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ISOGEOMETRIC ANALYSIS OF FREE VIBRATION OF TRUSSES AND PLANE STRESS

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Abstract. *Actually Isogeometric Analysis (IGA) have proven high accuracy and efficacy in dynamical problems, openning possibilities to improve the traditional FEM models. At the same time of IGA development, some steps forward in GFEM were done concerning also applications for dynamical problems. The aim of this paper is to test the response of IGA for free vibration problems of trusses and plane stress. Based on numerical applications, IGA models have their convergence and accuracy checked and compared with those developed in FEM and GFEM. The results shows high accuracy for IGA models, and reinforce its way as a promising tool.*

Keywords: *Isogeometric Analysis, GFEM, Free Vibration, Trusses, Plane Stress*

1 INTRODUCTION

Ten years after the introduction to Isogeometric Analysis (IGA) by Hughes et al. (2005) a large amount of applications and improvements were done in the context of numerical methods, where researches were motivated by a couple of advantages which the method promised. In dynamical analysis scenario, IGA presented high accuracy over Finite Element Method (FEM) (Cottrell et al., 2006) in the whole free vibration frequency sample.

At the same time in the extended versions of the classical FEM, mostly those based in Partition of Unity Method (PUM) (Melenk and Babuska, 1996), a couple of steps forward were done in dynamical analysis. Concerning this work scope, highlight the works of Arndt et al. (2010) and Torii and Machado (2012) whose applied Generalized Finite Element Method (GFEM) for the free vibration problem of bars and trusses. Naturally other kind of advances in the called "Enriched Methods" based in PUM were performed, as the Stable GFEM (Babuška and Banerjee, 2012) and the recent Orthonormalized GFEM (Sillem et al., 2014).

Due to the relative success of IGA from the results of the frequency error spectra for the free vibration problem of straight bars and beams (Cottrell et al., 2009, Rauen, 2014), this work aims to extend the vibration tests to other kinds of structural elements. The results of IGA, presented in form of free vibration frequencies and error spectra, are compared with those developed by Arndt et al. (2010), Torii and Machado (2012) and Torii (2012) for GFEM and classical FEM.

2 ISOGEOMETRIC ANALYSIS

Isogeometric Analysis is a FEM-like numerical method which reformulated the treatment of object geometry and mesh questions. Aiming to solve FEM mesh bottlenecks, which demands high computational costs, IGA works by means of NURBS (*Non Uniform Rational B-Splines*), which allow to connect CAD environment with FEA, since those functions are the same.

IGA follows the opposite way of Isoparametric Concept. Since FEM turns to find a set of functions to describe the mathematical problem, IGA aims to find a set of NURBS capable to describe object geometry perfectly (Cottrell et al., 2009).

2.1 NURBS Functions

NURBS is a family of B-Splines functions. It follows the recursive scheme of construction of Cox and de-Boor (De-Boor, 1972, Cox, 1972). This formulation constructs a base of n B-Splines with order p, where its behaviour depends on the called knot vector Ξ. The knot vector consists in a set of non decreasing coordinates, called knots. Given a polynomial degree p, a number of n shape functions and a knot vector $\Xi = \{\xi_1, \xi_2, \ldots, \xi_{n+p+1}\}$, B-Splines basis functions are defined by:

$$N_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i \leq \xi < \xi_{i+1}, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

for $p = 0$ and

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi), \tag{2}$$

for $p \geq 1$.

For IGA, a basic set of NURBS shape functions is defined by repeating the edge knots $p+1$ times. Some relevant NURBS properties are described in Hughes et al. (2005) and Piegl and Tiller (1997) which extensively contribute to the performance and optimization of IGA implementations. Figure 1 shows an example of NURBS shape functions with parameters $p = 2$, $n = 6$ and $\xi = \{0, 0, 0, 0.25, 0.5, 0.75, 1, 1, 1\}$.

