DESIGN OF OUTPUT FEEDBACK CONTROLLERS FOR CONSTRAINED LINEAR SYSTEMS USING DATA CLUSTER ANALYSIS

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Abstract— The design of output feedback controllers for linear systems subject to state and control constraints, additive disturbance and measurement noise, can be realized through the construction of an output-feedback controlled invariant set together with the multiparametric linear programming technique. This technique is commonly used to characterize the optimal control input as a piecewise affine (PWA) function over polyhedral regions. However, the multiparametric programming technique, when applied to multiple output systems, may require online high computational cost and high memory capacity. Within this context, this paper aims to use the K q-flat data cluster analysis algorithm to compute static output feedback control laws over a smaller number of polyhedral regions. An optimization problem is then proposed to compute sub-optimal output feedback control laws to further reduce the number of regions, as is illustrated by a numerical example.

Keywords— Control, constrained linear system, invariant sets, multiparametric programming, data cluster analysis.

1 Introduction

The static output feedback control is considered one of the most important classical control problems (Syrmos et al., 1997). This type of control has attracted the attention of many researchers in recent decades, see Chesi (2013). Among the several existing approaches, we have the concept of invariant positively polyhedral sets for controller design subject to linear constraints. A polyhedral set in state space is positively invariant if any trajectory, originated from this set, does not leave it. Based on this, Blanchini and Miani (2008) noticed that the system state can be restricted to a supremal controlled invariant polyhedral set by using a suitable state feedback control law. Then, such approach was extended to an output feedback structure and conditions were established to evaluate if a given polyhedral set is output-feedback controlled invariant (Dórea, 2009) (Araújo et al., 2013).

By using an output-feedback controlled invariant set, a control input sequence, which enforces the constraints, can be computed only by the knowledge of the system output. This sequence can be computed by using the multiparametric linear programming technique (mp-LP) (Johansen et al., 2006).

The mp-LP technique is commonly used in offline state feedback controller design, since it establishes a piecewise affine (PWA) explicit function over polyhedral regions which can be stored and used in online procedures (Kvasnica et al., 2004). However, the application of this technique can be very complex in design of static output feedback control for multiple-output systems (Dórea, 2009). Moreover, the solution given by the mp-LP technique can have high computational cost due to the number of regions that require great storage capacity, as well as high computational time in online procedures (Zeilinger et al., 2011).

Within this context, we have observed that there are few results available that address the solution of such problems in the design of output feedback controllers for multiple-output systems. Therefore, this paper aims to propose a new approach to output feedback controller design for linear systems subject to constraints. In addition, it establishes an offline solution with simple computations and less computational effort. For this purpose, the K q-flat data cluster analysis algorithm (Vidal, 2011) is used to calculate a PWA static output feedback control law over a smaller number of polyhedral regions. Firstly, the output feedback controlled invariant set is presented. Then, the application of the K q-flat algorithm in output feedback control design is described. Finally, a design of sub-optimal output feedback control is presented to significantly reduce the number of polyhedral regions. This approach is illustrated by a numerical example.

2 Output-Feedback Controlled Invariant Polyhedral Sets

Consider the linear, time-invariant, discrete-time system, defined as follows:

$$x(k+1) = Ax(k) + Bu(k) + Ed(k),$$
$$y(k) = Cx(k) + \eta(k) \quad (1)$$

where $k \in \mathbb{N}$ is the sample, $x \in \mathbb{R}^n$ is the state vector, subject to the constraints defined by, $\Omega = \{x : Gx \leq \mathbf{1}\}$, and $u(k) \in \mathbb{R}^m$ is the control input subject to the constraint defined by $\bar{\mathcal{U}} = \{u : U_c u \leq \mathbf{1}\}$($\mathbf{1}$ is a vector ones of appropriate dimension).

The disturbance d is assumed to be unknown but bounded to a compact set $\mathcal{D} = \{d : Dd \leq \mathbf{1}\}$ and the measurement noise η is assumed to be unknown but bounded to the set $\mathcal{N} = \{\eta : Q\eta \leq \mathbf{1}\}$.

Definition 1: Given $0 < \lambda < 1$ (contraction rate), the set $\Omega \subset \mathbb{R}^n$ is said to be λ-contractive controlled invariant with respect to the system (1) if $\forall x \in \Omega$, $\exists u \in \bar{\mathcal{U}} : Ax + Bu + Ed \in \lambda\Omega$ and $\forall d \in \mathcal{D}$ (Blanchini and Miani, 2008).

