H_∞ CONTROL FOR CONTINUOUS-TIME MARKOV JUMP LINEAR SYSTEMS WITH PARTIAL MODE INFORMATION†

C. C. Graciani Rodrigues*, Marcos G. Todorov*, Marcelo D. Fragoso*

*National Laboratory for Scientific Computing - LNCC/ CNPq
Av. Getúlio Vargas 333, Petrópolis, Rio de Janeiro, Brasil

Emails: caiocgr@lncc.br, todorov@lncc.br, frag@lncc.br

Abstract— In this paper we present new results for continuous-time Markov jump linear systems with partial observations on the jumping process. We assume the existence of a suitable detector which provides measurements of the Markov chain. Our aim is to design a detector-based H_∞ state feedback controller. The proposed design methods, which include an algorithm to treat a non-convexity issue, are given in terms of linear matrix inequalities and the results are applied to the control of an unmanned aerial vehicle and a power system.

Keywords— H_∞ control, Markov jump linear systems, Partial observations.

Resumo— Neste artigo, apresentamos novos resultados para sistemas lineares com salto Markoviano a tempo contínuo com observação parciais do processo de salto. Assumimos a existência de um detector que fornece informações da cadeia de Markov. Nosso objetivo é o projeto de um controlador com realimentação de estado baseado no detector. Os métodos de projeto propostos, incluindo um algoritmo para tratar a não-convexidade do problema, são estabelecidos em termos de desigualdades matriciais lineares e os resultados são aplicados no controle de um veículo aéreo não tripulado e de um sistema de potência.

Palavras-chave— Controle H_∞, Sistemas lineares com salto Markoviano, Observações parciais.

1 Introduction

Markov jump linear systems (MJLS) have been the subject of extensive research over the last few years and the associated literature is by now fairly extensive (see, e.g., (Mariton, 1990; Costa et al., 2005; Dragan et al., 2006; Costa et al., 2013), and references there in). This class of parameter-switching models has an intimate connection with systems which are sensitive to abrupt changes in their structure. On that account, MJLS can be used in a wide variety of real applications such as, for instance, in aircraft flight systems, communications networks, robotics, mathematical finance, etc. A fleeting glimpse of works which deal with application in this field can be found, e.g., in (Costa et al., 2005; Costa et al., 2013)(see also, (Costa et al., 2015)).

Regarding the H_∞ approach for MJLS, (de Souza and Fragoso, 1993; Pan and Başar, 1994; Fragoso et al., 1995), seem to be the first works dealing with H_∞-control. In the context of H_∞-*filtering*, works can be traced back at least to (de Souza and Fragoso, 2002a). Most of these works consider the setting in which the operation mode of the systems is completely accessible. The exceptions in the scenario of partial information of the Markov chain are the so-called mode-independent case (see, e.g., (de Souza et al., 2006)), the cluster setting (see, e.g., (Gonçalves et al., 2009)) and situations involving parametric uncertainty, in which the partial observation is related to incomplete knowledge of the transition probability matrix (see, e.g., (de Souza and Fragoso, 2002b)). For a recent overview on the state-of-the-art of MJLS with partial observations see, for instance, (Costa et al., 2013).

In this paper, we make some further foray on the H_∞ control problem for continuous-time MJLS. We consider the case in which *we do not have access to the Markov chain* but, instead, there is a *detector* that emits signals which provides information on this parameter.

Roughly, the contributions of the paper can be briefly summarized as follows:

1. We devise a detector-based H_∞ control approach for continuous-time MJLS with *partial information on the Markovian jump parameter.* The detector approach proposed here for the *continuous-time scenario* is inspired on (Fragoso and Costa, 2004; Costa et al., 2015), which treat, respectively, *mean-square stability of continuous-time* MJLS and a *control problem in discrete-time.* The main results come in the form of Theorem 2 and Algorithm 3.

2. In order to illustrate an application of the main results, we present a numerical example of the H_∞ control for an unmanned aerial vehicle model (see (Ducard, 2009)), as well the control of a power system model (see (Arrifano and Oliveira, 2004)).

An outline of the content of this paper is as follows. In Section 1.1 we provide the bare es-

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sentials of notations. The detector-based formulation is presented in Section 2, along with the H_∞ control problem in Section 2.1. In Section 3 we present numerical examples which illustrate the usefulness of the obtained result. The paper is concluded in Section 4 with some final comments.

