DELAY-DEPENDENT $\mathcal{D}$-STABILIZATION CONDITIONS FOR LINEAR UNCERTAIN DISCRETE-TIME SYSTEMS WITH TIME-DELAY IN THE STATES

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Abstract— We propose conditions of $\mathcal{D}$-stability analysis and $\mathcal{D}$-stabilization of linear discrete-time systems with time-delay in the states and polytopic uncertainties. The $\mathcal{D}$-stabilization problem consists to guarantee the regional allocation of the eigenvalues of the closed-loop systems in a $\mathcal{D}(\alpha, r)$ region, a circle with α being the center and r its the radius. The conditions are based on a Lyapunov-Krasovskii candidate function and they are formulated in terms of LMIs. To achieve our contribution, we obtain an equivalent system where the eigenvalues of the original systems located in $\mathcal{D}(\alpha, r)$ region are mapped in the unitary circle. Therefore, if the equivalent system is Schur stable, then the original system will be $\mathcal{D}(\alpha, r)$-stability. We show a numerical example to illustrate the efficiency of the proposed conditions and to compare them with similar results found in the literature.

Keywords: Linear discrete-time systems with time-delay in the states and polytopic uncertainties, $\mathcal{D}$-stability, Lyapunov-Krasovskii candidate function, and LMIs.

1 Introduction

Dynamic systems with time-delay are often found in industrial processes especially when there is transfer of mass, energy, and/or information. The time-delay usually causes performance deterioration and even loss of stability (Miranda and Leite, 2011), (Niculescu, 2001). Considering the control problem of linear discrete-time systems with time-delay in the state, we can find a lots of works that propose conditions for stability analysis and synthesis of controller for stabilization as we can see in (de Souza and Coutinho, 2014), (Xu et al., 2012), (Gao et al., 2013), (Liu et al., 2014), (Tascikaraoglu et al., 2014), and (Zhang et al., 2014).

Additionally to the stability and stabilization issues, the performance of the closed-loop system is a practical requirement that must be fulfilled by the controller design. Among the closed-loop performance criteria, like minimization of $\mathcal{H}_\infty$ guaranteed cost (Caldeira et al., 2011), exponential stability (da Silva Jr. et al., 2011), the allocation of auto structure is a quite attractive criteria because of engineering insights got with classical control theory can be easily inherited. Concerning such a criteria, it is not obvious how to handle the delay in the states since it leads to an augmentation of the number of eigenvalues. Most of the techniques found in the literature deals with the so called $\mathcal{D}$-stability and usually are restricted to the circular region on the complex plane with center at $(\alpha, 0)$ and radius r as we can see in the Figure 1, where the $\mathcal{D}$ region is inside the unitary circle for discrete-time systems.

In the $\mathcal{D}$-stabilization problem of systems with time-delay in the states, the interesting is to design a controller that assures that all closed-loop eigenvalues of the system are inside a $\mathcal{D}$-region of the complex plane. In this context, the work (Mao and Chu, 2009) proposes conditions for $\mathcal{D}$-stability analysis and synthesis of controllers for $\mathcal{D}$-stabilization of linear discrete-time systems

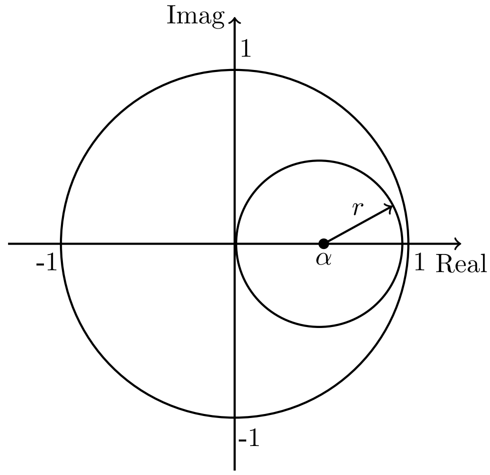

Figure 1: $\mathcal{D}$-stability region.