Consider now the set of *admissible outputs* associated to Ω as:

$$\mathcal{Y}(\Omega) = \{y : y = Cx + \eta \ for \ x \in \Omega, \eta \in \mathcal{N}\}$$

where $\mathcal{Y}(\Omega) \subset \mathbb{R}^p$ is the set of all values y that can be associated to $x \in \Omega$. This means that if $x(k) \in \Omega$, then $y(k) \in \mathcal{Y}(\Omega)$. Thus, for each $y \in \mathbb{R}^p$ is associated to the following set of states consistent with the measurement:

$$\mathcal{C}(y) = \{x : Cx = y - \eta \ for \ \eta \in \mathcal{N}\}$$

Definition 2: The set $\Omega \subset \mathbb{R}^n$ is said to be output feedback controlled invariant (or o.f.c.i) with respect to the system (1) if $\forall y \in \mathcal{Y}(\Omega)$, $\exists u \in \bar{\mathcal{U}} : Ax + Bu + Ed \in \lambda\Omega$, $\forall d \in \mathcal{D}$, $\forall x \in \Omega$, $\forall \eta \in \mathcal{N} : Cx = y - \eta$.

In Dórea (2009) a necessary and sufficient conditions were proposed to check if a polyhedral set Ω is output feedback controlled invariant (o.f.c.i) with contraction rate λ. This validation can be performed from the solution of several linear programming problems, preferably Ω is defined as supremal controlled invariant set (Blanchini, 1994).

Therefore, if Ω turns out to be o.f.c.i and the state measured $x(k)$ belongs to it, then, from the knowledge of output $y(k)$, it is possible to enforce $x(k+1) \in \lambda\Omega$, through the computation of suitable control input $u(k) \in \bar{\mathcal{U}}$, in spite of the disturbance $d(k) \in \mathcal{D}$ and noise $\eta(k) \in \mathcal{N}$.

3 Design of a Static Output Feedback Controller

Given an o.f.c.i polyhedron $\Omega = \{Gx \leq \mathbf{1}\}$, it is necessary to derive an output feedback control input $u(y(k), k)$ to enforce the state trajectory to belong to the polyhedron. This input can be computed online from the solution of the following linear programming problems (Dórea, 2009):

$$u(y(k), k) = \underset{\varepsilon,u}{argmin} \ \varepsilon$$
$$Subject \ to : \phi(y(k)) + GBu \leq \varepsilon\mathbf{1} - \delta, U_c u \leq \mathbf{1} \quad (2)$$

$$\delta_j = \underset{d}{max} \ \ G_j Ed$$
$$Subject \ to : \ \mathcal{D}d \leq \mathbf{1} \quad (3)$$

$$\xi^j(y) = arg \ \underset{x}{max} \ \ G^j Ax$$
$$Subject \ to : Gx \leq \mathbf{1}, -QCx \leq -Qy + \mathbf{1}. \quad (4)$$

$$\phi^j(y) = G^j A\xi^j(y) \quad (5)$$

where δ_j is the component of the vector δ, that represents the effects larger of disturbance and the superscript j refers to the j-th row of a vector or a matrix.

The problems described in equations (2) and (4) are, also, multiparametric linear programming problems (mp-LP) (Kvasnica et al., 2004) parametrized in y and can be used in the design of offline controllers (Dórea, 2009). This design is necessary to solve several multiparametric linear programming problems, where a PWA static output feedback control law, over optimal partition of $\mathcal{Y}(\Omega)$, is defined similarly to the design of state observers described in Dórea (2006).

The main drawback of the mp-LP technique in multiple-output systems is the considerable memory space required by the offline solution, as well as the high computational time to find the region that the measured output belongs to.

4 K q-flat Cluster Analysis Algorithm

The k q-flat (or affine subspace) algorithm is commonly used for high-dimensional data analysis. This algorithm was developed based on the popular K-means algorithm, where K is the number of affine subspaces (flats) contained in $\mathbb{R}^D$ with dimension $\mathbb{R}^q$ $(D < q)$ that replace centroids in data classification. Such subspaces can be described as:

$$S^i = \left\{e \in \mathbb{R}^D : e = \mu^i + U^i \bar{y}\right\}, i = 1, \cdots, K \quad (6)$$

where $\mu^i \in \mathbb{R}^D$ is the i-th average of the data points ($\mu^i = 0$ for linear subspaces), $U^i \in \mathbb{R}^{d \times q}$ is the i-th basis for S^i subspace and $\bar{y} \in \mathbb{R}^q$ is a low-dimensional representation for the point $e \in \mathbb{R}^D$ in the q dimensional coordinate system (Vidal, 2011).

