

DIGITAL LQG CONTROL FOR ACTIVE SUSPENSION SYSTEM OF ½ VEHICLE MODEL

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Abstract - This paper presents the design and simulation of a digital multivariable optimal control based on LQG technique (*Linear Quadratic Gaussian*) for an active suspension system of a light vehicle. The LQG technique was used to determine multivariate digital controller system. The dynamic model with and without the designed control were simulated using Simulink / MATLAB software. The results shown that the active suspension has advantages over passive suspension, since its presents reductions in displacement of the springs and increase in passenger comfort by reducing the vertical acceleration of the vehicle.

Keywords – Digital LQG (*Linear Quadratic Gaussian*) Control, Active Suspension, MATLAB/Simulink, Multivariable System.

1 Introduction

The suspension system has the fundamental function of supporting the chassis of the vehicle, as well as to provide comfort for passengers by rejecting road disturbances. Furthermore, the suspension system must provide adequate forces for chassis-wheels system in order to guarantee vehicle stability in cornering, accelerating and breaking situations (Heissing e Ersoy 2011).

Active suspension systems, unlike the passive systems, are designed to dynamically adjust the forces and energy flow according to control algorithm by independent actuators, improving comfort and performance.

Various control strategies can be applied for this problem, such as Nonlinear Optimal Control (A. M., et al. 2009) and Nonlinear Backstepping Control (Lin e Kanellakopoulos 1997). This paper aims to design and investigate the performance of a LQG (*Linear Quadratic Gaussian*) control with discrete approach for a half vehicle model. In order to simulate the model and to design and implement the controller the MATLAB/Simulink software were used.

2 System Dynamics Model

The system dynamics model of the chassis is consider being a rigid body with mass M , moment of inertia I and longitudinal velocity $V(t)$ at his center of mass (C.M.). The vehicle model presents an active suspension system with springs, dampers, actuators and tires (front and rear) and is submitted to a road disturbance.

The Figure 1 presents the active suspension for half vehicle scheme.

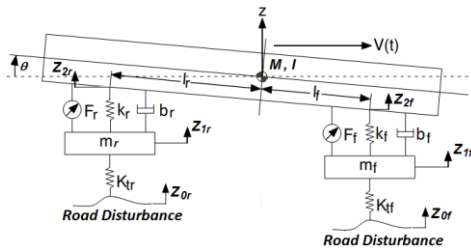


Figure 1– Half vehicle active suspension. (Ihsan, et al. 2009)

Model variables definitions with their respective units are presented in Table 1.

Table 1 – Dynamic model variables definition.

Variable	Definition	Unit
M	Mass of C.M.	kg
I	Moment of inertia (<i>pitch</i>)	kg.m ²
z	C.M. vertical displacement	m
z_{2f}	Front axle vertical displacement	m
z_{2r}	Rear axle vertical displacement	m
z_{1f}	Front tire vertical displacement	m
z_{1r}	Rear tire vertical displacement	m
z_{0f}	Road vertical disturbance (front axle)	m
z_{0r}	Road vertical disturbance (rear axle)	m
θ	<i>Pitch</i> angle at vehicle C.M.	rad
$V(t)$	C.M. longitudinal velocity	km/hr
l_r	Distance between rear axle and vehicle C.M.	m
l_f	Distance between front axle and vehicle C.M.	m
k_r	Rear spring coefficient	N/m
k_f	Front spring coefficient	N/m
b_r	Rear damper coefficient	N/m/s
b_f	Front damper coefficient	N/m/s
F_r	Rear actuator force	N
F_f	Front actuator force	N
m_r	Rear tire mass	kg
m_f	Front tire mass	kg
k_{tr}	Rear tire stiffness coefficient	N/m
k_{tf}	Front tire stiffness coefficient	N/m

From scheme of Figure 1 and applying Newton's second law to vehicle's body and to front and rear tires, we have the following set of equations (Rill 2007):

$$\begin{cases} M\ddot{z} = F_1 + F_2 \\ I\ddot{\theta} = F_2 l_r - F_1 l_f \\ m_f \ddot{z}_{1f} = -k_{tf}(z_{1f} - z_{0f}) - F_1 \\ m_r \ddot{z}_{1r} = -k_{tr}(z_{1r} - z_{0r}) - F_2 \end{cases} \quad (1)$$

Where,

$$F_1 = k_f(z_{1f} - z_{2f}) + b_f(\dot{z}_{1f} - \dot{z} + \dot{\theta}l_f) + F_f$$

$$F_2 = k_r(z_{1r} - z_{2r}) + b_r(\dot{z}_{1r} - \dot{z} - \dot{\theta}l_r) + F_r$$

Rearranging equations (1) we have the following motion equations (2) to (5).