with polytopic uncertainties and multiple time-delay in the states. We can find also works to treat with the $\mathcal{D}$-stabilization problem for descriptor discrete-time linear systems with polytopic uncertainties and time-delay in the states, as in (Cong et al., 2013) and (Ruizhi et al., 2010). In the (Minqing, 2013), a robust $\mathcal{D}$-stabilization problem is studied for a class of linear discrete-time systems with time-delay in the state subject to actuator failure, where the uncertainties are considered norm bounded. In the work (Silva et al., 2010), the presented conditions are an extension of the conditions shown in (Mao and Chu, 2009), where the robust guaranteed $\mathcal{D}$-stabilization problem is treated. In the same context, we can also cite the works (Teixeira et al., 2013), (Xiao and Su, 2012), and (Leite et al., 2011), that treat the robust $\mathcal{H}_\infty$ guaranteed cost problem and $\mathcal{D}$-stabilization problem and the last one treats also non-fragile control by static state feedback gains.

In this paper, we propose conditions for $\mathcal{D}$-stability analysis and for synthesis of $\mathcal{D}$-stabilization controllers of linear discrete-time systems with polytopic uncertainties and time-delay in the states. The main contributions of the manuscript are: *i)* a new form to represent a discrete-time systems with time-delay in the states from the change of the operator advance; and *ii)* new conditions to analysis the $\mathcal{D}$-stability and to synthesis controllers to $\mathcal{D}$-stabilize systems with polytopic uncertainties and time-delay in the states. The conditions are based on a Lyapunov-Krasovskii (L-K) candidate function and formulated in terms of LMIs. For we obtain the conditions, in the first moment it is necessary to obtain a equivalent system from of the variable v that maps the eigenvalues belong to the $\mathcal{D}(\alpha, r)$ region in the unitary circle. Finally, we present a numerical example to show the efficiency of the propose conditions. Furthermore, we compare our conditions with others found in the literature.

Notations: The symbol $\star$ represents the symmetric blocks in relation the diagonal. The scalar $c^* \in \mathbb{C}$ represents the complex conjugate of the complex number $c \in \mathbb{C}$. The matrices $\mathbf{I}$ and $\mathbf{0}$ denote, respectively, identity and null matrices of appropriate dimensions. For $d \in \mathbb{N}^*$ and $k \in \mathbb{N}$, $\varphi_{d,k} = \{x_{k-d}, x_{k-(d-1)}, \dots, x_k\}$ denotes a sequence of d vectors $x_j \in \mathbb{R}^n$, $j \in \mathcal{I}[-d, -1]$, where $\mathcal{I}[a, b]$ is the interval of the integer numbers starting with "a" and ending at "b". The symbol $\mathcal{D}(\alpha, r)$ represents a circle in the complex plane, with center α and radius r.

2 Problem statement

Consider the linear uncertain discrete system with time-delay in the state described by

$$x_{k+1} = A(\beta)x_k + A_d(\beta)x_{k-d} + B(\beta)u_k \qquad (1)$$

where $x_k \in \mathbb{R}^n$ is the state vector and $u_k \in \mathbb{R}^q$ is the control input. The delay is denoted by $d \in \mathbb{N}^*$, and it is continuous, unknown, and bounded by $\bar{d}$, that is, $1 \leq d \leq \bar{d}$ and $\varphi_{\bar{d},0}$ represents the sequence of the initial conditions of (1) assuring uniqueness to the solution. The matrices $A(\beta)$, $A_d(\beta)$, and $B(\beta)$ are uncertain and satisfy:

$$\begin{aligned}\begin{bmatrix} A(\beta) & A_d(\beta) & B(\beta) \end{bmatrix} = \sum_{j=1}^{N} \beta_j \\ \times \begin{bmatrix} A_j & A_{dj} & B_j \end{bmatrix}, \end{aligned} \qquad (2)$$

where N represents the number of vertices of the polytope. The uncertain vector $\beta \in \mathbb{R}^N$ belongs to Ω defined as

$$\Omega \equiv \left\{ \beta \in \mathbb{R}^N : \beta_j \geq 0, \sum_{j=1}^{N} \beta_j = 1 \right\}. \qquad (3)$$