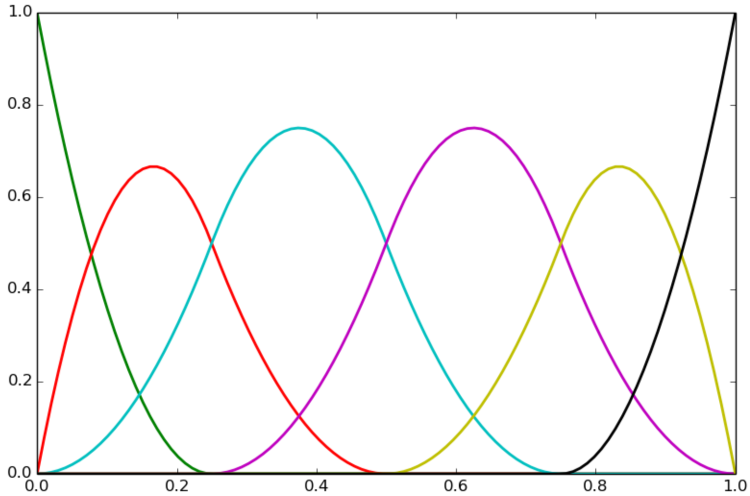

Figure 1: NURBS Shape Functions

2.2 NURBS Surfaces

NURBS surfaces are defined by functions product. Given $N_{i,p}(\xi)$ a set of NURBS with order p, n functions and knot vector Ξ, and $M_{j,q}(\eta)$ a set of NURBS defined by order q, m shape functions and knot vector H, the surface equation is defined by:

$$\tilde{N}_{i,j;p,q}(\xi, \eta) = N_{i,p}(\xi) M_{j,q}(\eta). \tag{3}$$

Figure 2 shows an example of a set of NURBS surfaces with the parameters $p = q = 2$, $n = m = 4$ and $\Xi = H = \{0, 0, 0, 0.5, 1, 1, 1\}$.

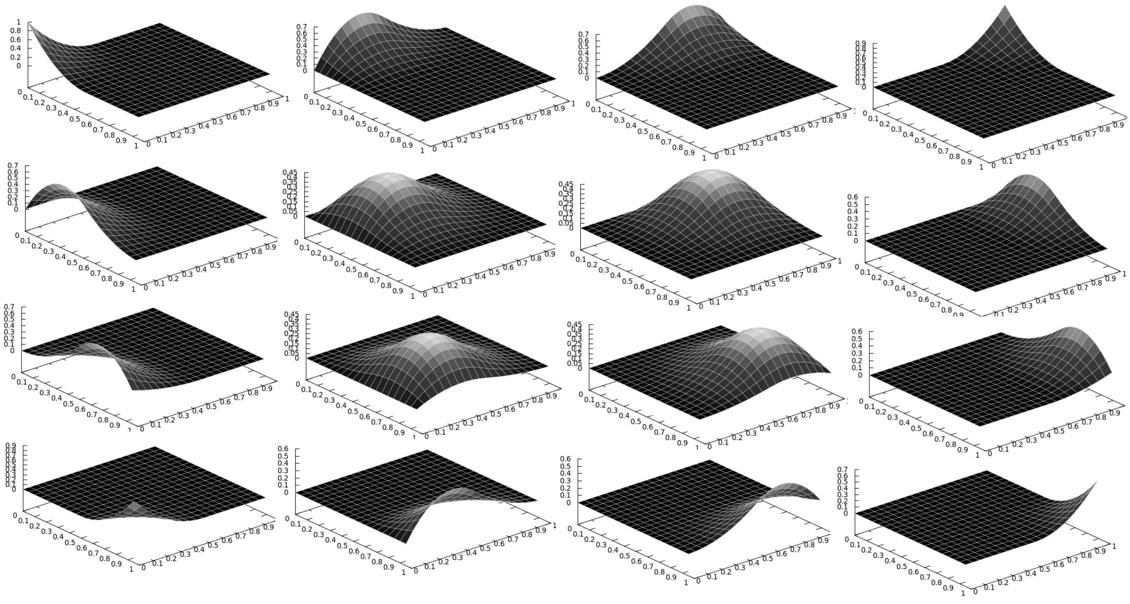

Figure 2: NURBS Surfaces

2.3 IGA Refinements

In the viewpoint of shape functions, IGA refinement could be seen as a set of modification in the functions parameters. Basically the innput parameters Ξ, n and p are modified and a new set of shape functions is created. Different kind of modifications describes the different kinds of refinements.