Equation (2) represents vehicle C.M. vertical displacement:

$$\ddot{z} = -\frac{b_f + b_r}{M}\dot{z} + \frac{l_f b_f - l_r b_r}{M}\dot{\theta} - \frac{k_f}{M}(z_{2f} - z_{1f}) + \frac{b_f}{M}\dot{z}_{1f} - \frac{k_r}{M}(z_{2r} - z_{1r}) + \frac{b_r}{M}\dot{z}_{1r} + \frac{F_f}{M} + \frac{F_r}{M} \quad (2)$$

The pitch angle dynamic equation is represented by equation (3):

$$\ddot{\theta} = \frac{l_f b_f - l_r b_r}{I}\dot{z} - \frac{l_f^2 b_f + l_r^2 b_r}{I}\dot{\theta} + \frac{k_f l_f}{I}(z_{2f} - z_{1f}) - \frac{b_f l_f}{I}\dot{z}_{1f} - \frac{k_r l_r}{I}(z_{2r} - z_{1r}) + \frac{b_r l_r}{I}\dot{z}_{1r} - \frac{l_f F_f}{I} + \frac{l_r F_r}{I} \quad (3)$$

The front tire vertical displacement dynamic equation is represented by equation (4):

$$\ddot{z}_{1f} = -\frac{k_{tf}}{m_f}z + \frac{b_f}{m_f}\dot{z} + \frac{k_{tf}l_f}{m_f}\theta - \frac{b_f l_f}{m_f}\dot{\theta} + \frac{k_f + k_{tf}}{m_f}(z_{2f} - z_{1f}) - \frac{b_f}{m_f}\dot{z}_{1f} + \frac{k_{tf}}{m_f}z_{0f} - \frac{F_f}{m_f} \quad (4)$$

The rear tire vertical displacement dynamic equation is represented by equation (5):

$$\ddot{z}_{1r} = -\frac{k_{tr}}{m_r}z + \frac{b_r}{m_r}\dot{z} - \frac{k_{tr}l_r}{m_r}\theta + \frac{b_r l_r}{m_r}\dot{\theta} + \frac{k_r + k_{tr}}{m_r}(z_{2r} - z_{1r}) - \frac{b_r}{m_r}\dot{z}_{1r} + \frac{k_{tr}}{m_r}z_{0r} - \frac{F_r}{m_r} \quad (5)$$

In order to formulate the problem in state-space representation the following states of the system were defined:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} z \\ \dot{z} \\ \theta \\ \dot{\theta} \\ z_{2f} - z_{1f} \\ \dot{z}_{1f} \\ z_{2r} - z_{1r} \\ \dot{z}_{1r} \end{bmatrix} \quad (6)$$

Thus, equation (7) represents the state-space matrix equation of the system:

$$\dot{\mathbf{x}} = [\mathbf{A}]\mathbf{x} + [\mathbf{B}_c]\mathbf{u} + [\mathbf{B}_d]\mathbf{d} \quad (7)$$

Where,

$$\mathbf{u} = [F_f \quad F_r]^T$$

$$\mathbf{d} = [z_{0f} \quad z_{0r}]^T$$

A regular passenger's vehicle was used as reference to model's data. The used data are presented in Table 2 (Keith e D. Geoff 2011).

Table 2 – Vehicle model data.

Variable	Value	Unit
M	1054	kg
I	1326	kg.m ²
l_r	1,422	m
l_f	0,953	m
k_r	20900	N/m
k_f	12800	N/m
b_r	1565	N/m/s
b_f	850	N/m/s
m_r	30	kg
m_f	30	kg
k_{tr}	97500	N/m
k_{tf}	97500	N/m

3 System Discretization

In order to design a discrete LQG controller the system was discretized with a sample time of 0.01 s. This sample time was selected after analysis of eigenvalues of matrix $\mathbf{A}$ (about 0.5 rad/s).