The following control law is assumed:

$$u_k = Kx_k, \qquad (4)$$

where $K \in \mathbb{R}^{q \times n}$. By applying control law (4) into (1) yields the uncertain closed-loop system

$$x_{k+1} = \hat{A}(\beta)x_k + A_d(\beta)x_{k-d}, \qquad (5)$$

where $\hat{A}(\beta)$ corresponds to $A(\beta) + B(\beta)K$ and

$$\hat{A}(\beta) = \sum_{j=1}^{N} \beta_j \hat{A}_j, \qquad (6)$$

with $\beta \in \Omega$ defined by (3).

Definition 1 *A linear uncertain discrete-time system with time-delay in the states (1)–(3) is said $\mathcal{D}(\alpha, r)$-stable if and only if all of its eigenvalues, $\forall \beta \in \Omega$, are inside the circle with center at $(\alpha, 0)$ and radius r of the complex plane.*

Following, we show the two main problems treat in this work.

Problem 1 *Determine if the closed-loop uncertain discrete-time system with time-delay in the state given by (5)–(6) is $\mathcal{D}(\alpha, r)$-stable.*

Problem 2 *Determine the gain K such that the uncertain discrete-time system with time-delay in the state (1) in closed-loop by the control law (4) is $\mathcal{D}(\alpha, r)$-stable.*

3 Preliminaries

The characteristic polynomial to the system (1) without control input is

$$F(z) = \det\left(z\mathbf{I} - A(\beta) - A_d(\beta)z^{-d}\right) \qquad (7)$$

where z represent the forward operator. We can replace z by $rv + \alpha$ as is done in (Mao and Chu, 2009) and (Xiao and Su, 2012), with r and $\alpha \in \mathbb{R}$ and $v \in \mathbb{C}$, and we get

$$\begin{aligned} F(z) = \det\left((rv + \alpha)\mathbf{I} - A(\beta)\right. \\ \left. - A_d(\beta)(rv + \alpha)^{-d}\right). \end{aligned} \qquad (8)$$

From (8), we can obtain

$$\begin{aligned} F(v) = \det\left(v\mathbf{I} - \left(\frac{A(\beta) - \alpha\mathbf{I}}{r}\right)\right. \\ \left. - \frac{A_d(\beta)}{r}\left(\frac{rv + \alpha}{v}\right)^{-d} v^{-d}\right). \end{aligned} \qquad (9)$$

Thus, the variable v is the new forward operator. Note that v maps the eigenvalues belong to the $\mathcal{D}(\alpha, r)$ region in the unitary circle. Furthermore, if we consider $\alpha = 0$ and $r = 1$, we have $v = z$ and $F(v) = F(z)$. Therefore, we can get an equivalent linear uncertain discrete-time system with time-delay in the state as

$$x_{k+1} = \frac{A(\beta) - \alpha\mathbf{I}}{r} x_k + \frac{A_d(\beta)}{r} \times \left(\frac{rv+\alpha}{v}\right)^{-d} x_{k-d} + \frac{B(\beta)}{r} u_k. \quad (10)$$

By using the control law (4) in (10), we obtain the following closed-loop linear uncertain discrete-time system with time-delay in the state as

$$x_{k+1} = \frac{\hat{A}(\beta) - \alpha\mathbf{I}}{r} x_k + \frac{A_d(\beta)}{r} \times \left(\frac{rv+\alpha}{v}\right)^{-d} x_{k-d}. \quad (11)$$

It is interested to say that the procedure used to obtain the equivalent systems (10) and (11) and these equivalent systems are contributions of this paper.

Note that if the closed-loop system (11) is Schur stable through the control law (4), then the closed-loop system (5) is $\mathcal{D}(\alpha, r)$-stable using the same control law. Therefore, to guarantee the $\mathcal{D}(\alpha, r)$-stability of the (5), we just need to guarantee the stability of the (11). In the sequel, we show a lemma adapted from (Mao and Chu, 2009), that is used to obtain the main results of this paper.