Isogeometric *h* refinement consists to change n and Ξ parameters only. With the increasing in the number of shape functions n there's a need also to add knot in Ξ. This results in a increasing in the number of shape functions with the same order p. Considering frequency error spectra for the free vibration of rods and beams (Cottrell et al., 2006), is proven that *h* refinements does not change the behaviour of normalized spectrum curves.

Cottrell et al. (2007) define the Isogeometric p refinement as the order increasing with continuity maintained. The number of shape functions n is also increased, but an important fact is related with knot vector: the multiplicity of the whole set of knots is also increased. Details of p refinement implementation are given by Cottrell et al. (2007) and Cottrell et al. (2009). Some comparisons isogeometric p refinement and other refinements developed by Rauen et al. (2013).

NURBS shape functions allows to control their continuities with parameters p and the multiplicity of knots. The concept of the k refinement is to increase polynomial degree without increase interior knots multiplicity. This gives a high continuity in element domain (Cottrell et al., 2007). Convergence rates in k refinement were proven higher than p refinement (Rauen et al., 2013, Rauen, 2014), due to inscrease continuity and shape functions smoothness with a lower number of shape functions.

3 ENRICHED METHODS

Every method which uses additional functions to increase the shape functions space of the classical FEM could be called enriched method (Arndt et al., 2011, Arndt, 2009).

The enriched methods presents the approximate solution given by:

$$u_h^e = u_{FEM}^e + u_{ENRICHED}^e \tag{4}$$

In matrix form, Eq. (4) defined by:

$$u_h^e = N^T q + \phi^T \bar{q} \tag{5}$$

where u_{FEM}^e is the FEM displacement field based on nodal degrees of freedom, $u_{ENRICHED}^e$ is the enriched displacement field based on field degrees of freedom, q is the FEM degrees of freedom vector and $\bar{q}$ is the field degrees of freedom vector. The vectors N and ϕ contain the classical FEM shape functions and the enriched shape functions, respectively. The vectors ϕ and $\bar{q}$ are defined by:

$$\phi^T(\xi) = [\ F_1 \quad F_2 \quad \dots \quad F_r \quad \dots \quad F_n\ ] \tag{6}$$

$$\bar{q} = [\ c_1 \quad c_2 \quad \dots \quad c_r \quad \dots \quad c_n\ ] \tag{7}$$

where F_r are the enrichment functions, c_r are the field degrees of freedom. Each enriched method is defined by a different set of shape functions.

The Composite Element Method (CEM) (Zeng, 1998a) uses the general analytical solution of vibration problems as enrichment functions, trigonometric functions appears to enrich the basis functions space. GFEM uses the PUM-based functions, leading to a set of methods more flexible to describe the basis functions space of a general phenomenon.

4 FREE VIBRATION VARIATIONAL FORMULATIONS

4.1 Truss Element

Classical Rod Formulation

Plane truss formulation is a generalization of bar element. First consider straight uniform bar, with cross sectional area A, Young Modulus E, specific mass ρ, lenght L and axial displacement $\bar{u}$, the differential equation for vibration of a straight bar can be written as:

$$\rho A \frac{\partial^2 \bar{u}}{\partial t^2} - \frac{\partial}{\partial x}\left(EA\frac{\partial \bar{u}}{\partial x}\right) = 0. \tag{8}$$

Its classical variational formulation is given by the orthogonalization between the diferential equation and a weight function w:

$$\rho A \int_0^L \frac{\partial^2 \bar{u}}{\partial t^2} w dx + EA \int_0^L \frac{\partial \bar{u}}{\partial x}\frac{\partial w}{\partial x} dx - EA\left[w\frac{\partial \bar{u}}{\partial x}\right]_0^L = 0. \tag{9}$$