Thus, the dynamic discrete equation of the system has the following structure:

$$\begin{cases} \mathbf{x}[n+1] = \Phi \mathbf{x}[n] + \Gamma \mathbf{u}[n] \\ \mathbf{y}[n] = \mathbf{C} \mathbf{x}[n] \end{cases} \quad (8)$$

Where,

$$\Phi = e^{AT_s}, \Gamma = \int_0^{T_s} e^{A\eta} d\eta B_c \text{ and}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Considering the states as:

$$\begin{bmatrix} x_1 \\ x_3 \\ x_5 \\ x_7 \end{bmatrix} = \begin{bmatrix} z \\ \theta \\ z_{2f} - z_{1f} \\ z_{2r} - z_{1r} \end{bmatrix}$$

The Figure 2 presents the equivalent discrete system, where u_n are inputs signals, x_n are the states and y_n are outputs signals.

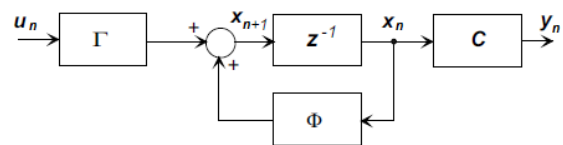


Figure 2 - ZOH equivalent discrete system.

4 Performance Specification

The discrete LQG controller is designed to satisfy the following performance specifications:

- States should have settling time bellow 1 second for a disturbance as step of 0.1 m;
- Center of mass maximum displacement z less than 0.01 m in module;
- $Pitch$ angle θ always less than 0.01 rad in module;
- In order to have the tires grounded is required that the difference between tire mass and chassis displacement is less than 0.01 m in module;
- Equivalent actuators force less than 2000 N in module.

5 Discrete LQG Theory

The LQG (*Linear Quadratic Gaussian*) regulator is composed by joining the LQR (*Linear Quadratic Regulator*) and the optimal estimator (*Kalman Stationary Filter*). According to *LQG Separation Theorem* (Davis e Zervos 1995), if the regulator and the estimator are both stable, then the whole system is stable.

5.1. LGR

The LQR (*Linear Quadratic Regulator*) problem is to find a feedback gain K for the states of the equation system (8) such as the control law $u[n] = -Kx[n]$ that minimizes the following value cost function (9):

$$J = \frac{1}{2} \sum_{n=0}^{\infty} (x^T[n]Qx[n] + u^T[n]Ru[n]) \quad (9)$$

Where Q and R are symmetric positive definite weight matrix related to the states x and to inputs u respectively.

As demonstrated in (Franklin, Powell e Workman, Digital Control of Dynamic Systems 2006), the solution for this optimal problem is given by the equation (11), using the solution of the *Algebraic Riccati Equation* (10):

$$P = \Phi^T (P - P\Gamma(R + \Gamma^T P\Gamma)^{-1}\Gamma^T P)\Phi + Q \quad (10)$$

$$K = (R + \Gamma^T P\Gamma)^{-1}\Gamma^T P\Phi \quad (11)$$

5.2. Discrete Kalman Stationary Filter

Consider the following discrete equation system:

$$\begin{cases} x[n+1] = \Phi x[n] + \Gamma u[n] + \Psi w[n] \\ y[n] = Cx[n] + v[n] \end{cases} \quad (12)$$

Where $w[n]$ and $v[n]$ are respectively the process uncertainties and the measurement noise. Both are temporally uncorrelated (Gaussian white noise), zero-mean random sequences with known covariance:

$$\mathbb{E}\{v[k]w^T[j]\} = 0$$

$$\mathbb{E}\{v[k]v^T[j]\} = \begin{cases} 0, & k \neq j \\ R_v, & k = j \end{cases}$$

$$\mathbb{E}\{w[k]w^T[j]\} = \begin{cases} 0, & k \neq j \\ R_w, & k = j \end{cases}$$

Where R_v and R_w are measurement noise covariance matrix and process noise covariance matrix respectively.