1.1 Notation

Throughout this paper we shall denote by $\mathbb{R}^n$ the n-dimensional real Euclidean space and by $\mathbb{R}^{n\times m}$ the set of $n\times m$ real matrices. The Euclidean vector norm is represented by $\|\cdot\|$. We also denote by I_n the $n\times n$ identity matrix, and by $\mathbb{R}^+$ the interval $[0,\infty)$. The superscript $'$ will denote transpose, $\mathrm{Her}(\cdot)\triangleq(\cdot)+(\cdot)'$ and the symbol $*$ represents a symmetric block in the linear matrix inequality (LMI). For a real matrix V, $V>0(V<0)$ means that V is symmetric and positive definite (negative definite). Let $(\Omega,\mathscr{F},P)$ be a complete probability space carrying its natural filtration $\{\mathscr{F}_t, t\in\mathbb{R}^+\}$, as usual augmented by all null sets in the P-completion of $\mathscr{F}$. In addition we denote by $L_2^{n_w}(\Omega,\mathscr{F},P)$ the space of all $\mathscr{F}_t$-adapted stochastic processes $w=\{w(t);t\in\mathbb{R}^+\}$ in $\mathbb{R}^{n_w}$ such that $\|w\|_2<\infty$, where

$$\|w\|_2 \triangleq \left(\int_0^\infty E[\|w(t)\|^2]dt\right)^{1/2},$$

with $E[\cdot]$ standing for the usual mathematical expectation.

Let $x=\{x(t);t\in\mathbb{R}^+\}$ in $\mathbb{R}^{n_x}$ denote the state variable, $z=\{z(t);t\in\mathbb{R}^+\}$ in $\mathbb{R}^{n_z}$ the system's controlled output and $w=\{w(t);t\in\mathbb{R}^+\}\in L_2^{n_w}(\Omega,\mathscr{F},P)$ an exogenous disturbance input. We also define $\theta=\{(\theta(t),\mathscr{F}_t),t\in\mathbb{R}^+\}$ as a time-homogeneous Markov process with right continuous trajectories, taking values on the finite set $\mathcal{S}\triangleq\{1,\dots,N\}$. For small $h>0$ we have that

$$P(\theta(t+h)=j\mid\theta(t)=i)=\begin{cases}\lambda_{ij}h+o(h), & i\neq j,\\ 1+\lambda_{ii}h+o(h), & i=j,\end{cases} \tag{1}$$

where $\Lambda=[(\lambda_{ij})]$ is the homogeneous $N\times N$ transition rate matrix of $\{\theta\}$ with $\lambda_{ij}\geq 0$, $i\neq j$ and $\lambda_i=-\lambda_{ii}=\sum_{j:j\neq i}\lambda_{ij}$. Here $o(h)$ denotes an infinitesimal of higher order than h.

2 The detector-based H_∞ approach

The interest in this paper is on the situation where the process θ cannot be known precisely, but instead a measurement $\hat\theta$, which is provided by a *suitable detector*, is available at all times. In this context, we *extend* the detector formulation introduced by (Costa et al., 2015) in the discrete-time setup, and consider a finite set $\mathcal{D}\triangleq\{1,\dots,M\}$ and subsets $\mathcal{D}_i\subset\mathcal{D}$ such that a signal $\hat\theta(t)\in\mathcal{D}_i$ is emitted whenever the Markov chain $\theta(t)=i$, independently of all previous and present values of the other processes.

For the context of H_∞ analysis we consider the following continuous-time MJLS

$$\Sigma_w^{cl}:\begin{cases}\dot x(t)=A_{\theta(t)\hat\theta(t)}x(t)+J_{\theta(t)\hat\theta(t)}w(t)\\ z(t)=C_{\theta(t)\hat\theta(t)}x(t)+L_{\theta(t)\hat\theta(t)}w(t),\end{cases} \tag{2}$$

with $A_{i\ell}\in\mathbb{R}^{n_x\times n_x}$, $J_{i\ell}\in\mathbb{R}^{n_x\times n_w}$, $C_{i\ell}\in\mathbb{R}^{n_z\times n_x}$, $L_{i\ell}\in\mathbb{R}^{n_z\times n_w}$, for $i\in\mathcal{S}$ and $\ell\in\mathcal{D}$.

Assumption 1 *The joint process* $(\theta(t),\hat\theta(t))=\{(\theta(t),\hat\theta(t),\mathscr{F}_t);t\in\mathbb{R}^+\}$ *satisfies a homogeneous Markov property.*

Assumption 2 *We set* $\widehat{\mathscr{F}_t}$ *as the* σ*-field generated by* $\{x(s),\theta(s),w(s),\hat\theta(\tau);0\leq s\leq t, 0\leq\tau<t\}$*, and we assume that*

$$P(\hat\theta(t)=\ell\mid\widehat{\mathscr{F}_t})=\alpha_{\theta(t)\ell},\quad \ell\in\mathcal{D}, \tag{3}$$

holds for some prespecified emission probabilities $\{\alpha_{i\ell}\geq 0;\ i\in\mathcal{S},\ \ell\in\mathcal{D}_i\}$ *satisfying* $\sum_{\ell=1}^M\alpha_{i\ell}=1$ *and* $\sum_{i\in\mathcal{S}}\alpha_{i\ell}>0$*, for each* $i\in\mathcal{S}$ *and* $\ell\in\mathcal{D}$*.*

