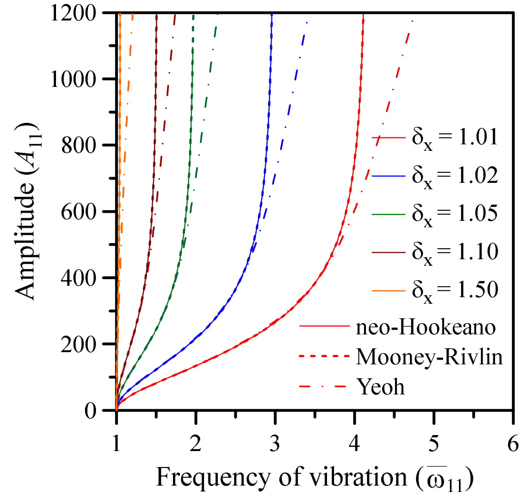


Figure 6. Frequency of vibration (rad/s) versus amplitude for the square membrane (k_o = 1) with uniform equi-biaxial stretching (k_f = 1) and δ_x = variable.

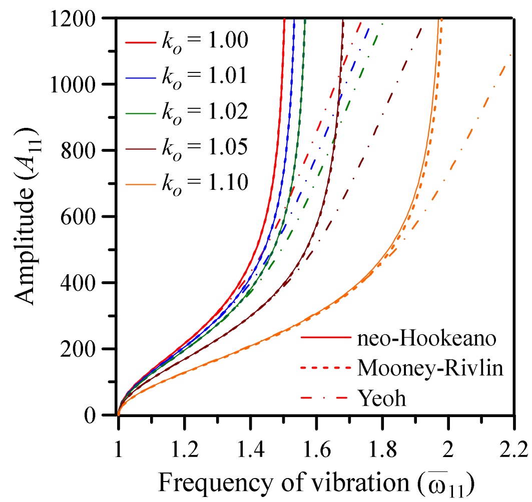


Figure 7. Frequency of vibration (rad/s) versus amplitude for the variable undeformed geometry relation (k_o) with uniform equi-biaxial stretching (k_f = 1) and δ_x = 1.10.

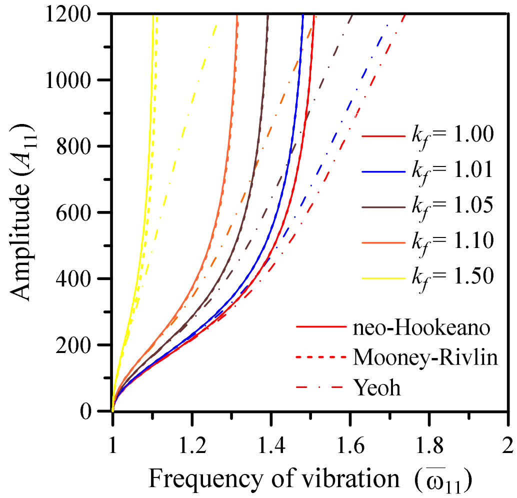
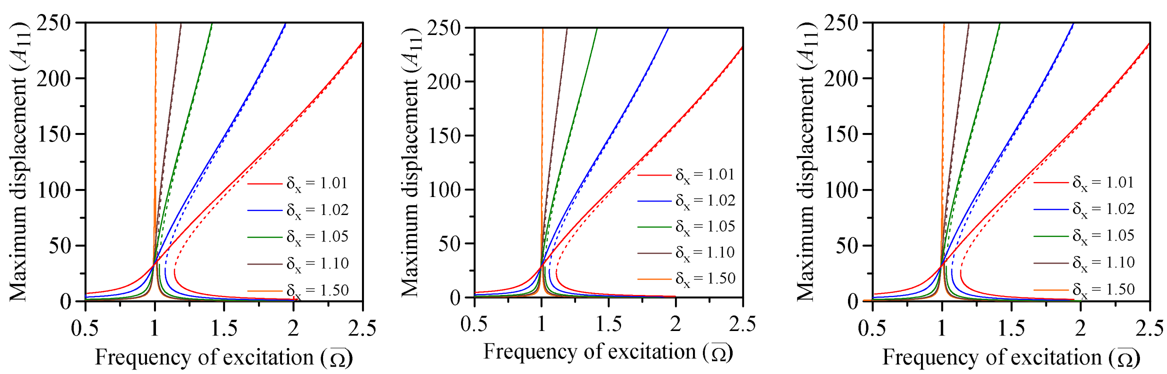


Figure 8. Frequency of vibration (rad/s) versus amplitude for the square membrane (k_o = 1) and the variable deformed geometry relation (k_f) with δ_x = 1.10.