Lemma 1 *Consider X and Y real matrices and f a complex scalar with $f^*f \leq 1$. Then, for any symmetric positive definite matrix Λ,*

$$XfY - Y^Tf^*X^T \leq X\Lambda X^T + Y^T\Lambda f^*fY \quad (12)$$

Proof: Consider the following inequality

$$\left(X\Lambda^{1/2} - Y^Tf^*\Lambda^{-1/2}\right) \times \left(\Lambda^{1/2}X^T - \Lambda^{-1/2}fY\right) \geq \mathbf{0}, \quad (13)$$

where X and Y are real matrices, f is a complex scalar, with $f^*f \leq 1$, and Λ is any real matrix and symmetric positive definite. Thus,

$$X\Lambda X^T - XfY - Yf^*X^T + Y^T\Lambda^{-1}f^*fY \geq \mathbf{0}, \quad (14)$$

that can be rearranged to obtain the inequality (12). □

4 Main results

Consider the following Lyapunov-Krasovskii (L-K) candidate function, $V(x_k, \beta) : \mathbb{R}^n \times \Omega \to \mathbb{R}^+$:

$$V(\varphi_{\bar{d},k}, \beta) = x_k^T P(\beta) x_k + \sum_{i=k-d}^{k-1} x_i^T S(\beta) x_i + \sum_{i=k-\bar{d}}^{k-1} x_i^T R(\beta) x_i, \quad (15)$$

where $P(\beta) = \sum_{j=1}^N \beta_j P_j$, $S(\beta) = \sum_{j=1}^N \beta_j S_j$, and $R(\beta) = \sum_{j=1}^N \beta_j R_j$ are required to be positive definite.

Following (Stojanović et al., 2007), the $V(\varphi_{\bar{d},k}, \beta) > 0$ is a L-K function if

$$\Delta V(\varphi_{\bar{d},k}, \beta) = V(\varphi_{\bar{d},k+1}, \beta) - V(\varphi_{\bar{d},k}, \beta) < 0. \quad (16)$$

Then, the asymptotically stability of the (11) for given K and $\beta \in \Omega$ is sufficient when the inequality (16) is verified.

Based on the L-K candidate function (15) and the results showed in the previous section, we present in the following two theorem. In the first one, we show an analysis conditions that tests the $\mathcal{D}(\alpha, r)$-stability of the linear uncertain discrete-time system with time-delay in the state (5)–(6) for a given controller gain K. In the second theorem, we develop a synthesis conditions that allows to compute the controller gain K to guarantee the $\mathcal{D}(\alpha, r)$-stability of the system (1)–(2) through the control law (4).

4.1 Robust analysis

Theorem 1 *Suppose given gain matrix K and that exist symmetric positive definite matrices $P_j \in \mathbb{R}^{n\times n}$, $S_j \in \mathbb{R}^{n\times n}$, $R_j \in \mathbb{R}^{n\times n}$, Λ_j, for $j \in \mathcal{I}[1, N]$, and matrices $F_i \in \mathbb{R}^{n\times n}$, for $i \in \mathcal{I}[1, 5]$, verifying the LMIs (17). Then, the closed-loop linear uncertain discrete-time system with time-delay in the states (5)–(6) is $\mathcal{D}(\alpha, r)$-stable.*

Proof: We can rewrite (17) as Π_j, where Π_j is shown in (18). By considering that $(r - |\alpha|)^{-2\bar{d}} \geq \left|\frac{rv+\alpha}{v}\right|^{-2d}$, for $|v| \geq 1$ (see (Mao and Chu, 2009) for details), and the Lemma 1, we obtain $\Phi_j \leq \Pi_j$, where Φ_j is shown in (19). Note that $\Phi(\beta) = \sum_{j=1}^N \beta_j \Phi_j$ and pre and post-multiplying $\Phi(\beta)$ by $\zeta_k^T = \begin{bmatrix} x_{k+1}^T & x_k^T & x_{k-d}^T & x_{k-\bar{d}}^T & \omega_k \end{bmatrix}$ and its transpose, respectively, we get

$$\Xi_k = x_{k+1}^T P(\beta) x_{k+1} + x_k^T \left[S(\beta) - P(\beta) + R(\beta)\right] x_k + x_{k-d}^T \left[-S(\beta)\right] x_{k-d} + x_{k-\bar{d}}^T \left[-R(\beta)\right] x_{k-\bar{d}} < 0. \quad (20)$$