Therefore $\alpha_{i\ell}$ represents the probability that the detector will emit the signal $\ell\in\mathcal{D}$ given that $\theta(t)=i$. For later use, let us also define the stochastic matrix

$$\Upsilon\triangleq\begin{bmatrix}\alpha_{11} & \dots & \alpha_{1M}\\ \vdots & \ddots & \vdots\\ \alpha_{N1} & \dots & \alpha_{NM}\end{bmatrix}. \tag{4}$$

Notice that $\mathcal{D}_i$ represents the possible signals that can be emitted from the detector whenever $\theta(t)=i$. For each $i\in\mathcal{S}$ we consider integer numbers $\tau^i\geq 1$, $1\leq\ell_j^i\leq M$ and $1\leq j\leq\tau^i$, with $M=\tau^1+\dots+\tau^N$ such that $\mathcal{D}_i\triangleq\{\ell_1^i,\dots,\ell_{\tau^i}^i\}\subset\mathcal{D}$. Thus τ^i represents the number of elements in $\mathcal{D}_i$ and ℓ_j^i, $j=1,\dots,\tau^i$, the elements in $\mathcal{D}_i$.

Definition 1 *System* (2) *is said to be* internally mean-square stable *(MSS) if, for an arbitrary initial state* $x_0\in\mathbb{R}^{n_x}$ *such that* $E[\|x_0\|^2]<\infty$*, we have*

$$\lim_{t\to\infty}E[\|x(t)\|^2]=0, \tag{5}$$

in which $x=\{x(t);t\in\mathbb{R}^+\}$ *stands for the unique solution to* $\dot x(t)=A_{\theta(t)\hat\theta(t)}x(t)$, $x(0)=x_0$*.*

Definition 2 *Assume that system* (2) *is MSS. Given a scalar* $\gamma>0$*, the system* (2) *is said to have an* H_∞ *performance level* γ *if it satisfies*

$$\|\Sigma_w^{cl}\|_\infty\triangleq\sup_{w\in L_2^{n_w}(\Omega,\mathscr{F},P)}\left\{\frac{\|z\|_2}{\|w\|_2};w\neq 0\right\}<\gamma, \tag{6}$$

with $x(0)=0$*.*

Remark 1 *The detector-based formulation covers some of the better studied particular scenarios considered in the MJLS literature. Next, we briefly present how to access these particular settings from the aforementioned detector.*

1. *The scenario with complete observations of the jump process is equivalent to the situation where $\hat{\theta} \equiv \theta$. In our context, it can be determined by considering $M = N$ and $\Upsilon = I_N$.*

2. *On the other hand, it is also possible to represent the set up with the information concerning the jump process totally unknown, i.e., the mode-independent case. Here, it suffices to suppose $M = 1$ and $\alpha_{i1} = 1$ for $i \in \mathcal{S}$.*

3. *The so-called cluster observations (see (do Val et al., 2002)) corresponds to the situation where only part of the jumping process is accessible and further arranged on disjoint sets. Under the referenced formulation, we define $g : \mathcal{S} \to \mathcal{D}$ such that $g(i) = j$ for all $i \in \mathcal{S}_j$. The set $\mathcal{S}_j$ represents the cluster to which i belongs, even if the state i itself is unknown. Then, given the information provided by the detector, it suffices to consider $\mathcal{D}_i = \{g(i)\}$ and $\alpha_{ig(i)} = 1$, with $M \le N$, so that $\hat{\theta}(t)$ would indicate to which cluster $\theta(t)$ belongs.*

Definition 3 *We say that $i, j \in \mathcal{S}$ are* distinguishable *by the detector* (3) *whenever $\mathcal{D}_i \cap \mathcal{D}_j = \emptyset$. Otherwise, $\mathcal{D}_i \cap \mathcal{D}_j \neq \emptyset$ and they are said to be* indistinguishable *by this detector.*

The preceding definition allows a partition of the state space of the Markov jump process $\mathcal{S}$, in accordance to the zero pattern of the matrix Υ defined in (4). More precisely, this can be done through the set-valued map

$$i \mapsto \{j \in \mathcal{S};\quad \mathcal{D}_i \cap \mathcal{D}_j \neq \emptyset\}. \tag{7}$$

In general, some of the sets in the image of (7) will coincide, because $M \le N$ by hypothesis. Letting $N(\Upsilon)$ stand for the number of such distinct sets, we obtain $1 \le N(\Upsilon) \le M \le N$ along with a partition $\mathcal{S}_1, \ldots, \mathcal{S}_{N(\Upsilon)}$ of $\mathcal{S}$:

$$\bigcup_{s=1}^{N(\Upsilon)} \mathcal{S}_s = \mathcal{S}, \text{ with } \mathcal{S}_s \cap \mathcal{S}_\upsilon = \emptyset \quad \text{whenever } s \neq \upsilon.$$

For later use, we define the set $\mathcal{R} \triangleq \{1, \ldots, N(\Upsilon)\}$, along with the function $\psi : \mathcal{S} \to \mathcal{R}$ such that $\psi(i) = \upsilon$ if and only if $i \in \mathcal{S}_\upsilon$.