The iterative K q-flat algorithm has been used by many researchers for data analysis. This iterative K q-flat algorithm aims to partition a given data set $X = (p^1, \cdots, p^N) \subseteq \mathbb{R}^D$ into K clusters $K_1 \cdots, K_i$ (Zhang et al., 2009) by using an objective function that represents the sum of the squared distances from each data point p to its own subspace.

$$J = \sum_{j=1}^{N} \sum_{i=1}^{K} dist(i,j)^2$$

$$dist(i,j) = \left\| p^j - \mu^i - U^i \bar{y}^j \right\| \qquad (7)$$

where, $dist(i,j)$ is defined as the dissimilarity measure between the data p^j and its representation in the affine subspace or q-flat S^i, $e^j = U^i \bar{y}^j + \mu^i$.

The point $\bar{y}^j$ is the projection of the point $p^j \in \mathbb{R}^D$ in a new coordinate system of dimension q. Such projection is defined as follows (Shlens, 2003):

$$\bar{y}^j = {U^i}^T p_m^j \qquad (8)$$

where, p_m is the data subtracted from the average of the used points μ^i.

In the K q-flat algorithm, an initial partition is given to fit the affine subspaces. This initial partition can be obtained randomly or by the K-means algorithm (Vidal, 2011). Then, a PCA model (Principal Component Analysis) is determined to define to K affine subspaces. Finally, the data p are projected in K affine subspaces and the objective function is minimized. By iterating these steps until convergence, a refined estimate of the affine subspaces and partition is obtained (Shlens, 2003).

5 Design of a Static Output Feedback Control Law Using K q-flat Algorithm

The application of the K q-flat algorithm is proposed, in this paper, to determine a PWA static output feedback control law that enforce the state trajectory to the o.f.c.i set. To this end, an algorithm (Algorithm 1) is proposed to partition an optimal data set in clusters. Then, the parameters of the static output feedback control law are defined from the following matrix equation:

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} U_y^i \\ U_u^i \end{bmatrix} \cdot \bar{y} + \begin{bmatrix} \mu_y^i \\ \mu_u^i \end{bmatrix},\ with\ \bar{y} \in \mathbb{R}^n \qquad (9)$$

where y, u, U^i and μ^i are respectively the output vector, the optimal control input associated to y, and the parameters that form the subspace S^i.

The representation of the static output feedback control law is obtained by:

$$\bar{y} = {U_y^i}^{-1}(y - \mu_y^i) \qquad (10)$$

Replacing equation (10) in equation (9), the following expression is obtained:

$$u = U_u^i {U_y^i}^{-1} y + \mu_u^i - U_u^i {U_y^i}^{-1} \mu_y^i \qquad (11)$$

From equation (11) the static output feedback control law $u(y(k)) = F^i y(k) + g^i$ is defined, with $F^i = U_u^i {U_y^i}^{-1}$ and $g^i = \mu_u^i - U_u^i {U_y^i}^{-1} \mu_y^i$.

Algorithm 1:

1: Choose a regular grid of equally spaced points $H_y = \{y^j, j = 1, \cdots, N\}$ which covers the whole desired feasible region $\mathcal{Y}(\Omega)$.
2: For each y^j belonging to the grid, compute the associated optimal control input $u(y^j)$, by solving the optimization problems described in equations (2), (3) and (4).
The cloud of points H in $\mathbb{R}^{n+m}$ is defined as:

$$H = \left\{ p^j = \begin{bmatrix} y^j \\ u(y^j) \end{bmatrix}, j = 1, \cdots, N \right\}$$

3: Establish an initial partition of H and apply the K q-flat algorithm;
4: Store the parameters (U^i and μ^i), that describe the affine subspaces S^i.
5: Use the parameters of each affine K-subspace S^i to compute a control law (Equation (11)).