Therefore, we have $\Delta V(x_k, \beta) = \Xi_k$ and we can conclude that the feasibility of (17) guarantees the negativity of $\Delta V(x_k, \beta)$, the Lyapunov-Krasovskii's theorem is verified, and it is guaranteed the $\mathcal{D}(\alpha, r)$-stability of the close-loop linear uncertain discrete-time system with time-delay in the state (5)–(6) for given K. □

4.2 Robust $\mathcal{D}(\alpha, r)$-stabilization

Theorem 2 *Suppose that exist symmetric positive definite matrices $\tilde{P}_j \in \mathbb{R}^{n\times n}$, $\tilde{S}_j \in \mathbb{R}^{n\times n}$, $\tilde{R}_j \in \mathbb{R}^{n\times n}$, and $\tilde{\Lambda}_j$, for $j \in \mathcal{I}[1, N]$, matrices $U \in \mathbb{R}^{n\times n}$ and $Y \in \mathbb{R}^{q\times n}$ verifying the LMIs (21). Then, the robust state feedback gain is given by*

$$K = YU^{-1} \quad (22)$$

$$\begin{bmatrix} P_j - F_1 - F_1^T & (1/r)F_1\left(\hat{A}_j - \alpha\mathbf{I}\right) - F_2^T & -F_3^T & -F_4^T & (1/r)F_1A_{dj} \\ \star & \begin{pmatrix} S_j - P_j + R_j \\ +(1/r)F_2\left(\hat{A}_j - \alpha\mathbf{I}\right) + (1/r)\left(\hat{A}_j - \alpha\mathbf{I}\right)^T F_2^T \end{pmatrix} & (1/r)\left(\hat{A}_j - \alpha\mathbf{I}\right)^T F_3^T & (1/r)\left(\hat{A}_j - \alpha\mathbf{I}\right)^T F_4^T & (1/r)F_2A_{dj} \\ \star & \star & -S_j + \Lambda_j & \mathbf{0} & (1/r)F_3A_{dj} \\ \star & \star & \star & -R_j & (1/r)F_4A_{dj} \\ \star & \star & \star & \star & -(r - |\alpha|)^{2\bar{d}}\Lambda_j \end{bmatrix} < \mathbf{0} \quad (17)$$

$$\Pi_j = \begin{bmatrix} P_j - F_1 - F_1^T & (1/r)F_1\left(\hat{A}_j - \alpha\mathbf{I}\right) - F_2^T & -F_3^T & -F_4^T \\ \star & \begin{pmatrix} S_j - P_j + R_j \\ +(1/r)F_2\left(\hat{A}_j - \alpha\mathbf{I}\right) + (1/r)\left(\hat{A}_j - \alpha\mathbf{I}\right)^T F_2^T \end{pmatrix} & (1/r)\left(\hat{A}_j - \alpha\mathbf{I}\right)^T F_3^T & (1/r)\left(\hat{A}_j - \alpha\mathbf{I}\right)^T F_4^T \\ \star & \star & -S_j & \mathbf{0} \\ \star & \star & \star & -R_j \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{I} \\ \mathbf{0} \end{bmatrix} \Lambda_j \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{I} \\ \mathbf{0} \end{bmatrix}^T + \begin{bmatrix} (1/r)F_1A_{dj} \\ (1/r)F_2A_{dj} \\ (1/r)F_3A_{dj} \\ (1/r)F_4hA_{dj} \end{bmatrix}$$
$$\times (r - |\alpha|)^{-2\bar{d}} \Lambda_j^{-1} \left[(1/r)A_{dj}^T F_1^T \quad (1/r)A_{dj}^T F_2^T \quad (1/r)A_{dj}^T F_3^T \quad (1/r)A_{dj}^T F_4^T\right] < \mathbf{0} \quad (18)$$