The last term of eq. (9) vanishes by boundary condition applying. The known eigenvalue problem arise when the classical solution of the differential equation, given by

$$\bar{u}(x,t) = e^{i\omega t}u(x) = ((cos(\omega t) + i.sen(\omega t))\, u(x). \tag{10}$$

is applied. The system turns into:

$$B(u,w) = \lambda F(u,w) \tag{11}$$

where $B(u,w)$ and $F(u,w)$ are bilinear forms and λ is the eigenvalue related with the natural vibration frequencies. The matricial form for Eq.(11) is:

$$\mathbf{K}\mathbf{u}^{\mathbf{h}} = \lambda^h \mathbf{M}\mathbf{u}^{\mathbf{h}} \tag{12}$$

where u^h are the approximate eigenvectors, related with the natural vibration modes and λ^h are the approximate eigenvalue related with the natural vibration frequencies. $\mathbf{K}$ and $\mathbf{M}$ are, respectively, the stiffness and mass matrix. The isogeometric numerical expressions for an element of those matrix are written as:

$$K_{ij} = EA \int_0^L \frac{\partial N_{i,p}}{\partial x}\frac{\partial N_{j,p}}{\partial x}\, dx \tag{13}$$

$$M_{ij} = \rho A \int_0^L N_{i,p}\, N_{j,p}\, dx \tag{14}$$

Transformation Matrix

Truss elements are global defined by a linear transformation from the original bar element. The global stiffness and mass matrix are given by:

$$\mathbf{K_G} = \mathbf{T^T K T} \tag{15}$$

$$\mathbf{M_G} = \mathbf{T^T M T} \tag{16}$$

where $\mathbf{K_G}$ and $\mathbf{M_G}$ are the global transformed matrix and $\mathbf{T}$ is the transformation matrix (Bathe, 1996).

In the element domain, NURBS generates field degrees of freedom. Nodal degrees of freedom are separated, aiming to facilitate boundary condition imposing. The transformation matrix can be expressed by:

$$\mathbf{T} = \begin{bmatrix} cos(\gamma) & sen(\gamma) & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & cos(\gamma) & sen(\gamma) & 0 & 0 & \dots & 0 \end{bmatrix} \tag{17}$$

where

$$cos(\gamma) = \frac{u_j - u_i}{Le} \tag{18}$$

$$sen(\gamma) = \frac{v_j - v_i}{Le} \tag{19}$$

$$Le = \sqrt{(u_j - u_i)^2 + (v_j - v_i)^2} \tag{20}$$

and u_i, u_j, v_i and v_j are the displacements in the global domain. Figure 3 shows the transformation scheme.

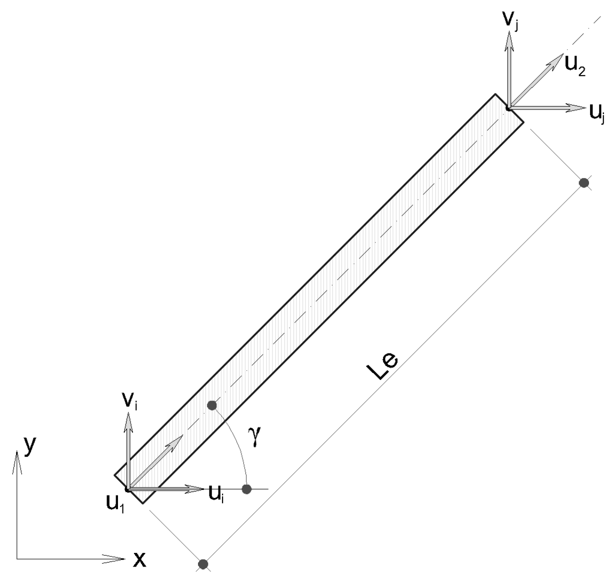

Figure 3: Truss Element Transformation

4.2 Plane Stress

The free vibration of the plane stress phenomena is given by a set of differential equations, described as:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = \rho \frac{\partial^2 \bar{u}}{\partial t^2} \tag{21}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = \rho \frac{\partial^2 \bar{v}}{\partial t^2} \tag{22}$$

where ρ is the specific mass, $\sigma_x, \sigma_y, \tau_{xy}$ are the stresses and $\bar{u}, \bar{v}$ are the horizontal and vertical displacements.