Remark 2 *The sets of distinguishable operating modes just defined are somewhat related to the clusters introduced in (do Val et al., 2002). The main difference between these two approaches is that, in our case, besides the sets we are endowed with additional parameters (the matrix Υ), which can be taken into account for analysis and synthesis purposes. Informally, the emission probabilities can be seen as relative weights which tell the controller how to proceed in the scenario of partial observations.*

We introduce the following lemma, which will allow us to deal with the H_∞ control problem that will be formulated in Section 2.1. This result is a bounded real lemma (BRL) for continuous-time MJLS with partial information on the Markovian parameters. (For the proof, see (Graciani Rodrigues et al., 2015). See also (Todorov and Fragoso, 2008).)

Lemma 1 *(BRL) Given $\gamma > 0$, system* (2) *is internally MSS with $\|\Sigma_w^{cl}\|_\infty < \gamma$ if and only if there exists $P_i > 0$ in $\mathbb{R}^{n_x \times n_x}$, such that*

$$\begin{bmatrix} \mathcal{T}_i(\mathbf{P}) & * & * \\ \sum_{\ell \in \mathcal{D}_i} \alpha_{i\ell} J'_{i\ell} P_i & -\gamma^2 I_{n_w} & * \\ \bar{C}_i & \bar{L}_i & -I_{n_z \tau^i} \end{bmatrix} < 0, \tag{8}$$

where

$$\mathcal{T}_i(\mathbf{P}) \triangleq \mathrm{Her}\left\{P_i \sum_{\ell \in \mathcal{D}_i} \alpha_{i\ell} A_{i\ell}\right\} + \sum_{j \in \mathcal{S}} \lambda_{ij} P_j,$$

$$\bar{C}_i \triangleq \begin{bmatrix} \sqrt{\alpha_{i\ell_1^i}} C'_{i\ell_1^i} & \cdots & \sqrt{\alpha_{i\ell_{\tau^i}^i}} C'_{i\ell_{\tau^i}^i} \end{bmatrix}',$$

$$\bar{L}_i \triangleq \begin{bmatrix} \sqrt{\alpha_{i\ell_1^i}} L'_{i\ell_1^i} & \cdots & \sqrt{\alpha_{i\ell_{\tau^i}^i}} L'_{i\ell_{\tau^i}^i} \end{bmatrix}'.$$

are satisfied for each $i \in \mathcal{S}$.

In addition to BRL, the following notation is used in the sequel:

$$\widetilde{\mathcal{H}}_i(\mathbf{U}) \triangleq \sum_{\ell \in \mathcal{D}_i} \alpha_{i\ell} U_{\ell\psi(i)},\ \mathcal{H}_i(\mathbf{V}) \triangleq \sum_{\ell \in \mathcal{D}_i} \alpha_{i\ell} V_{i\ell}, \tag{9a}$$

$$\mathbb{D}_i(\mathbf{P}) \triangleq \mathrm{diag}(P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_N), \tag{9b}$$

$$\Pi_i \triangleq \begin{bmatrix} \lambda_{i1}^{1/2} I & \cdots & \lambda_{i(i-1)}^{1/2} I & \lambda_{i(i+1)}^{1/2} I & \cdots & \lambda_{iN}^{1/2} I \end{bmatrix}, \tag{9c}$$

$$\varphi(\mathbf{K}) \triangleq \begin{bmatrix} K'_1 & \cdots & K'_N \end{bmatrix}'. \tag{9d}$$

2.1 H_∞ Control

In this section we discuss the static state feedback scenario with partial information on the Markov chain. Concerning this matter, we deal with the following controlled continuous-time MJLS:

$$\mathcal{G}_u : \begin{cases} \dot{x}(t) = A_{\theta(t)} x(t) + B_{\theta(t)} u(t) + J_{\theta(t)} w(t), \\ z(t) = C_{\theta(t)} x(t) + D_{\theta(t)} u(t) + L_{\theta(t)} w(t), \end{cases} \tag{10}$$

with $A_i \in \mathbb{R}^{n_x \times n_x}$, $B_i \in \mathbb{R}^{n_x \times n_u}$, $J_i \in \mathbb{R}^{n_x \times n_w}$, $C_i \in \mathbb{R}^{n_z \times n_x}$, $D_i \in \mathbb{R}^{n_z \times n_u}$, $L_i \in \mathbb{R}^{n_z \times n_w}$, for $i \in \mathcal{S}$. Here, $u = \{u(t); t \in \mathbb{R}^+\}$ is the control variable in $\mathbb{R}^{n_u}$ and we assume that the state $x(t)$ is completely known while on the other hand there is only partial information regarding $\theta(t)$.