$$\Phi_j = \begin{bmatrix} P_j - F_1 - F_1^T & (1/r)F_1\left(\hat{A}_j - \alpha\mathbf{I}\right) - F_2^T & -F_3^T & -F_4^T \\ \star & \begin{pmatrix} S_j - P_j + R_j \\ +(1/r)F_2\left(\hat{A}_j - \alpha\mathbf{I}\right) + (1/r)\left(\hat{A}_j - \alpha\mathbf{I}\right)^T F_2^T \end{pmatrix} & (1/r)\left(\hat{A}_j - \alpha\mathbf{I}\right)^T F_3^T & (1/r)\left(\hat{A}_j - \alpha\mathbf{I}\right)^T F_4^T \\ \star & \star & -S_j & \mathbf{0} \\ \star & \star & \star & -R_j \end{bmatrix} + \begin{bmatrix} (1/r)F_1A_{dj} \\ (1/r)F_2A_{dj} \\ (1/r)F_3A_{dj} \\ (1/r)F_4A_{dj} \end{bmatrix} \left(\frac{rv + \alpha}{v}\right)^{-d} \left[\mathbf{0} \quad \mathbf{0} \quad \mathbf{I} \quad \mathbf{0}\right]$$
$$+ \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{I} \\ \mathbf{0} \end{bmatrix} \left(\overline{\frac{rv + \alpha}{v}}\right)^{-d} \left[(1/r)A_{dj}^T F_1^T \quad (1/r)A_{dj}^T F_2^T (1/r)A_{dj}^T F_3^T \quad (1/r)A_{dj}^T F_4^T\right] < \mathbf{0} \quad (19)$$

$$\begin{bmatrix} \tilde{P}_j - U - U^T & +(1/r)((A_j - \alpha\mathbf{I})U + BY) & \mathbf{0} & \mathbf{0} & (1/r)A_{dj}U \\ \star & \tilde{S}_j - \tilde{P}_j + \tilde{R}_j & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \star & -\tilde{S}_j + \tilde{\Lambda}_j & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & -\tilde{R}_j & \mathbf{0} \\ \star & \star & \star & \star & -(r - |\alpha|)^{2\bar{d}}\tilde{\Lambda}_j \end{bmatrix} < \mathbf{0} \quad (21)$$

ensuring that the linear uncertain discrete-time system with time-delay in the state (1)–(2) in closed-loop by control law (4) is $\mathcal{D}(\alpha, r)$-stable.

Proof: Pre- and post-multiplying the LMIs (21) by $\mathfrak{T}^T$ and $\mathfrak{T}$, respectively, where $\mathfrak{T} = \text{diag}\{U^{-1}, U^{-1}, U^{-1}, U^{-1}, U^{-1}\}$ and $U^{-1} = F_1^T$, and considering that $Y = KU$, $\tilde{P}_j = U^T P_j U$, $\tilde{S}_j = U^T S_j U$, $\tilde{R}_j = U^T R_j U$, and $\tilde{\Lambda}_j = U^T \Lambda_j U$, we obtain (23). Replacing $A_j + BK$ by $\hat{A}_j$ in (23) and considering that $F_i = \mathbf{0}$, for $i \in \mathcal{I}[2, 4]$, in (17), we can obtain the first one by the second. Therefore, the feasibility of (21) guarantees the negativity of $\Delta V(x_k, \beta)$, the Lyapunov-Krasovskii's theorem is verified, and it is guaranteed the $\mathcal{D}(\alpha, r)$-stability of the linear uncertain discrete-time system with time-delay in the state (1)–(3) in closed-loop through control law (4) with the controller matrix K given by (22).