The eigenvalue formulation is obtained by the description of eqs.(22) in terms of strain and then to apply variational techniques. Formulation details of the free vibration of plane stress element are developed by Reddy (1993). Those applications, followed by substitution of the basic solution, leads to the expression:

$$h \iint_\Omega \left[\frac{\partial w_1}{\partial x} \left(c_{11} \frac{\partial u}{\partial x} + c_{12} \frac{\partial v}{\partial y} \right) + \frac{\partial w_1}{\partial y} c_{33} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] dx\, dy - \rho\lambda \iint_\Omega w_1 u \, dx\, dy = 0, \quad (23)$$

$$h \iint_\Omega \left[\frac{\partial w_2}{\partial x} c_{33} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + \frac{\partial w_1}{\partial y} \left(c_{12} \frac{\partial u}{\partial x} + c_{22} \frac{\partial v}{\partial y} \right) \right] dx\, dy - \rho\lambda \iint_\Omega w_2 v \, dx\, dy = 0. \quad (24)$$

where h is the element thickness, ω_1, ω_2 are weighting functions and $c_{11}, c_{12}, c_{22}, c_{33}$ are the components c_{ij} of the constitutive matrix, which is given by:

$$\mathbf{C} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}. \quad (25)$$

The application of Galerkin's Method turns the system an aproximate generalized eigenvalue problem, given by:

$$\mathbf{K}\Delta^h = \lambda^h \mathbf{M}\Delta^h \quad (26)$$

where

$$\Delta^h = \begin{Bmatrix} u^h \\ v^h \end{Bmatrix} \quad (27)$$

and u^h, v^h are the horizontal and vertical plane stress displacements.

The matrixes $\mathbf{K}$ and $\mathbf{M}$ are given by the expressions:

$$\mathbf{K} = h \int_\Omega \mathbf{B^T C B} \, dx\, dy \quad (28)$$

$$\mathbf{M} = \rho h \int_\Omega \mathbf{H^T H} \, dx\, dy \quad (29)$$

and the matrixes $\mathbf{H}$ and $\mathbf{B}$ being defined as:

$$\mathbf{H}^T = \begin{bmatrix} \tilde{N}_{1,1:p} & 0 & \tilde{N}_{1,2:p} & 0 & \dots & \tilde{N}_{N,M:p} & 0 \\ 0 & \tilde{N}_{1,1:p} & 0 & \tilde{N}_{1,2:p} & \dots & 0 & \tilde{N}_{N,M:p} \end{bmatrix}, \quad (30)$$

and

$$\mathbf{B} = \mathbf{DH} \quad (31)$$

where $\mathbf{D}$ is an operator given by:

$$\mathbf{D} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \quad (32)$$

5 EXAMPLES

5.1 Seven Bar Truss

The seven bar truss free vibration problem was originally proposed by Zeng (1998a) (Figure 4). This modelling uses the parameters: cross sectional area $A = 0.001m^2$, specific mass $\rho = 8000\ kg/m^3$ and Young Modulus $E = 210GPa$.

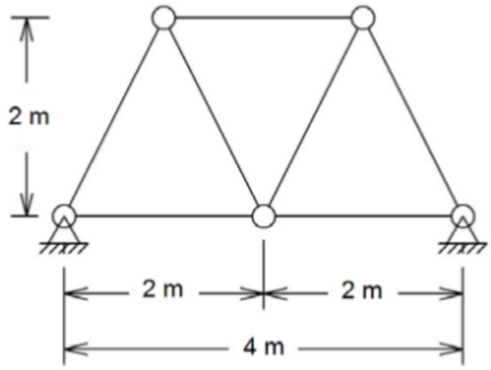

Figure 4: Seven Bar Truss (Zeng, 1998a)

FEM, GFEM and CEM Natural Vibration Frequencies

Table 1 shows the free vibration for the seven bar truss for approach using FEM, GFEM and CEM with 5 enrichment functions (Arndt et al., 2010, Arndt, 2009). The results obtained from Zeng (1998a) for CEM with 1 and 2 enrichment functions are also shown in Table 1.