The *Kalman Filter* is the solution of the optimal problem for equation (12) states estimator. Consider the estimator structure as shown in equations (13) and (14),

$$\hat{x}[n] = \bar{x}[n] + L_c[n](y[n] - C\bar{x}[n]) \quad (13)$$

$$\bar{x}[n] = \Phi\bar{x}[n-1] + \Gamma u[n-1] \quad (14)$$

The *Kalman Stationary Filter* gain L_c is given in an equivalent manner to the LQR gain K , from *Algebraic Riccati Equation* solution (16), as shown in equation (15):

$$L_c = MC^T(CMC^T + R_v)^{-1} \quad (15)$$

Where,

$$M = \Phi(M - MC^T(R_v + CMC^T)^{-1}CM)\Phi^T + \Psi R_w \Psi^T \quad (16)$$

Figure 3 presents the discrete LQG control functional block diagram.

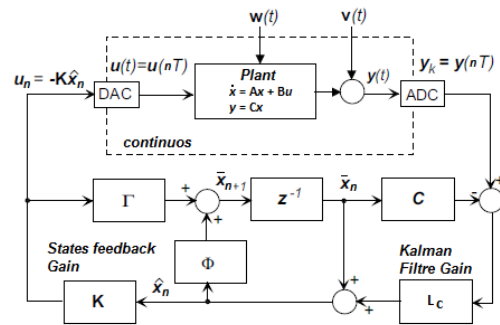


Figure 3 - Discrete LQG control functional block diagram.

6 Discrete LQG Controller Design

Following the model discretization, as shown in Section 2, the optimal states feedback gain K is computed according to equations (10) and (11).

However, as the designed matrices Q and R are weakly related to system performance, the *pincer* method mentioned in (Franklin, Powell e Workman, Digital Control of Dynamic Systems 2006) was used in order to meet the state's settling time requirement.

The *pincer* method consists in multiply the system states by a factor α in such a way to ensure the states decay with a rate of at least $1/\alpha^n$. Thus, the system matrices turn in to $\alpha\Phi$ and $\alpha\Gamma$, with

$$\alpha = 100 \frac{T_s}{t_s} \quad (17)$$

Where T_s is the sample time and t_s is the settling time required for major critical state.

In this paper the designed controller it was used $t_s = 0,8$ s and $T_s = 0,01$ as shown in Section 3, thus:

$$\alpha = 1,0593$$

Furthermore, in order to limit the states, as shown in Section 4, the *Bryson* rule was used as mentioned in (Franklin, Powell e Emami-Naeini, Feedback Control of Dynamic Systems 2002), where it is suggested that the matrices Q and R should be chosen as diagonals, such as:

$$Q_{ii} = \frac{1}{\max \text{value of } x_i^2}, i = 1, 2, \dots, k.$$

$$R_{jj} = \frac{1}{\max \text{value of } u_j^2}, j = 1, 2, \dots, r.$$

Therefore the matrices Q and R were selected as follow:

$$Q = \begin{bmatrix} 10^5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10^5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10^5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 10^5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R = \begin{bmatrix} 10^{-3} & 0 \\ 0 & 10^{-3} \end{bmatrix}$$

The matrix Q has been selected to limit the process states of interest and the matrix R to not limit the control effort, once it has been noticed in simulations that the efforts involved met the performance specifications.

Thus, using the MATLAB® software, the feedback gain matrix K was computed by the command,

$$K = \text{dlqr}(\alpha * \Phi, \alpha * \Gamma, Q, R)$$

For the states estimator, where the states x_1, x_3, x_5 and x_7 were considered to be the measure outputs, were selected the following power spectral density matrices for the measurement noise (R_v), for the process uncertainties (R_w) and for process noise weighting Ψ respectively:

$$R_v = 0,001 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad R_w = 0,1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$B_d \cong \Psi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{k_{tf}}{m_f} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{k_{tr}}{m_r} \end{bmatrix}$$

Thus, using the MATLAB®, the *Kalman Filter* gain matrix L_C is computed by the command:

$$L_c = \text{dlqe}(\Phi, \Psi, C, R_w, R_v);$$

In order to show stability, the close loop poles are computed considering the *Kalman Filer* and the feedback gains:

$$\lambda = \begin{bmatrix} 0,732 \pm 0,493i \\ 0,672 \pm 0,424i \\ 0,939 \pm 0,081i \\ 0,937 \pm 0,060i \\ 0,137 \pm 0,747i \end{bmatrix}, \quad \begin{bmatrix} 0,539 \pm 0,336i \\ 0,959 \pm 0,049i \\ -0,562 \\ 0,9297 \end{bmatrix}$$

7 Simulink Suspension Simulation

In order to validate the designed controller and to check the control law's performance, the active and the passive suspension systems were simulated in MATLAB/Simulink® software.