Our aim is to design a *detector-based* state feedback controller of the form

$$u(t) = K_{\hat{\theta}(t)} x(t), \tag{11}$$

with $K_\ell \in \mathbb{R}^{n_u \times n_x}$, for $\ell \in \mathcal{D}$, such that the resulting closed-loop system

$$\mathcal{G}_u^{cl}: \begin{cases} \dot{x}(t) = A_{\theta(t)\hat{\theta}(t)}x(t) + J_{\theta(t)}w(t), \\ z(t) = C_{\theta(t)\hat{\theta}(t)}x(t) + L_{\theta(t)}w(t), \end{cases} \quad (12)$$

where $A_{i\ell} \triangleq A_i + B_i K_\ell$ and $C_{i\ell} \triangleq C_i + D_i K_\ell$, is internally mean-square stable with $\|\mathcal{G}_u^{cl}\|_\infty < \gamma$. Furthermore, we note that system (12) is a particular case of (2) with $J_{i\ell} = J_i$ and $L_{i\ell} = L_i$. The design comes in the form of the next theorem, which is the first main result of the paper.

Theorem 2 *Given $\gamma > 0$, there is a controller such as (11) which internally stabilizes system (12) and guarantees $\|\mathcal{G}_u^{cl}\|_\infty < \gamma$ whenever there are $X_i > 0$ in $\mathbb{R}^{n_x \times n_x}$ for all $i \in \mathcal{S}$, as well as $V_{\ell\upsilon} \in \mathbb{R}^{n_u \times n_x}$, $G_\upsilon \in \mathbb{R}^{n_x \times n_x}$, for $\ell \in \mathcal{D}$, $\upsilon \in \mathcal{R}$, along with scalars $\varepsilon_i > 0$, such that the following coupled matrix inequalities are feasible*

$$\Delta_i(\mathbf{X}) + \mathrm{Her}\left\{ \begin{bmatrix} \mathcal{Z}_i(\mathbf{G},\mathbf{V}) \\ 0 \\ \mathcal{W}_i(\mathbf{G},\mathbf{V}) \\ -G_{\psi(i)} \\ 0 \end{bmatrix} \begin{bmatrix} \varepsilon_i I_{n_x} \\ 0 \\ 0 \\ I_{n_x} \\ 0 \end{bmatrix}' \right\} < 0, \quad (13)$$

where

$$\Delta_i(\mathbf{X}) \triangleq \begin{bmatrix} \lambda_{ii}X_i & * & * & * & * \\ J_i' & -\gamma^2 I_{n_w} & * & * & * \\ 0 & \bar{L}_i & -I_{n_z\tau^i} & * & * \\ X_i & 0 & 0 & 0 & * \\ \Pi_i' X_i & 0 & 0 & 0 & -\mathbb{D}_i(\mathbf{X}) \end{bmatrix}, \quad (14)$$

$$\mathcal{Z}_i(\mathbf{G},\mathbf{V}) \triangleq A_i G_{\psi(i)} + B_i \widetilde{\mathcal{H}}_i(\mathbf{V}), \quad (15)$$

$$\mathcal{W}_i(\mathbf{G},\mathbf{V}) \triangleq \begin{bmatrix} \sqrt{\alpha_{i\ell_1^i}}(C_i G_{\psi(i)} + D_i V_{\ell_1^i \psi(i)}) \\ \vdots \\ \sqrt{\alpha_{i\ell_{\tau^i}^i}}(C_i G_{\psi(i)} + D_i V_{\ell_{\tau^i}^i \psi(i)}) \end{bmatrix}, \quad (16)$$

for each $i \in \mathcal{S}$. Furthermore, the gains of such a controller are given by

$$K_\ell = V_{\ell\upsilon} G_\upsilon^{-1}, \quad (17)$$

for each $\upsilon \in \mathcal{R}$ and $\ell \in \bigcup_{i \in \mathcal{S}_\upsilon} \mathcal{D}_i$.

