# COMPUTATIONAL MODEL FOR DYNAMIC STABILITY OF CIRCULAR THIN AND THICK ANNULAR PLATES SUBJECTED TO NONCONSERVATIVE LOADING

**Joaquín Leonel Sánchez Salas**

**Raul Rosas e Silva**

jlssalas10@hotmail.com

raul@puc-rio.br

Department of Civil Engineering Pontifical Catholic University of Rio de Janeiro

Rua Marquês de São Vicente 225, Edifício Cardeal Leme - sala 301, Gávea - CEP 22451-900

Rio de Janeiro – Brasil

**Glauco Jose de Oliveira Rodrigues**

glauco.rodrigues@oi.com.br

Department of Civil Engineering Rio de Janeiro State University

***Abstract.*** *This paper allows us to study the effect of the level of load conservativeness on the stability of circular plates. A stability analysis of thin and thick circular and annular plates structures subject to a combination of nonconservative tangential follower loads, using the dynamic criterion is performed. The initial tangent stiffness and mass matrices are obtained using the Rayleigh-Ritz method and energy principles. The model allows was implemented in MAPLE18. The results for the nonconservative loading present interesting comparisons illustrating the effect of shear deformation and inertial rotations in thickness and thin circular annular plates.*

***Keywords:*** *Circular annular plate; stability of circular plates; nonconservative tangential follower loads; Rayleigh-Ritz method.*

# 1 INTRODUCTION

Many engineering structural elements that are used in the civil, aeronautic, marine and machining industries are frequently subjected to significant axial loads. In order to assure the safety of such structures it is often required to include in the analysis the phenomenon of buckling. Results for buckling analysis of thin plates are encountered in classical texts (Timoshenko and Gere, 1961). More recently, Wang (1993) studied the axisymmetric buckling of radially loaded static circular Mindlin plates with internal concentric ring supports. In certain situations, unstable vibrations (flutter) may be of concern. Suanno and Silva (1990) showed that the influence of shear deformation is more significant for the flutter load than for the static buckling load, in the case of beams.

This study considers the buckling problem of simplify supported and clamped thin plate theory of Kirchhoff and thick circular plate theory of Reissner-Mindlin. For such special configurations, the Rayleigh-Ritz method with global functions is a feasible technique. The combination of internal functions and end (nodal) functions is an interesting approach because it allows simple consideration of different boundary conditions. A modified version of the energy method is required to include the follower loads which are non-conservative. The flutter loads for thin and thick plates are compared, considering shear deformation and rotatory inertia. The results

# 2 MATHEMATICAL FORMULATION

## 2.1 Description of the geometric parameters

Consider in Fig. 1 a homogeneous isotropic annular plate of constant thickness $h$ with inner and outer radius denoted by $b$ and $a$ respectively. The origin of the coordinate system is taken at center of the plate in the mid-surface. Thus, the plate geometry and dimensions are defined by a cylindrical coordinate system $(r, \theta, z)$.

## 2.2 The governing equation of motion

Based on the assumptions of Mindlin-Reissener plate theory, the displacement field of the circular plate may be described as follows:

$$\begin{aligned} u &= u_0 + z\psi_r \\ v &= v_0 + z\psi_\theta \\ w &= w_0 \end{aligned} \tag{1}$$

In the above equations, the terms $u_0, v_0$ and $w_0$ are the radial, circumferential, and transverse displacement components of the mid-surface of the circular plate, respectively; and $\psi_r$ and $\psi_\theta$ are rotations in the radial and circumferential directions, respectively. In simple bending analysis, $u_0$ and $v_0$ may be set equal to zero.