Figure 4 to Figure 5 presents the suspension system model in Simulink block diagrams implementation. Figure 4 presents an overview diagram of the active suspension with Digital LQG Controller implemented in a MATLAB Function (figure's top) and the passive suspension (figure's bottom).

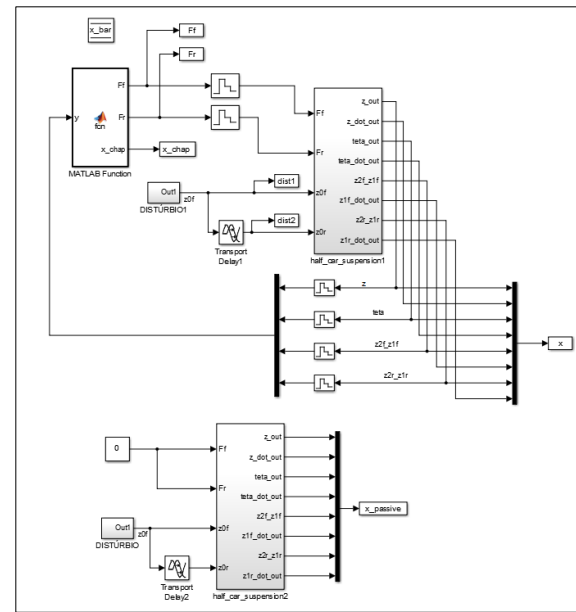


Figure 4 – Simulink block diagram – Overview.

The suspension models were implemented as shown in Figure 5 according to equations (1).

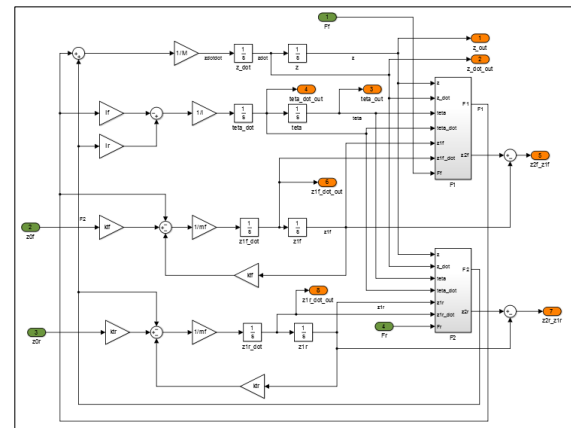


Figure 5 – Simulink block diagram – Suspension Model.

Road disturbance (z_{of} and z_{or}) are model's inputs and presents a delay (*Transport Delay*) between front axle (z_{of}) and rear axle (z_{or}) according to vehicle's lon-

itudinal velocity V and to axles distance (axle track), as shown in equation (18):

$$\text{Delay} = \frac{l_f + l_r}{V} \quad (18)$$

For system performance's analysis, two types of disturbances were selected: step of 0.1 m (see Figure 6) and sinusoidal with amplitude of 0.1 m (see Figure 7).

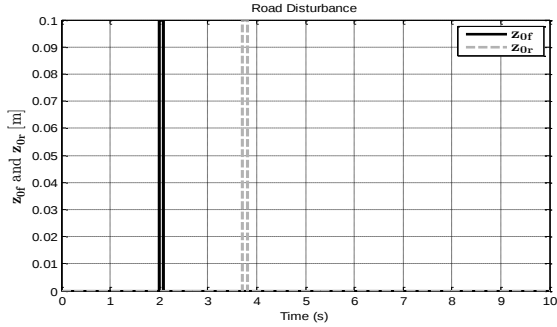


Figure 6 – Step disturbance for vehicle speed of 5 km/h.

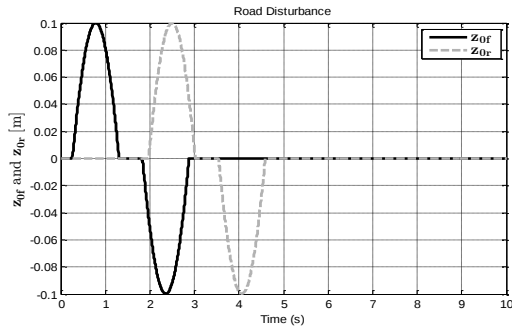


Figure 7 – Sinusoidal disturbance for vehicle speed of 5 km/h.