5.1 *Validation of the Static Output Feedback Control Law*

The validation of the control law shall be performed to verify its ability to ensure the positive invariance of $\{Gx \leq \mathbf{1}\}$, i.e., if it is able to keep the state trajectory completely contained in the constraint set. First, polyhedral regions $\{G_y^i y \leq \rho_y^i\}$ are formed from the clusters obtained by application of Algorithm 1. These regions are convex and can be obtained through the convex hull of the points belonging to each cluster. Furthermore, gaps between such polyhedral regions, can be filled by using the proposed method in Goebel and Allgower (2013). Then, for each set $\{G_y^i y \leq \rho_y^i\}$ a polyhedral region, associated to the states, is defined by:

$$G_{x,\eta}^i \begin{bmatrix} x \\ \eta \end{bmatrix} \leq \rho_{x,\eta}^i$$

where:

$$G_{x,\eta}^i = \begin{bmatrix} G_y^i C & G_y^i \\ G & 0 \\ 0 & Q \end{bmatrix} \rho_{x,\eta}^i = \begin{bmatrix} \rho_y^i \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix}$$

The one-step admissible set to $\{Gx(k) \leq \mathbf{1}\}$ is defined as follows (Blanchini and Miani, 2008):

$$x(k) : Gx(k+1) \leq \lambda \mathbf{1} \qquad (12)$$

Substituting the proposed static output feedback control law and the equation (1) in equation (12), the following expression is obtained:

$$G(A + BF^iC)x(k) + GBF\eta(k) \leq \lambda\mathbf{1} - \delta - GBg^i \tag{13}$$

$$U_cF^iCx(k) + U_cF^i\eta(k) \leq \mathbf{1} - U_cg^i \tag{14}$$

The positive invariance of Ω under the proposed control law can be checked by using extended Farkas lemma described in Hennet and Castelan (1993) or by the following geometrical condition:

$$\left\{ G^i_{x,\eta} \begin{bmatrix} x \\ \eta \end{bmatrix} \leq \rho^i_{x,\eta} \right\} \subseteq \left\{ P^i_{x,\eta} \begin{bmatrix} x \\ \eta \end{bmatrix} \leq T^i_{x,\eta} \right\}$$

where:

$$P^i_{x,\eta} = \begin{bmatrix} G(A + BF^iC) & GBF^i \\ U_cF^iC & U_cF^i \end{bmatrix}$$

$$T^i_{x,\eta} = \begin{bmatrix} \lambda\mathbf{1} - \delta - GBg^i \\ \mathbf{1} - U_cg^i \end{bmatrix}$$

Therefore, we have that $\forall x(k) \in \{Gx \leq \rho\}$, the control law $u(y(k), k) = F^iy(k) + g^i$ is such that $x(k+1) \in \Omega$.

It has been observed that for high order systems, the PWA static output feedback control law obtained by using Algorithm 1, does not always ensure the positive invariance of the output feedback controlled invariant set. Thus, a refinement of the partition resulting from the application of the Algorithm 1 is used to solve this problem: the K q-flat algorithm is applied to the regions $\left\{G^i_y y \leq \rho^i_y\right\}$ for which the previously described test for positive invariance failed. This refinement is repeated until Ω is positively invariant under the computed control laws.

5.2 An Initialization Algorithm for the K q-flat

Static feedback control laws can be computed by using the K q-flat algorithm as described above. However, the positive invariance of the output feedback controlled invariant set is not assured if a random initial partition of the cloud H or the K-means algorithm is used. Therefore, the following initialization algorithm (Algorithm 2) is proposed to establish a suitable initial partition. In this algorithm, the initial number of clusters is defined according to the distribution of the data in H, as well as the choice of the two tolerances tol_1 and tol_2. The tolerance tol_1 is used to regulate the quantity of data that will be used to compute the affine subspace and the tolerance tol_2 is used to evaluate if a given point belongs to the computed affine subspace. If the values of these tolerances are too small, then the number of initial clusters will be large. On the other hand, the final partition will be more likely to result in a PWA law which guarantees positive invariance of the set Ω.