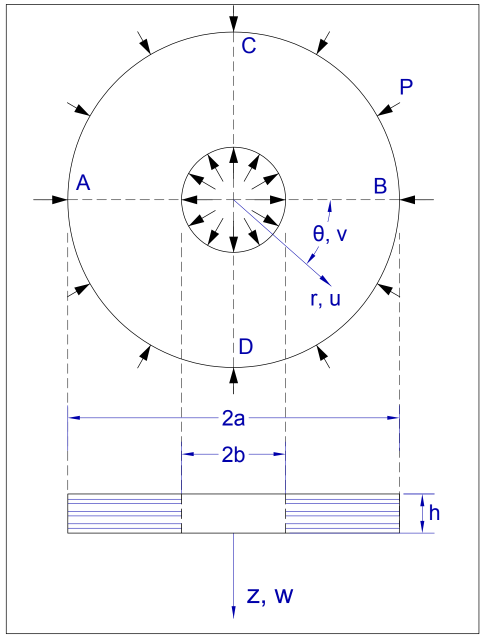


**Figure 1. Geometry and dimensions of an annular plate in cylindrical coordinate system**

## 2.3 Strain displacement relations

For small deflections, the strain-displacement relations may be described as follows:

$$
\begin{aligned}
&\varepsilon_{rr} = \frac{\partial u}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{u}{r} + \frac{\partial v}{r\,\partial\theta}, \quad \varepsilon_{zz} = 0, \\
&\varepsilon_{r\theta} = \frac{\partial v}{\partial r} + \frac{\partial u}{r\,\partial\theta} - \frac{v}{r}, \quad \varepsilon_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}, \quad \varepsilon_{\theta z} = \frac{\partial v}{\partial z} + \frac{\partial w}{r\,\partial\theta}
\end{aligned}
\tag{2}
$$

Where $\varepsilon_{rr}$ and $\varepsilon_{\theta\theta}$ are the normal strains and $\varepsilon_{rz}, \varepsilon_{rz}$ and $\varepsilon_{\theta z}$ are the engineering shear strains.

## 2.4 Hooke's law

The stress-strain relations for the elastic plate can be written as:

$$
\begin{aligned}
&\sigma_{rr} = \frac{E}{(1-\nu^2)}\left(\varepsilon_{rr} + \nu\,\varepsilon_{\theta\theta}\right), \quad \sigma_{\theta\theta} = \frac{E}{(1-\nu^2)}\left(\varepsilon_{\theta\theta} + \nu\,\varepsilon_{rr}\right), \quad \sigma_{r\theta} = \frac{E}{2\,(1+\nu)}\,\varepsilon_{r\theta} \\
&\sigma_{rz} = k^2\,\frac{E}{2\,(1+\nu)}\,\varepsilon_{rz}, \quad \sigma_{\theta z} = k^2\,\frac{E}{2\,(1+\nu)}\,\varepsilon_{\theta z}
\end{aligned}
\tag{3}
$$

Where $E$ and $\nu$ are the Young´s modulus and Poisson`s ratio, respectively; $k^2$ denotes the transverse shear correction factor. In the present analysis, this coefficient is adopted as $k^2 = \pi^2/12$ (Reddy, 2007).

## 2.5 Rayleigh-Ritz method for analysis

The problem is to determine the buckling load of the thick circular plate under uniform in-plane radial compression.

The plate model contains three degrees of freedom for both end nodes. The accuracy of the Rayleigh-Ritz model is controlled by the number of functions (N) used to provide the description of the displacement field, in addition to the basic linear functions (for the thick plate model) or cubic functions (for the thin plate model). There is considerable freedom in the choice of the additional functions, except that they should be zero at the nodes (the thin plate functions must also have zero derivative at the nodes).

$$\begin{aligned} w_n^{(1)} &= \sum_{i=1}^{N} c_i w_i(x) \psi r^{(1)} = \sum_{i=1}^{N} d_i \psi r_i \, \psi\theta^{(1)} = \sum_{i=1}^{N} e_i \psi\theta_i \\ w_n^{(2)} &= \sum_{i=1}^{N} f_i w_i(x) sen\theta \psi r^{(2)} = \sum_{i=1}^{N} j_i \psi r_i \cos\theta \psi\theta^{(2)} = \sum_{i=1}^{N} s_i \psi\theta_i \, sen\theta \end{aligned} \tag{4}$$

Where the coefficients (*c, d, e, f, j, s*) in Eq. (4) are weighting terms associated with their shape functions $w_n, \psi r$ and $\psi\theta$ which correspond to the transverse displacement and the two rotations at each point along the radius in the mid surface. The indexes $^{(1),(2)}$ correspond to the polynomial functions in the radial directions and trigonometric functions in the circumferential directions for displacements.