The behavior of the square membrane under external lateral harmonic pressure is now investigated. Figure 9 shows the undamped nonlinear resonance curve for increasing values of δ_x, considering a normalized excitation amplitude $P_0 = 1$ and the three materials. In Fig. 9 and hence forth, dashed lines represent unstable responses, while solid lines represent stable responses. Due to the hardening characteristics of the problem, the curves bend to the right towards higher frequencies and veers upward following the backbone curve. A saddle-node bifurcation at the point of vertical tangency along the non-resonant branch of the resonance curve leads to a broad region where two competing stable solutions occur.



(a) **(b)** **(c)**

Figure 9. Resonance curves of the harmonically forced, undamped membrane. Forcing magnitude Po = 1. (a) neo-Hookeano (b) Mooney-Rivlin (c) Yeoh.

A comparison of the three materials for resonance curve of the square membrane is shown in Figure 10, considering δ_x = 1.05. As can be seen, for the neo-Hookean and Mooney-Rivlin material models, the responses overlap each other and for large vibration amplitudes the results of Yeoh materials are different from the other two materials.

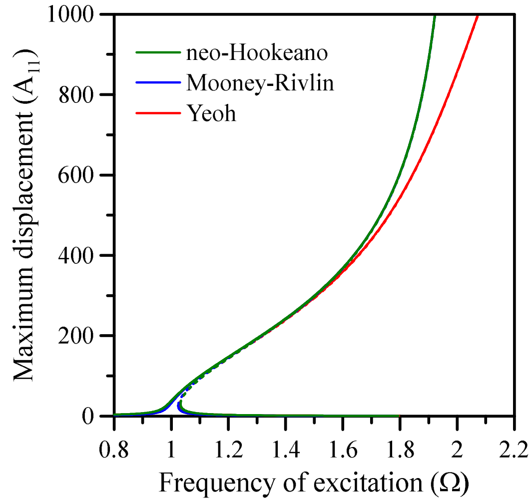


Figure 10. Resonance curves of the harmonically forced, undamped membrane. Forcing magnitude Po = 1 and stretching δ_x = 1.05.

The effect of damping is taken into account in Fig. 11, where the resonance curves for δ_x = 1.05, $k_o = k_f = 1$, and increasing values of the damping parameter ζ are depicted. It is possible to observe that the damping ratio has a strong influence on the resonance curve, progressively reducing the maximum amplitude response and, consequently, the region where two stable equilibrium states (attractors) coexist for a given set of system parameters. This phenomenon, known as multistability, leads to a high degree of complexity in the behavior of the membrane due to the "interaction" among the different attractors.

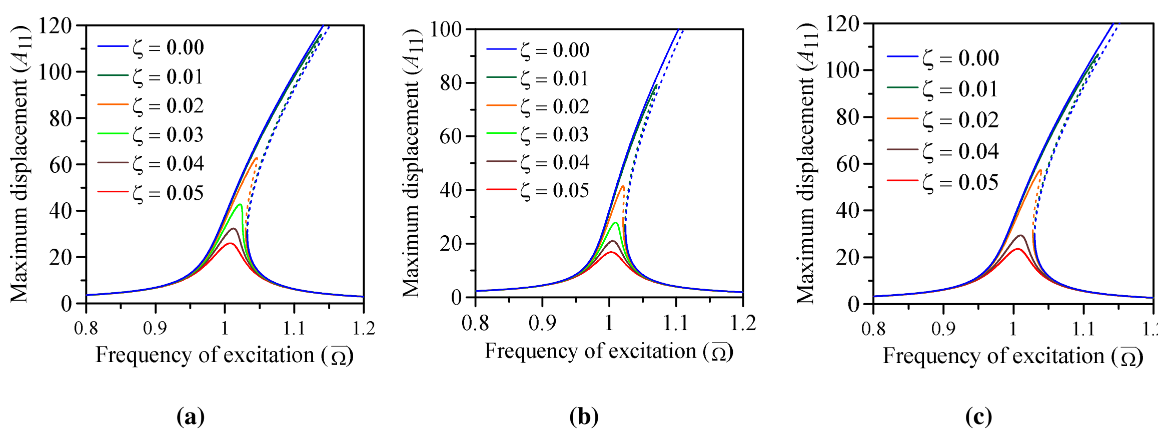


Figure 11. Resonance curves of the harmonically forced, damped membrane. Forcing magnitude Po = 1. (a) neo-Hookeano (b) Mooney-Rivlin (c) Yeoh.