Algorithm 2:

1: Randomly choose a point $\bar{p}$ in the cloud of points H;
2: Assign $i = 1$;
3: Compute the distance between the chosen point $\bar{p}$ and all the other points $p^j \in H$ $(j = 1, \cdots, N)$: $d_1 = \left\|\bar{p} - p^j\right\|$;
4: Use an auxiliary set Q to place all points $p^j \in H$ such that $d_1 < tol_1$;
5: Compute the parameters U and μ of the q-dimensional affine subspace by using the set Q and the PCA;
6: Project the point $p^j \in H$ in a new coordinate system $\bar{y}^j = U^T p^j_m$;
7: Compute the representation of $p^j \in H$ in the affine subspace $e^j = U\bar{y}^j + \mu$;
8: Compute the distance between the point $p^j \in H$ and its representation in the affine subspace: $d_2 = \left\|p^j - b^j\right\|$;
9: Define a set C^i to transfer from H all points p^j such that $d_2 < tol_2$;
10: Increment $i = i + 1$;
11: Repeat the steps 1 to 9 until $H = \emptyset$.

5.3 Sub-optimal Control with Complexity Reduction

In order to reduce the complexity of output feedback controllers for multiple-output systems, we propose a sub-optimal controller over a smaller number of regions. In this approach, we consider that the admissible values of $u(y(k))$ belong to a set defined by the equation (2) and that any control input $u(y(k))$, contained in the constrained set, can be chosen. Sequentially, we observed that by using the problem (2) it is possible to compute a sub-optimal static output feedback control law by using the following linear programming problem:

$$\begin{aligned} &\underset{\tilde{F}^i, \tilde{g}^i, \varepsilon^i}{min} \ \varepsilon^i \\ &Subject\ to: GB(\tilde{F}^iy^j + \tilde{g}^i) - \varepsilon^i\mathbf{1} \leq -\delta - \phi(y^j) \\ &U_c(\tilde{F}^iy^j + \tilde{g}^i) \leq \mathbf{1} \end{aligned} \tag{15}$$

$$\forall y^j \in \left\{G^i_y y \leq \rho^i_y\right\} \cap H_y$$

where, ε^i is the contraction rate, $\tilde{F}^i$ and $\tilde{g}^i$ are the parameters of the sub-optimal control law.

In this design, we applied to the cloud of points H the Algorithm 1 together with the proposed initialization algorithm (Algorithm 2), where the tolerances tol_1 and tol_2 are tuned to establish a given number of clusters. Then, a piecewise affine static output feedback control law is computed through the solution of the linear programming problem (15), which checks the positive invariance of Ω. In other words, the PWA static output feedback control law isn't anymore defined by the equation (11), now, it is defined by the equation (15), where the set of all the $y^j \in H_y$ belonging to region $\{G_y^i y \leq \rho_y^i\}$ are considered.

Therefore, we observed that by using the proposed approach it is possible to reduce the number of polyhedral regions and to compute sub-optimal control laws $u(y(k)) = \tilde{F}^i y(k) + \tilde{g}^i$, such that $Ax(k) + Bu(k) + Ed(k) \in \Omega, \forall x(k) \in \Omega$.

6 Numerical Example

The proposed approach was tested on a linear system with multiple-input and multiple-output (MIMO) represented by:

$$A = \begin{bmatrix} 0.43 & 0.08 & 0.35 & 0.31 \\ 0.54 & 0.28 & 0.66 & 0.10 \\ 0.61 & 0.07 & 0.49 & 0.37 \\ 0.44 & 0.73 & 0.39 & 0.36 \end{bmatrix}, B = \begin{bmatrix} 0.34 & 0.77 \\ 0.17 & 0.32 \\ 0.00 & 0.62 \\ 0.86 & 0.35 \end{bmatrix},$$

$$E = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} C = \begin{bmatrix} 0.51 & 0.46 & 0.63 & 0.54 \\ 0.86 & 0.26 & 0.56 & 0.39 \end{bmatrix}.$$

The initial state $x(0)$ belongs to the polyhedron $\Omega = \{x : |x| \leq \mathbf{1}\}$. The disturbances and measurement noise are limited to the constraints sets: $\mathcal{D} = \{d : |d| \leq 0.1\}$ and $\mathcal{N} = \{\eta : |\eta| \leq 0.5\}$.

A controlled invariant λ-contractive polyhedron, with $\lambda = 0.95$, was obtained using the algorithms proposed in Dórea and Hennet (1999). Then, it was checked to be o.f.c.i by using the conditions proposed in Dórea (2009), with $\lambda = 0.95$. Sequentially, the polyhedron of admissible outputs $\mathcal{Y}(\Omega)$ was obtained together with the offline solution of the linear multiparametric problems, described in Dórea (2009). This offline solution was obtained by the mp-LP function (Multi-Parametric Toolbox (MPT) for Matlab® software) (Kvasnica et al., 2004). The mp-LP function obtained the explicit representation PWA of $u(y(k))$, defined over 581 polyhedral regions as shown in figure 1.