Proof: Since $V_{\ell\psi(i)} = K_\ell G_{\psi(i)}$ and $\widetilde{\mathcal{H}}_i(\mathbf{V}) = \sum_{\ell \in \mathcal{D}_i} \alpha_{i\ell} K_\ell G_{\psi(i)}$ follows from (17), we promptly obtain $A_i G_{\psi(i)} + B_i \widetilde{\mathcal{H}}_i(\mathbf{V}) = \mathcal{H}_i(\mathbf{A}) G_{\psi(i)}$ and $\mathcal{W}_i(\mathbf{G},\mathbf{V}) = \bar{C}_i G_{\psi(i)}$. Therefore, LMI (13) can be written as

$$\Delta_i(\mathbf{X}) + \mathrm{Her}\left\{ \begin{bmatrix} \mathcal{H}_i(\mathbf{A}) \\ 0 \\ \bar{C}_i \\ -I_{n_x} \\ 0 \end{bmatrix} G_{\psi(i)} \begin{bmatrix} \varepsilon_i I_{n_x} \\ 0 \\ 0 \\ I_{n_x} \\ 0 \end{bmatrix}' \right\} < 0.$$

Notice that[1] $\ker \begin{bmatrix} \mathcal{H}_i(\mathbf{A})' & 0 & \bar{C}_i' & -I & 0 \end{bmatrix} = \mathrm{im}(W_i)$, where

$$W_i \triangleq \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ \mathcal{H}_i(\mathbf{A})' & 0 & \bar{C}_i' & 0 \\ 0 & 0 & 0 & I \end{bmatrix}.$$

[1] Here ker and im denote the kernel and image of matrices, respectively.

Hence, bearing in mind that the columns of W_i are linearly independent, from Finsler's lemma (Boyd et al., 1994), the existence of such $G_{\psi(i)}$ yields the following:

$$0 > W_i' \Delta_i(\mathbf{X}) W_i = \begin{bmatrix} \widetilde{\mathcal{T}}_i(\mathbf{X}) & J_i & X_i \bar{C}_i' & X_i \Pi_i \\ J_i' & -\gamma^2 I_{n_w} & \bar{L}_i' & 0 \\ \bar{C}_i X_i & \bar{L}_i & -I_{n_z\tau^i} & 0 \\ \Pi_i' X_i & 0 & 0 & -\mathbb{D}_i(\mathbf{X}) \end{bmatrix},$$

where $\widetilde{\mathcal{T}}_i(\mathbf{X}) \triangleq \mathrm{Her}(\mathcal{H}_i(\mathbf{A})X_i) + \lambda_{ii}X_i$. Using then Schur's complement, followed by the congruence transformation $\mathrm{diag}(X_i^{-1}, I, I)(\cdot)\mathrm{diag}(X_i^{-1}, I, I)$, we get (8) with $P_i \equiv X_i^{-1}$ and data as in (12). Therefore, the result follows from Lemma 1. □

Remark 3 *In the same vein of the partition $\mathcal{S}_1, \ldots, \mathcal{S}_{N(\Upsilon)}$ for $\mathcal{S}$, which is a consequence of Definition 3, by letting $\mathscr{D}_\upsilon \triangleq \bigcup_{i \in \mathcal{S}_\upsilon} \{\ell \in \mathcal{D}; \alpha_{i\ell} > 0\}$, for $\upsilon \in \mathcal{R}$, we have that $\cup_{s=1}^{N(\Upsilon)} \mathscr{D}_s = \mathcal{D}$, with $\mathscr{D}_s \cap \mathscr{D}_\upsilon = \emptyset$ whenever $s \neq \upsilon$, which guarantees the consistency of the bivariate design formula (17).*

Algorithm 3 *Given a desired precision $\varpi > 0$ and scalars $\varepsilon_i^{(0)} > 0$, $i \in \mathcal{S}$, let $t = 0$ and consider the following steps.*

1. *Letting $\varepsilon_i \equiv \varepsilon_i^{(t)}$, minimize $\gamma > 0$ subject to the LMIs (13), and let $\bar{\gamma}^{(t)} \leftarrow \gamma$, $G_\upsilon^{(t)} \leftarrow G_\upsilon$, $V_{\ell\upsilon}^{(t)} \leftarrow V_{\ell\upsilon}$ store the obtained numerical values for all $\upsilon \in \mathcal{R}$ and $\ell \in \cup_{i \in \mathcal{S}_\upsilon} \mathcal{D}_i$.*

2. *Letting $G_\upsilon \equiv G_\upsilon^{(t)}$ and $V_{\ell\upsilon} \equiv V_{\ell\upsilon}^{(t)}$, minimize $\gamma > 0$ subject to the LMIs (13) (with decision variables X_i and ε_i, for $i \in \mathcal{S}$). Let $\gamma^{(t)} \leftarrow \gamma$, $X_i^{(t)} \leftarrow X_i$, $\varepsilon_i^{(t)} \leftarrow \varepsilon_i$ store the obtained numerical values for all $i \in \mathcal{S}$.*