4 CONCLUSION

In this work the mathematical modeling for the nonlinear vibration analysis of a pre-stretched hyperelastic rectangular membrane under finite deformations is presented. The membrane material is described by the Mooney-Rivlin, neo-Hookean and Yeoh constitutive models and the equations of motion of the stretched membrane are then obtained. By analytically solving the linearized equations of motion, the vibration modes and frequencies of the hyperelastic membrane are obtained, and the influence of material parameters and stretching ratios on the natural frequencies is analyzed. For a neo-Hookean material, for each pair of wavenumbers, the frequency increases as the initial stretching ratios increase and converge to a constant upper boundary large stretching ratios. For the Mooney–Rivlin and Yeoh material the natural frequencies increase continuously with the initial pre-stretch.

The results highlight the influence of the stretching ratios on the natural frequencies, the nonlinear frequency-amplitude relation and bifurcation diagrams. The linear normal modes are used, together with the Galerkin method, to obtain reduced order models of the nonlinear dynamic response. It is shown that a lightly stretched membrane displays a highly nonlinear response, the nonlinearity decreases as the stretching ratio increases, and the response becomes practically linear for a highly pre-stretched membrane.

ACKNOWLEDGEMENTS

The authors acknowledge the financial support of the Brazilian research agencies CAPES, CNPq, FAPERJ-CNE and FAPEG.

REFERENCES

Bower, A. F., 2010. *Applied Mechanics of Solids.* CRC Press, Taylor and Francis Group, Boca Raton.

Boyce, M.C., Arruda, E.M, 2000. Constitutive models of rubber elasticity. A review. *Rubber Chemistry and Technology*, vol. 73, pp. 504–523.

Fu, Y.B., Ogden, R.W., 2001. *Nonlinear Elasticity – Theory and Applications*, London Mathematical Society, Lecture Notes Series 283, Cambridge University Press, Cambridge.

Green, A.E., Adkins, J.E., 1970. *Large Elastic Deformation*, Oxford University Press, London.

Libai, A., Simmonds, J. G.,1998. *The Nonlinear Theory of Elastic Shells*, Cambridge University Press, Cambridge.

Ogden, R.W.,1984. *Non-Linear Elastic Deformations*, Ellis Horwood Ltd., Chichester.

Pascalis, R., 2010. *The Semi-Inverse Method in solid mechanics: Theoretical underpinngs and novel applications*. PhD thesis, Université Pierre et Marie Curie – Universitá del Salento.

PRADO, Z. J. G. N., 2001. *Acoplamento e interação modal na instabilidade dinâmica de cascas cilíndricas*. PhD thesis, Pontifícia Universidade Católica do Rio de Janeiro, Rio de Janeiro.

Przybylo, P.A., Arruda, E.M., 1998. Experimental investigations and numerical modeling of incompressible elastomers during nonhomogeneous deformations. *Rubber Chemistry and Technology*, vol. 71, pp 730–740. doi: 10.5254/1.3538501

Saccomandi, G., Ogden, R.W., 2004. *Mechanics and Thermomechanics of Rubberlike Solids.* CISM Courses and Lectures *No. 452. Springer, Wien.*

Sasso, M., Palmieri, G., Chiappini, G., Amodio, D., 2008. Characterization of hyperelastic rubber-like materials by biaxial and uniaxial stretching tests based on optical methods. *Polymer Testing*, vol. 27, pp. 995–1004. doi: 10.1016/j.polymertesting.2008.09.001

Selvadurai, A.P.S., 2006. Deflections of a rubber membrane, *Journal Mech. Phys. of Solids*, vol. 54, n. 6, pp. 1093-1119.

Selvadurai, A.P.S., Shi, M., 2012. Fluid pressure loading of a hyperelastic membrane. *International Journal of Non-Linear Mechanics,* vol. 47, pp 228–239. doi: 10.1016/j.ijnonlinmec.2011.05.011.

Soares, R. M.; Gonçalves, P. B., 2014. Large-amplitude nonlinear vibrations of a Mooney-Rivlin rectangular membrane. *Journal of Sound and Vibration,* vol. 333, pp. 2920-2935.

Soares, R.M., 2009. *Dynamic Analysis of Circular Hyperelastic Membranes.* PhD Thesis, Pontifical Catholic University of Rio de Janeiro, Rio de Janeiro, Brazil.

Soares, R.M.; Gonçalves, P.B., 2012, Nonlinear vibrations and instabilities of a stretched hyperelastic annular membrane, *Int. Journal Solids and Structures*, vol. 49, pp. 514-526.

Soares, R.M.; Gonçalves, P.B., 2014, Large-amplitude nonlinear vibrations of a Mooney-Rivlin rectangular membrane., *Journal of Sound and Vibration*, vol. 333, pp. 2920-2935.

Treloar, L.R.G., 1975. *Rubber Elasticity*, Oxford University Press, London.