Furthermore, in order to verify the performance and robustness of the designed control law, both systems, active and passive suspension, were simulated changing the vehicle velocity from 5 km/h to 40 km/h and the vehicle mass from 1054 kg to 1454 kg.

Figure 8 to Figure 10 presents the step response of the road disturbance for active and passive systems for vehicle's mass of 1054 kg and velocity of 5 km/h. Displacements are in meter, *pitch* angle in rad and the control effort u in Newton.

In each figure are shown the active and the passive response as well as the value of the estimated variable by the Kalman Filter.

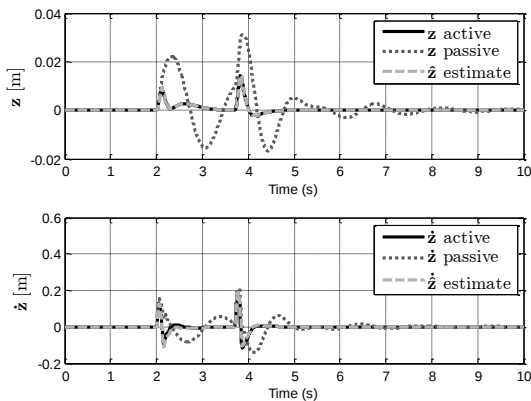


Figure 8 – Step response for 5 km/h - z and $\dot{z}$.

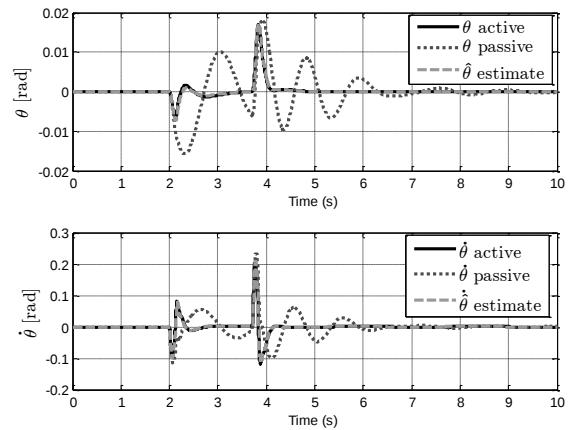


Figure 9 – Step response for 5 km/h - θ and $\dot{\theta}$.

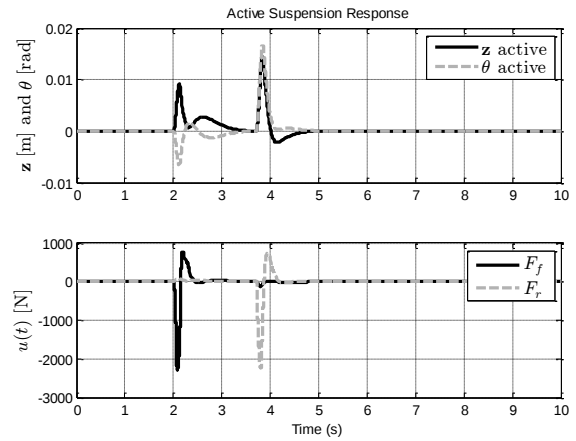


Figure 10 – Step response for 5 km/h - z , θ and $u(t)$.

Figure 11 to Figure 13 presents the sinusoidal response of nominal active and passive systems (mass of 1054 kg) for a sinusoidal disturbance with velocity of 5 km/h.

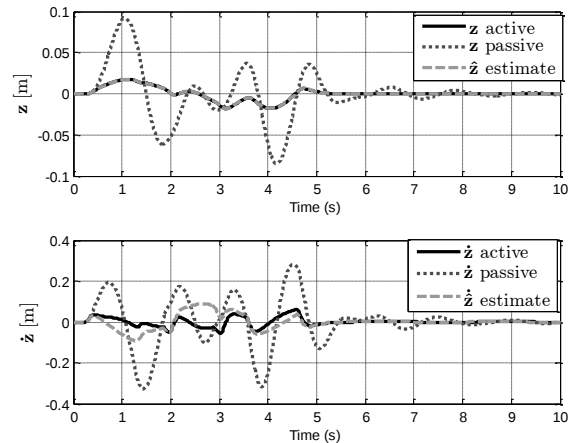


Figure 11 – Sinusoidal response for 5 km/h - z and $\dot{z}$.