The boundary conditions are imposed by adding adequate spring stiffness values at the diagonals of the stiffness submatrix associated with the nodal degrees of freedom, zero representing a free displacement and a very large value (penalty approach) corresponding to a restraint. Herein the approximation to be used for the displacement field is polynomial in the radial direction and trigonometric in the circumferential direction. The parameters are displacements and rotations in $r = a$ and $r = b$ ("nodal" displacements), plus the magnitudes of an arbitrary number of polynomial functions of increasing order which are zero at the ends. This approach satisfies exactly the essential boundary conditions (displacements and rotations) at $r = a$ and $r = b$, and is assured to converge for other boundary conditions.

## 2.6 Equation of motion

For free vibration, the kinetic energy $T$ and the strain energy $U$ of an elastic annular plate are expressed as

$$T = \frac{1}{2}\int_b^a \int_0^{2\pi} \int_{-h/2}^{h/2} \rho\left(\dot{u}^2 + \dot{v}^2 + \dot{w}^2\right) r dr d\theta dz \tag{5}$$

And

$$U = \frac{1}{2}\int_b^a \int_0^{2\pi} \int_{-h/2}^{h/2} \left(\sigma_{rr}\varepsilon_{rr} + \sigma_{\theta\theta}\varepsilon_{\theta\theta} + \sigma_{rz}\varepsilon_{rz} + \sigma_{\theta z}\varepsilon_{\theta z} + \sigma_{r\theta}\varepsilon_{r\theta}\right) r dr d\theta dz \tag{6}$$

Where $\rho$ is the specific mass of the plate material and the upper dot represents the differentiation with respect to the time variable t.

The potential energy corresponding to the conservative component $V$ is given by

$$V = \frac{1}{2}\int_b^a \int_0^{2\pi} \int_{-h/2}^{h/2} \left(N_r \phi_r^{\,2} + N_\theta \phi_\theta^{\,2}\right) r dr d\theta dz \tag{7}$$

Where $N_r$ and $N_\theta$ are the stresses depend to the uniform compressive load $P$ is considered along its edge.

## 2.7 Formulation of the eigenvalue problem

Let $\Pi$ be the energy functional given by

$$\Pi = U + V - T \tag{8}$$

For linear small-strain assuming harmonic motion, the global displacement vectors can be written as,

$$x = \bar{x}e^{i\omega t} \tag{9}$$

$$\bar{x} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{10}$$

Here $\bar{x}$ is a nonzero vector of constants to be determined, $\omega$ is the natural circular frequency. Substituting Eq. (5), (6) and (7) into the expressions (8) leads to the governing eigenvalue equation of the form

$$\left(\overline{K} + \lambda_{cr}\left(\overline{G} + \overline{G}_L^{\,s}\right) - \omega^2 \overline{M}\right)\bar{x} = 0 \tag{11}$$

In the above expression, $\overline{K}$ is the stiffness matrix, $\overline{G}$ is the geometric matrix, $\overline{G}_L^{\,s}$ is the symmetric load matrix, $\overline{M}$ is the mass matrix, $\lambda_{cr}$ is the parameter load and $\omega$ is the frequency. Clearly, this matrix cannot be obtained from the energy functional, but results from consideration of equilibrium in the deformed configuration.