3. *If $\bar{\gamma}^{(t)} - \gamma^{(t)} < \varpi$, proceed to the next step. Otherwise, increment $t \leftarrow t + 1$ and go back to step 1.*

4. *Return the controller (17) with $V_{\ell\upsilon} \equiv V_{\ell\upsilon}^{(t)}$, $G_\upsilon \equiv G_\upsilon^{(t)}$, along with the guaranteed H_∞ cost $\gamma_* = \gamma^{(t)}$.*

Upon convergence, the algorithm returns a mean square stabilizing controller which guarantees a terminal H_∞ cost γ_* that is at least as good as the one obtained by solving the LMIs (13) with $\varepsilon_i \equiv \varepsilon_i^{(0)}$ (i.e., the first run of step 1).

Remark 4 *The preceding algorithm yields a decreasing sequence of guaranteed H_∞ costs of the form $\bar{\gamma}^{(0)} > \gamma^{(0)} \geq \bar{\gamma}^{(1)} > \gamma^{(1)} \geq \cdots \geq \bar{\gamma}^{(t_*)} \geq \gamma_*$, where t_* denotes the number of necessary iterates for a particular set of parameters. Bearing in mind the convergence criterion in step 3, it is not difficult to see that $(t_* + 1)\varpi \leq \bar{\gamma}_0$, so that the maximum number of iterates cannot be larger than $\bar{\gamma}^{(0)}/\varpi$.*

Corollary 4 *Given $\gamma > 0$, Υ invertible and $D_i \equiv 0$ for all $i \in \mathcal{S}$ in (10), there exists a controller such as (11) which internally stabilizes system (12), with $C_{i\ell} \equiv C_i$, and guarantees $\|\mathcal{G}_u^{cl}\|_\infty < \gamma$ whenever there are $X_i > 0$ in $\mathbb{R}^{n_x \times n_x}$ and $Q_i \in \mathbb{R}^{n_u \times n_x}$, for all $i \in \mathcal{S}$, as well as $G_\upsilon \in \mathbb{R}^{n_x \times n_x}$, $\upsilon \in \mathcal{R}$, along with scalars $\varepsilon_i > 0$, such that the LMIs (13) are feasible with $\Delta_i(\mathbf{X})$ as in (14) and $\mathcal{Z}_i(\mathbf{G}, \mathbf{Q}) = A_i G_{\psi(i)} + B_i Q_i$, instead of (15), for each $i \in \mathcal{S}$. In addition, a controller $\mathbf{K} = (K_1, \ldots, K_N)$ is given by*

$$\varphi(\mathbf{K}) = (\Upsilon \otimes I_{n_u})^{-1} \varphi(\mathbf{Q}\mathbf{G}^{-1}), \qquad (18)$$

where $\otimes$ denotes the Kronecker product and $\varphi(\cdot)$ as in (9d).

Proof: Bearing in mind Theorem 2, assume $\widetilde{\mathcal{H}}_i(\mathbf{V}) \equiv Q_i$ in (15). Then, since Υ is invertible and bearing in mind (18), the result follows. $\square$

3 Numerical Examples

In this Section we will present a numerical evaluation of the result presented in the previous section. The numerical simulations were obtained using the solver SeDuMi in Matlab 7.8.0.347 (R2009a). The examples represent systems governed by a Markov chain with $\mathcal{S} = \{1, 2, 3\}$ and emission probability matrix given by

$$\Upsilon = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho & 1-\rho \\ 0 & 1-\rho & \rho \end{bmatrix}, \qquad (19)$$

where $\rho \in [0, 1]$, is a prespecified parameter that determines the quality of the estimator. In this situation, $\rho = 1$ corresponds to the case with perfect information of the Markov jump process, whereas for $\rho = 0.5$ we recover the case of cluster observations in the sets $\{1\}$ and $\{2, 3\}$.

3.1 Aerial Vehicle

Consider system (10) as a linearized model of the lateral dynamics of a small unmanned aerial vehicle in steady flight (see (Ducard, 2009, Appendix E)). We assume that failures affect the dynamics, then let $J_1 \equiv 0_{4\times 2}$, $L_i = -I_2$, $D_i = I_2$, along with

$$A_i = \begin{bmatrix} -11.4540 & 2.7185 & -19.4399 & 0 \\ 0.5068 & -2.9875 & 23.3434 & 0 \\ 0.0922 & -0.9957 & -0.4680 & 0.3256 \\ 1 & 0.0926 & 0 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} b_i^{11} & b_i^{12} \\ b_i^{21} & b_i^{22} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2 = J_3 = \begin{bmatrix} 0 & 0 \\ b_i^{21} & b_i^{22} \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$B_3 = J_2 = \begin{bmatrix} b_i^{11} & b_i^{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad C_i \equiv \begin{bmatrix} I_2 & 0_{2\times 2} \end{bmatrix},$$