Table 1: Seven Bar Truss Natural Vibration Frequencies developed in FEM, GFEM and CEM (Arndt et al., 2010)

i	FEM (7e) 6 dgof	CEM (7e 1c) 13 dgof	CEM (7e 2c) 20 dgof	CEM (7e 5c) 41 dgof	GFEM (7e) 34 dgof	Adap. GFEM $1 \times 6dgof + 2 \times 34dgof$
	ω_i (rad/s)	ω_i (rad/s)	ω_i (rad/s)	ω_i (rad/s)	ω_i (rad/s)	ω_i (rad/s)
1	1683.521413	1648.516148	1648.258910	1647.811939	1647.785439	1647.784428
2	1776.278483	1741.661466	1741.319206	1740.868779	1740.840343	1740.839797
3	3341.375203	3119.123132	3113.835167	3111.525066	3111.326191	3111.322715
4	5174.353866	4600.595156	4567.688849	4562.562379	4561.819768	4561.817307
5	5678.184561	4870.575795	4829.702095	4824.125665	4823.253509	4823.248678
6	8315.400602	7380.832845	7379.960217	7379.515018	7379.482416	7379.482322
7		8047.936309	7532.305498	7506.784243	7499.144049	
8		8272.611818	8047.936313	8047.936297	8047.936312	
9		11167.56472	9997.484917	9931.261415	9922.385851	

In vibration problems, which leads to an eigenvalue problem, truncation errors occurs in upper bound (see proofs in Carey and Oden (1984) and Arndt (2009)). By this statement, the results in Table 1 shows that adaptive GFEM presents the best approach for the free vibration truss problem, followed by GFEM and CEM with 5 enrichment functions.

IGA Natural Vibration Frequencies

Table 2 shows the results for IGA with polynomial degrees $p = 2$, 3 and 4. Routines uses non repeated interior knots, generating high continuity in interior, typical procedure from k refinement.

Table 2: Seven Bar Truss Natural Vibration Frequencies developed in IGA

i	IGA $p = 2$ 13 dgof	IGA $p = 3$ 20 dgof	IGA $p = 4$ 34 dgof
	ω_i (rad/s)	ω_i (rad/s)	ω_i (rad/s)
1	1648.06320092	1647.78555214	1647.78444027
2	1741.21025463	1740.84082235	1740.83980945
3	3117.08579405	3111.40375945	3111.32274788
4	4597.86055319	4562.60137052	4561.81765304
5	4869.00145652	4825.08885595	4823.24943613
6	7430.30350284	7429.37528256	7379.54133569
7	8100.92593226	7502.19348783	7499.17033141
8	8273.78472987	8100.92593226	8047.99557886
9	11364.16244158	10086.2045829	9923.15407939

Results show an accurate approximation between IGA $p = 4$ and adaptive GFEM. Even with inicial results with higher accuracy than FEM and CEM with 1 and 2 enrichments, the most refined results can't reach adaptive GFEM accuracy.

5.2 Unitary Square Steel Plate

The example developed above and ilustrated by fig. 5 consists in a stell plate with $L_x = L_y = 1m$ and $h = 0.1m$. Material properties are: $\rho = 8000kg/m^3$, $E = 210GPa$ and $\nu = 0.3$.

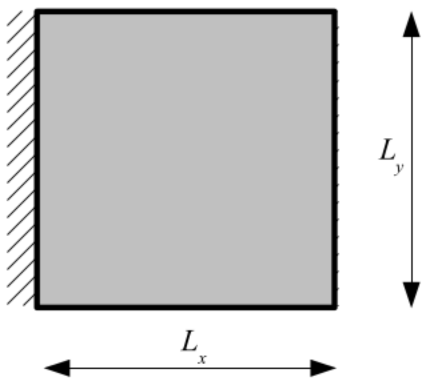

Figure 5: Square Steel Plate (Torii, 2012)

Natural Vibration Frequencies and Errors

Table 3 shows the percentual errors for models developed in Hierarchical FEM (HFEM) and GFEM by Torii (2012) and IGA with polynomial degrees $p = 3$ and $p = 5$. The results used to determine the errors is a refined model developed in HFEM with degree $p = 9$ also, by Torii (2012).