# 3 RESULTS AND DISCUSSION

For the next examples for an annular plate with a = 9.0 m, b = 3.0 m, $E = 20GPa, \nu = 0.3$ is considered. The first two vibration mode shapes of the annular plate for clamped inner edge and outer edges free are presented in Fig. 2. Kirchhoff and Reissner–Mindlin theories give mode shapes which differ just slightly. The first two modes are most influential in the analysis of flutter under non-conservative tangential follower loads.

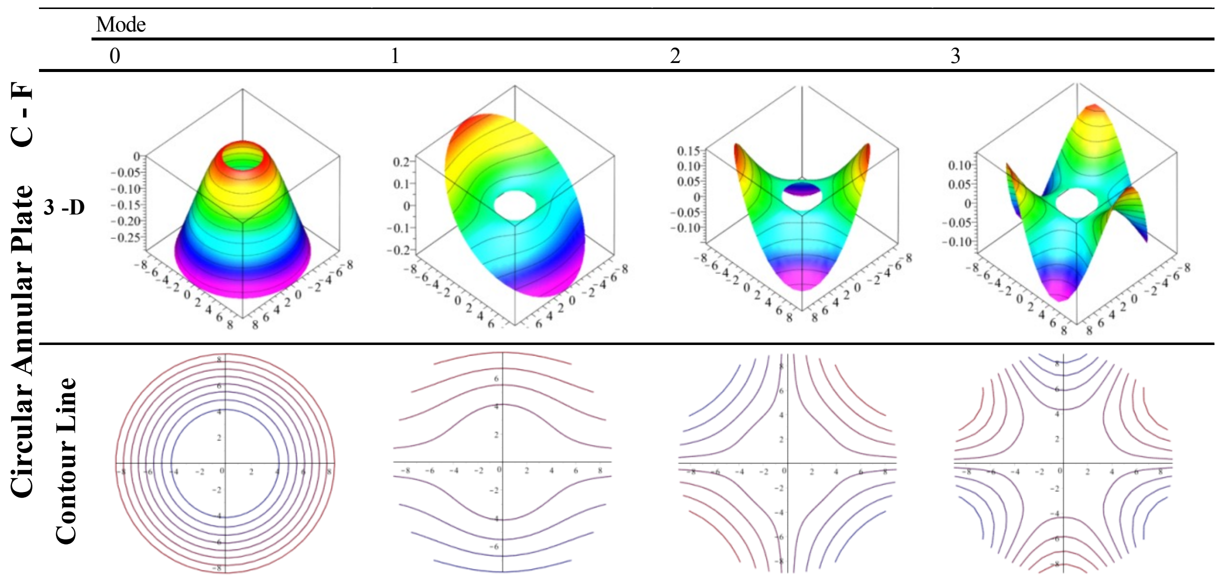


**Figure 2. Deformed mode shape of an annular plate with clamped the inner and outer edge free**

The variation of the first two frequencies of plates subjected to uniform radial load of increasing magnitude is depicted in Figs. 3a and 3b. The load is compressive but remains tangential to the mid-surface during buckling (follower load), hence the problem is non-conservative. Contrary to the static case, where the lowest frequency decreases to zero at buckling load, here the first frequency increases with increasing load. Flutter occurs when the first two frequencies coalesce, just prior to become complex conjugate pairs. The presence of an imaginary positive part implies a vibration of increasing amplitude in time.