where $b_i^{11} = 78.4002$, $b_i^{12} = -2.7282$, $b_i^{21} = -3.4690$, $b_i^{22} = 13.9685$, for $i \in \mathcal{S}$. In this case, for the transition rate matrix consider $\Lambda = \frac{1}{10}\begin{bmatrix} -1 & 1 & 0 \\ 1 & -10 & 9 \\ 0 & 10 & -10 \end{bmatrix}$. Thus, after solving the LMI (13), with $\varepsilon_i \equiv 10$, we obtained for each $\rho \in [0, 1]$ the guaranteed H_∞ costs depicted in Fig. 1. The figure shows that the guaranteed H_∞ cost in the case of cluster observations ($\rho = 0.5$) is close (about 77.28%, actually) to the worst case scenario of no information on the Markov jump process (traced line), which coincides with (Morais et al., 2015, Theorem 2). Further, for $\rho \neq 0.5$ we obtain a whole range of possible scenarios, including the case of complete observation of the jump process.

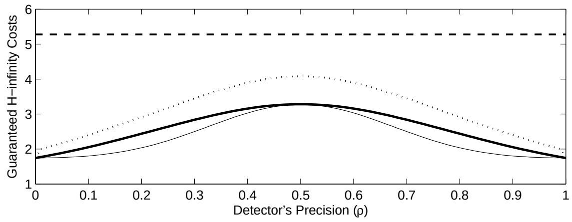

Figure 1: Comparison between the guaranteed H_∞ costs computed via Theorem 2 (dotted line) and Algorithm 3 (thick solid line), along with the actual closed-loop H_∞ norms (thin solid line), vis-à-vis the mode-independent (traced line).

As for the performance of Algorithm 3, the reduction of guaranteed H_∞ costs vis-à-vis Theorem 2 ranged from 0% (for $\rho \in \{0, 1\}$) to about 24% (for ρ close to 0.5). In all cases, the algorithm converged in 16 iterations or less for the adopted precision ($\varpi = 10^{-3}$).

3.2 Power System

In order to illustrate the result from Corollary 4, we consider a single-machine-infinite-bus power system with parameters borrowed from (Arrifano and Oliveira, 2004). Thus, for $i \in \mathcal{S}$, consider $A_i = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -0.6250 & -39.2699 \\ 0 & a_i & -0.4127 \end{bmatrix}$, $B_i = \begin{bmatrix} 0 \\ 0 \\ b_i \end{bmatrix}$, where $a_1 = 1.8792$, $a_2 = 1.5530$, $a_3 = 1.2580$, $b_1 = 0.1190$, $b_2 = 0.1082$, $b_3 = 0.0974$, and $L_i \equiv \begin{bmatrix} 1 & 1 & l_i \end{bmatrix}'$, with $l_1 = 0.4860$, $l_2 = 0.8913$, $l_3 = -0.7621$. Also, we assume $J_i \equiv B_i$, $C_1 = C_2 \equiv 0_{3\times 3}$, $C_3 \equiv I_3$, along with $\varepsilon_i \equiv 1$. We also consider (19) with $\rho = 0.55$ and $\Lambda = \frac{1}{100}\begin{bmatrix} -10 & 10 & 0 \\ 6 & -10 & 4 \\ 0 & 10 & -10 \end{bmatrix}$. In this case, we obtained the controller $K_1 = 10^3 \times \begin{bmatrix} 1.3715 & 1.3895 & -5.2356 \end{bmatrix}$, $K_2 = 10^6 \times \begin{bmatrix} -1.3250 & -1.4728 & 1.4167 \end{bmatrix}$, $K_3 = 10^6 \times \begin{bmatrix} 1.8631 & 2.0710 & -1.9935 \end{bmatrix}$, with guaranteed H_∞ cost $\gamma = 1.6717$.

4 Concluding remarks

In this paper we have presented a new approach to H_∞ control for continuous-time Markov jump linear systems (MJLS), in a scenario of partial observations of the Markov jump process (θ). The distinguishing feature of this approach is that, inspired by (Costa et al., 2015; Graciani Rodrigues et al., 2015), we have assumed that the partial observations are provided by a suitable *detector* ($\hat{\theta}$) of the Markov chain. The main results (Theorem 2 and Algorithm 3) feature an efficient method, based on linear matrix inequalities, for designing detector-based H_∞ controllers for continuous-time MJLS. Numerical simulations show that, given a suitable detector along with the parameters in (4), the proposed design can yield much less conservative results than mode-independent approach.

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