$\square$

5 Numerical example

Consider the problem of $\mathcal{D}(\alpha, r)$-stabilization of the linear uncertain discrete-time system with time-delay in the state described by (1), where

$$A = \begin{bmatrix} 0.27 + 0.9\epsilon_1 & 0.9 \\ 0.18 & 1.08 - 0.9\epsilon_1 \end{bmatrix},$$
$$A_d = \begin{bmatrix} 0.08 - 0.8\epsilon_2 & 0.24 \\ -0.016 & 0.08 + 0.8\epsilon_2 \end{bmatrix},$$
$$B = \begin{bmatrix} 1 + \epsilon_3 \\ 0.6 - \epsilon_3 \end{bmatrix}, \quad (24)$$

$|\epsilon_1| \leq 0.1$, $|\epsilon_2| \leq 0.05$, and $|\epsilon_3| \leq 0.1$, and the time-delay is $d \in \mathcal{I}[1, 3]$, that is, $\bar{d} = 3$. This system is not stable in open-loop and the uncertain parameters given above yields a polytopic representation with eight vertex made up with the combinations of the extremum values of ϵ_1, ϵ_2, and ϵ_3.

Firstly, we used Theorem 2 to design the controller matrix, K, that $\mathcal{D}(\alpha, r)$-stabilize the system (1) with $\alpha = 0$ and $r = 0.67$, where this radius is the smallest computed. Therefore, this value of the radius is the minimum that the synthesis conditions find solution. The controller matrix obtained is

$$K = -\begin{bmatrix} 0.2668 & 1.8338 \end{bmatrix}. \quad (25)$$

The achieved eigenvalues are shown in Figure 2, where the eigenvalues are shown for all values that the time-delay can assume and all of them are inside the $\mathcal{D}(0, 0.67)$ region. Furthermore, we can note that how bigger the delay, closer the eigenvalues are of the limits of the $\mathcal{D}(0, 0.67)$ region limits.

In the last, we compare the results developed here with the conditions present in (Mao and Chu, 2009). For this, we consider three different values to α and compute the minimums of the radius. With the results we built the Table 1. Note that for the system treated in this example, the conditions developed in this paper obtain radius less than the conditions present in (Mao and Chu, 2009) for all values of α used. This shows that the conditions developed here are less conservative than conditions developed in (Mao and Chu, 2009).

Table 1: The minimums of radius for different α.

α	-0.1	0	0.1
Theorem 2	0.74	0.67	0.73
(Mao and Chu, 2009)	0.80	0.72	0.77

6 Conclusions

Conditions of $\mathcal{D}$-stability analysis and $\mathcal{D}$-stabilization linear discrete-time systems with time-delay in the states and polytopic uncertainties are proposed in this paper. The conditions were based on a Lyapunov-Krasovskii candidate function and they were in terms of LMIs. Initially, we obtained an equivalent system where the eigenvalues of the original systems located in $\mathcal{D}(\alpha, r)$ region, with α is a center of the circle and r is the radius, are mapped in the unitary circle. Therefore, by guaranteeing the stability of the equivalent system, we ensure the $\mathcal{D}(\alpha, r)$-stability of the original system. A numerical example was presented, where we demonstrated the efficient of the proposed conditions and compared these proposed conditions with similar results found in the literature.

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$$\begin{bmatrix} P_j - F_1 - F_1^T & +(1/r)F_1\left(A_j - \alpha\mathbf{I} + BK\right) & \mathbf{0} & \mathbf{0} & (1/r)F_1A_{dj} \\ \star & S_j - P_j + R_j & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \star & -S_j + \Lambda_j & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & -R_j & \mathbf{0} \\ \star & \star & \star & \star & -\left(r - |\alpha|\right)^{2\bar{d}}\Lambda_j \end{bmatrix} < \mathbf{0} \qquad (23)$$

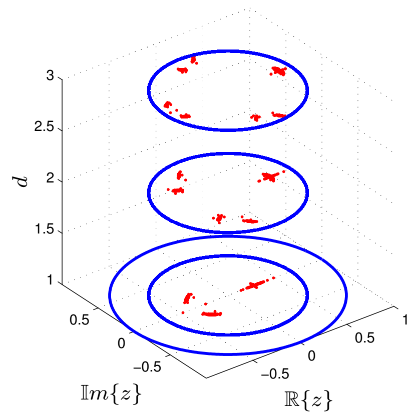

Figure 2: Eigenvalues for $d \in \mathcal{I}[1,3]$.

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