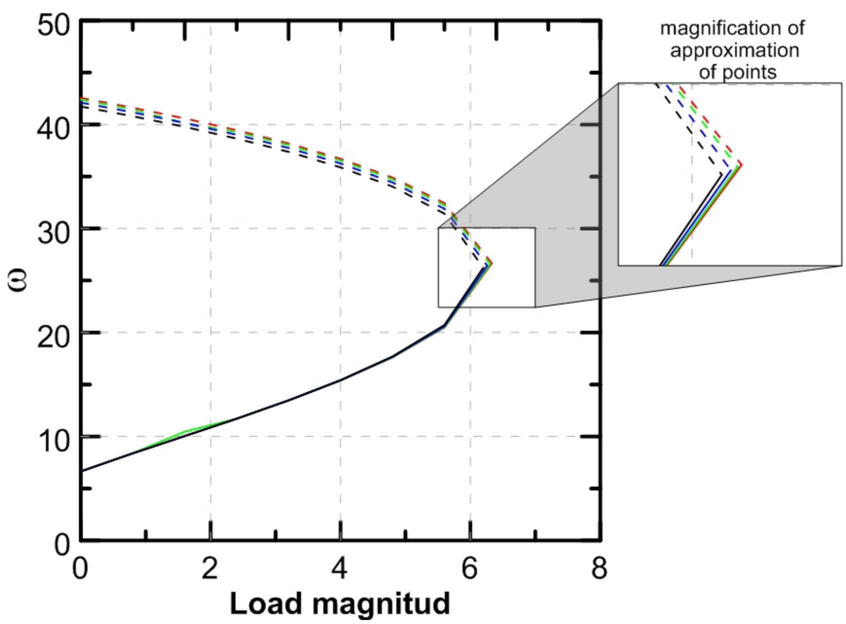


**Figure 3a. Coalescence of the first two frequencies of the thin circular annular plate.**

In Fig. 3a thin plates are considered (no shear deformation). The coalescence of the first two frequencies is reached when the structure is subjected to a load magnitude equal to approximately 6,33 times the magnitude of the static buckling load. Very little change in the load of coalescence (flutter load) relative to the static buckling load (dashed lines of different colors) is observed when the thickness is varied from 0.20 to 0.80 m. It should be remarked that such small changes should disappear when rotatory inertia effects are removed.

In Fig. 3b a similar plot is presented, but the plates are considered according to thick plate theory (shear deformation and rotatory inertia included). The second frequency is particularly affected as the thicknesses are varied from 0,20m to 0,80m because of the higher shear deformation content in the second mode. Hence, the coalescence and consequently the flutter load are considerably changed, with respect to the static buckling load, with this ratio varying from approximately 6.3 to 4.14 .