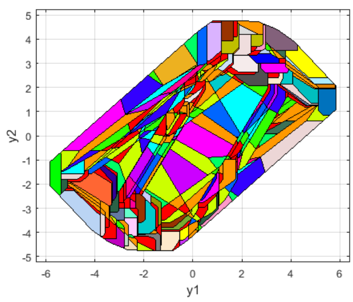

Figure 1: Polyhedral regions given by mp-LP solution

The initialization algorithm applied in H with $tol_1 = 0.6$ and $tol_2 = 0.1$ returned as solution $K = 7$ for $q = 2$. After that, K q-flat algorithm was used. However, for this case, only two calculated laws were admissible with respect to the positive invariance of Ω. Consequently, the constraints were not satisfied. Based on this, a refinement from the application of the Algorithm 1 was performed and 203 clusters together with new static output feedback control laws were obtained, as shown in figure 2(a).

In order to reduce the complexity of the offline solution for multiple-output systems and the number of regions, a suboptimal controller was designed. In this design the Algorithms 1 and 2, with $K = 7$ and $q = 2$, were used as shown in figure 2(b). Furthermore, for each polyhedral region, a suboptimal control law (formed by $\tilde{F}^i$ and $\tilde{g}^i$) was computed by the optimization problem defined in equation (15).

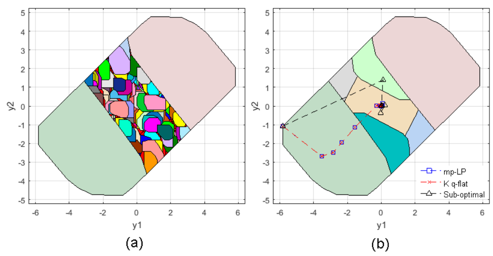

Figure 2: Polyhedral regions obtained by:(a) The refined algorithm K q-flat.(b) The design of suboptimal controllers and the output trajectories.

In figure 3 output trajectories are shown, starting from the output $y = [-5.83, \ -1.07]^T$. Such trajectories were obtained by using the mp-LP solution and the proposed approaches, without considering the disturbance and measurement noise.

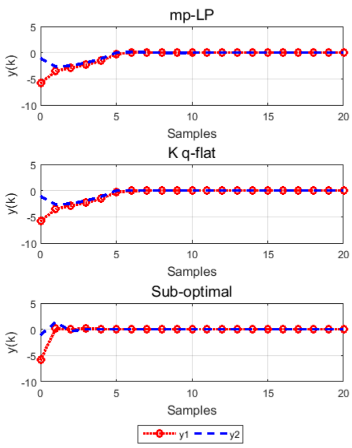

Figure 3: Output Trajectories.

The time required, by iteration, to establish the control action $u(y(k), k)$ in online algorithm (Tondel et al., 2003) is shown in table 1. It is apparent that the time obtained, with application of the proposed algorithms was smaller, compared to the results obtained by using a mp-LP solution. Therefore, the polyhedral region, which contains the current output $y(k)$, can be rapidly identified to compute the control action $u(y(k), k)$.

Table 1: Simulation time by iteration.

Method	$\#R_i$	$t_{seconds}$
mp-LP	581	1.7×10^{-1}
K q-flat	203	7.4×10^{-3}
Sub-optimal	7	7.8×10^{-4}

7 Conclusion

In this paper we have proposed the application of data cluster analysis in design of output feedback control for linear systems. This application was performed using K q-flat algorithm, where static output feedback control laws over polyhedral regions were computed. Such laws were able to respect the constraints in the state and control variables. In addition, we proposed a design of suboptimal control to reduce complexity and computational cost given by mp-LP technique.

The main advantage of the application of the K q-flat algorithm for offline control design is the capacity of replacing the offline solution defined by the mp-LP technique in an offline solution with a smaller number of polyhedral regions and low computational cost in online procedures, since a high storage capacity is not required.

The main limitation of this approach is the complexity of the data distribution in high dimensional space. Future research will address the improvement of this algorithms in high dimensional data cluster analysis and the extension of the K q-flat algorithm to the design of dynamic output feedback controllers.

Acknowledgment

This work was supported in part by CNPq and the LAUT-UFRN.

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