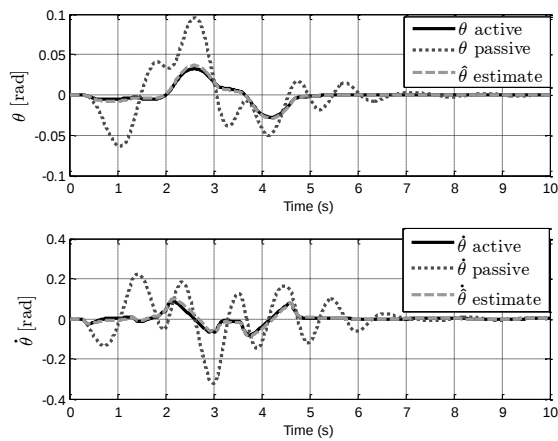


Figure 12 - Sinusoidal response for 5 km/h - θ and $\dot{\theta}$.

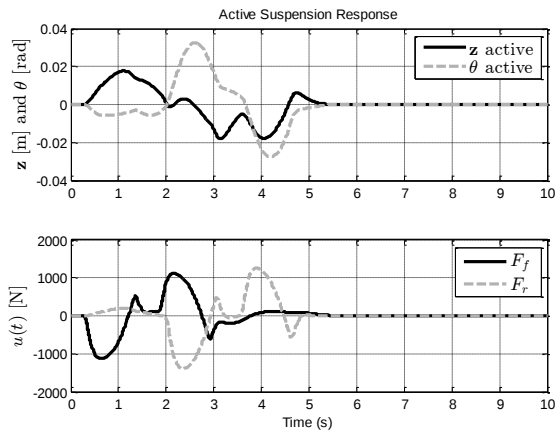


Figure 13 – Sinusoidal response for 5 km/h - z , θ and $u(t)$.

Figure 8 to Figure 13 shows that active system presents significant fewer oscillations than the passive one. In addition, the control-states present a faster decrease rate, turning the active system into a faster system.

It is also observed that the estimated states by the designed Kalman Filter have presented a good approximation when compared to the real values. Regarding the control effort, it's noted that the maximum actuators forces are about 2200 N, near to specified value in Section 4.

Simulated systems responses were compared by using RMS (Root Mean Square) values of the states. Table 3 presents the perceptual difference between active and passive systems of the main states for all simulated conditions. The negative values in Table 3 shows that the active system had a better performance than passive system, since in all simulated situations the main states were reduced.

Table 3 – Perceptual difference between active and passive system for step and sinusoidal response and for mass variation response.

States	z	$\dot{z}$	θ	$\dot{\theta}$
Active vs Passive – Step Response				
5 km/h	-68,5%	-35,1%	-43,6%	-27,2%
10 km/h	-50,5%	-20,0%	-67,5%	-34,9%
20 km/h	-66,7%	-28,8%	-49,0%	-36,0%
40 km/h	-73,1%	-44,5%	-19,3%	-14,7%
Active vs Passive – Sinusoidal Response				
5 km/h	-75,6%	-82,8%	-63,9%	-71,7%
10 km/h	-60,3%	-61,2%	-74,2%	-81,3%

20 km/h	-70,8%	-79,5%	-53,7%	-61,7%
40 km/h	-72,7%	-80,3%	-30,7%	-49,5%
Active vs Passive – Step Response for 5 km/h				
1254 kg	-71,9%	-40,9%	-44,1%	-25,0%
1454 kg	-73,8%	-45,7%	-46,6%	-24,5%

8 Conclusion

In this paper it was designed a Digital Multivariable LQG control for an active suspension. The feedback and the Kalman Filter Estimator gains were computed in order to meet the performance specifications.

By using MATLAB/Simulink® software the active and the passive systems were simulated in many different conditions and for all this conditions the active system was able to meet the specifications and presented a better performance than passive one.

Furthermore it could be noted that the designed control was able to maintain the performance even with vehicle's parameters changes (mass and velocity), showing control robustness.

For future works is intend to study and implement a Discrete LQG/LTR (Loop Transfer Recovery) control and connect the suspension model into a full car model.

9 References

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