Table 3: Square Steel Plate Vibration Frequencies

i	ω_i (rad/s)	HFEM	GFEM π	IGA $p = 3$	IGA $p = 5$
1	$3372, 13$	$0, 057055$	$0, 063529$	$0, 035073$	$0, 010609$
2	$8092, 72$	$0, 018804$	$0, 020749$	$0, 011647$	$0, 003539$
3	$9079, 09$	$0, 010432$	$0, 013191$	$0, 005292$	$0, 001138$
4	$14427, 23$	$0, 002301$	$0, 002959$	$0, 001192$	$0, 000260$
5	$15558, 23$	$0, 030022$	$0, 034655$	$0, 017811$	$0, 005016$
6	$16511, 96$	$0, 000493$	$0, 000658$	$0, 000291$	$0, 000067$
7	$20812, 77$	$0, 015192$	$0, 017391$	$0, 006618$	$0, 000988$
8	$21911, 61$	$0, 036135$	$0, 042266$	$0, 020378$	$0, 005238$
9	$24194, 74$	$0, 012133$	$0, 013001$	$0, 005003$	$0, 000499$

Frequency Spectrum

Figure 6 shows graphically the same results shown in Table 3.

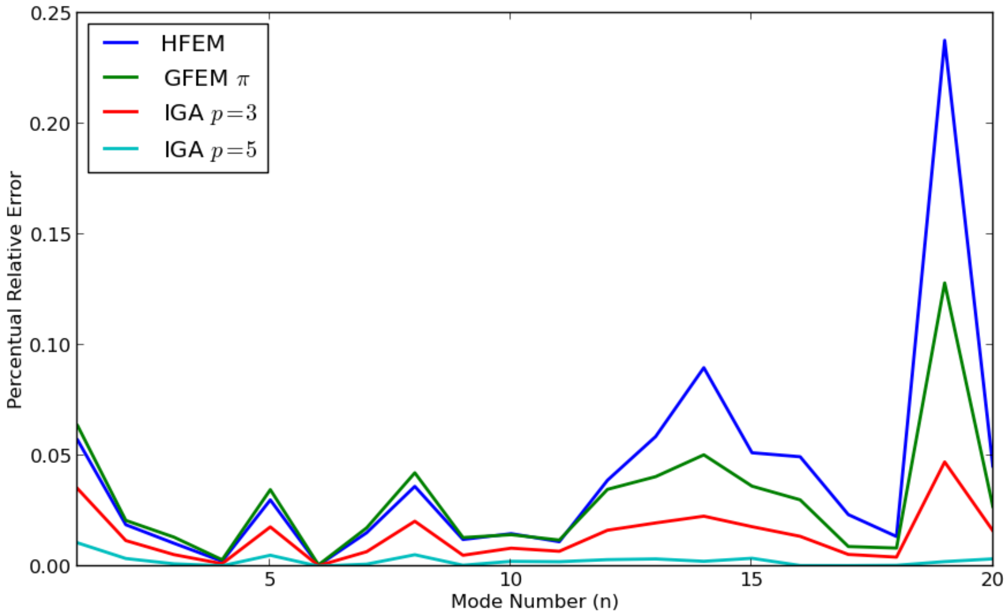

Figure 6: Steel Square Plate Frequency Error Spectra

The plane state frequency spectrum shows a set of more accurate curves for IGA models followed, in accuracy, by GFEM and HFEM.

6 CONCLUSIONS

This work aimed to test the efficiency of Isogeometric Analysis as approach to the free vibration problem of trusses and plane state. The results show accurate behaviour of IGA shape functions if compared with FEM, HFEM and CEM for both problems. Adaptive GFEM results for the truss problem are superior, concerning the upper bound error approximation.

Concerning bidimensional problems like plane state, IGA presented some advantages mostly by the fact of global directly defined NURBS functions in physical space. Results shows highlights in accuracy, giving feasibility in future IGA modelling.

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