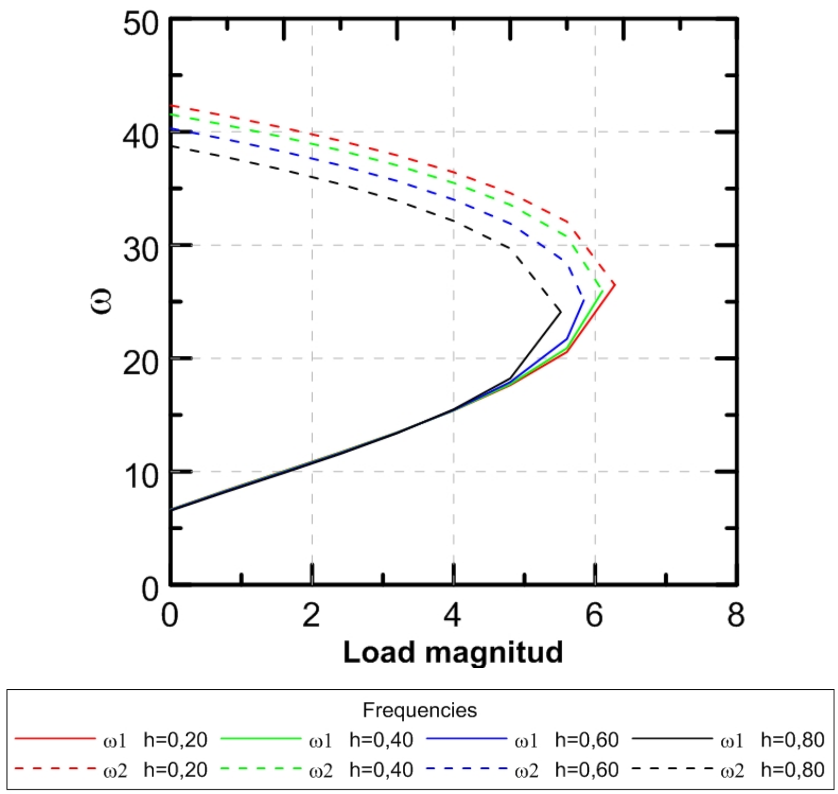


**Figure 3. b) Coalescence of the first two frequencies of the circular thick annular plate.**

Such effects are further illustrated in Fig. 4 The continuous line indicates the results obtained according to thin plate theory, while the dashed line corresponds to the results considering shear deformation and rotational inertia (thick plate theory). The vertical axis correspond to the flutter load divided by the static critical load, in each theory. The drop in the relative load value is more accentuated for the thick plate (approximately 6.32 to 5.51) than for the thin plate (6.33 to 6.20). Again, this is mostly due to the dynamics involved the second vibration mode which is most affected by the shear deformations. In the case of thin plates, the variation is due only to rotatory inertia, usually disregarded.

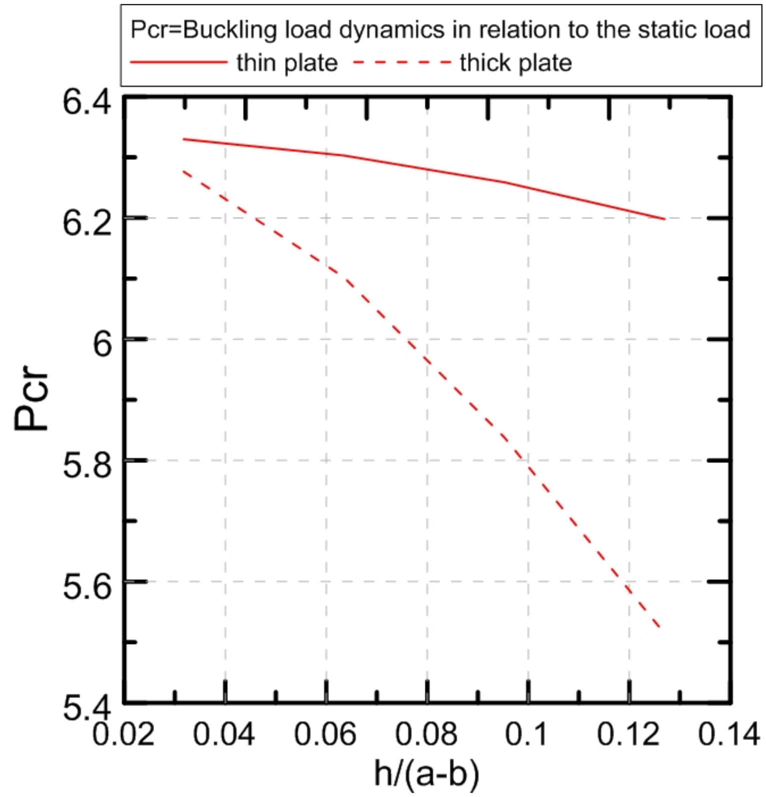


**Figure 4. Flutter load for circular annular plate**

# 4 CONCLUSIONS

The paper presented the flutter problem of annular plates under uniform radial compressive load acting along the outer edge and remaining tangent to the mid-surface when buckling occurs. The results are most interesting when thin and thick models are considered, as illustrated in Fig. 4.

In the case of the thin theory, the relation between the flutter load and the static buckling load depends slightly on the thickness/radius ratio, due to the effect of rotatory inertia. In the case of thick theory, the above mentioned relation varies significantly with the thickness/radius ratio. The phenomenon is explained (Fig. 3) by the effect of shear deformation, which differs for the first and second modes of vibration involved in the flutter load, causing a pronounced change in the point of coalescence of the frequencies.

It is intended to extend the study of annular plates to more general configurations, allowing for thickness variation in a three-dimensional model. In addition to the theoretical study of buckling problems, the discretization Rayleigh-Ritz type approach described here can be applied as a useful tool for practical applications, in preliminary design or verification of final designs, for instance foundation strucutures of wind energy generators.

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