RECURSIVE LINEAR QUADRATIC REGULATOR SUBJECT TO MARKOVIAN JUMP LINEAR SYSTEMS IN A ROBOTIC APPLICATION SYSTEM FOR REHABILITATION

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Abstract— In this paper, a recursive regulator for linear discrete-time systems, which are subject to Markovian jumps is studied under consideration of a rehabilitation application with an exoskeleton. Using a penalty function and robust weighted least-squares methods, tuning parameters can be dispensed and permits the application in online processes. This kind of regulator is a novelty in rehabilitation robotics and will ensure new options and variety of control strategies in this are of control problem. The performance is compared with an established method based on offline calculation by solving linear matrix inequalities for H_∞ controllers.

Keywords— Markovian jump linear system, linear systems, transition probabilities, optimal control

1 INTRODUCTION

This paper deals with the quadratic optimal control problem for finite horizon, discrete time Markov Jump Linear Systems (MJLS), when the state variable x_k and the jump parameter, θ_k are observed at all times. The law of optimal control is determined by minimizing the expected value of a quadratic cost function restricted to the dynamic model in a finite horizon. The widely known solution in the literature is recursively given by a set of coupled Riccati equations, and is called Linear Quadratic Regulator (LQR). The proposed wording for the optimal control problem in question is slightly different, but equivalent, see (Cerri et al., 2010), (Cerri, 2013) and (Terra et al., 2014).

The minimization is performed not only in terms of the input variable u_k, but also in function with x_{k+1}. An alternative procedure is adopted to solve the minimization problem with constraint, which is based on combination of optimal solution of weighted least squares problems and the technical penalty functions. Under this solution approach, the system closed-loop, the optimal control law, the feedback gain matrix and the coupled Riccati equations are presented in an alternative structure of matrix blocks in a symmetrical arrangement. The obtained recursive solution is equivalent to the classical solution and justifies the use of this technique to the problem of question.

Similar approaches were plenty discussed in literature as there are (Matei et al., 2008), (do Val et al., 1999) and (Costa and Val, 2002). However we want to introduce this specific kind of controller due to its effectiveness and innovation, especially in the area of robotic rehabilitation systems which demand a fast and precise adaptation during every phase of the treatment.

2 Recursive Linear Quadratic Regulator for Markovian Jump Linear Systems

We consider a Markovian Jump Linear System of the form

$$x_{k+1} = F_{\theta_k,k}x_k + G_{\theta_k,k}u_k \text{ , } k = 0,\dots,N-1 \qquad (1)$$

with $x_k \in \mathbb{R}^n$ as the state vector, $u_k \in \mathbb{R}^m$ as the vector of the control input and $F_{\theta_k,k} \in \mathbb{R}^{n\times n}$ and $G_{\theta_k,k} \in \mathbb{R}^{n\times m}$ are parameter matrices, in which θ_k represents the jump parameter admitting values for every instance of k in the finite set $\Theta := \{1,\dots,s\}$. The process $\{\theta_k\}_{k=0}^{N-1}$ is modeled as a discrete time Markov process with finite states.

The transition probability matrix (TPM) is defined by $\mathbb{P} = [p_{ij}] \in \mathbb{R}^{s\times s}$ and its entries satisfy

$$Prob[\Theta_{k+1} = j|\Theta_k = i] = p_{ij}, \;\; Prob[\Theta_0 = i] = p_i,$$
$$\sum_{j=1}^{s} p_{ij} = 1, 0 \le p_{ij} \le 1. \qquad (2)$$

Further, the initial conditions x_0 and θ_0 are known. Assuming that the jump parameter θ_k and the state x_k are available for every instance k, we can define for every $k = 0,\dots,N-1$ the auxiliary quadratic term

$$\mathfrak{L}_{\theta_k}(x_k,u_k) := \begin{bmatrix} x_k \\ u_k \end{bmatrix}^T \begin{bmatrix} Q_{\theta_k,k} & 0 \\ 0 & R_{\theta_k,k} \end{bmatrix} \begin{bmatrix} x_k & u_k \end{bmatrix}, \quad (3)$$

with $Q_{\theta_k,k} \succ 0$ and $R_{\theta_k,k} \succ 0$ are weighting matrices of compatible dimensions. The quadratic cost function, bonded by the initial conditions ($\theta_0 = i, x_0$), is defined by

$$\mathscr{J}(\theta,x,u) := \sum_{k=0}^{N-1} \mathfrak{L}_{\theta_k}(x_k,u_k) + x_N^T P_{\theta_N,N} x_N, \qquad (4)$$

and being $P_{\theta_N,N} \succ 0$.

By the classic formulation of the optimal quadratic problem, the objective is to find a optimal control sequence $\mathscr{U}^* = \{u_0^*, \ldots, u_{N-1}^*\}$ which minimizes the expected value of the quadratic cost function (4) subject to the restrictions of (1), *i.e.*

$$\min_{u_k, k=0,\ldots,N-1} \mathbb{E}\{\mathscr{J}(\theta, x, u)|\mathscr{O}_0\}, \tag{5}$$

subject to $x_{k+1} = F_{\theta_k,k}x_k + G_{\theta_k,k}u_k, k = 0, \ldots, N-1$ such that $\mathscr{O}_0 = \{(\theta_0 = i, x_0)\}$. The solution of the problem (5) is recursive and consists of the linear quadratic regulator, whose equations are given by:

$$u_k^* = K_{\theta_k,k}x_k^*, k = 0, \ldots, N-1$$

with $\theta_k \in \Theta$, and

$$K_{i,k} = -(R_{i,k} + G_{i,k}^T \Psi_{i,k+1} G_{i,k})^{-1} G_{i,k}^T \Psi_{i,k+1} F_{i,k},$$

and $P_{i,k}$ for every $i \in \Theta$ calculated back/forward iteratively in time by the coupled Riccati equations

$$\begin{aligned} \Psi_{i,k+1} &:= \sum_{j=1}^{s} P_{j,k+1} p_{ij}, \\ P_{i,k} &= F_{i,k}^T \big(\Psi_{i,k+1} - \Psi_{i,k+1} G_{i,k} (R_{i,k} \\ &\quad + G_{i,k}^T \Psi_{i,k+1} G_{i,k})^{-1} G_{i,k}^T \Psi_{i,k+1}\big) F_{i,k} + Q_{i,k}, \end{aligned}$$

with the initial condition $P_{i,N} \succeq 0$. For the problem (5), the minimization is realized just by the control input u_k. The optimal process $\{x_{k+1}^*\}_{k=0}^{N-1}$ is determined by (1) due to the knowledge of the optimal control sequence $\{u_k^*\}_{k=0}^{N-1}$.

Equivalently, it is possible to consider simultaneously x_{k+1} and u_k as variables of the problem restricted minimization. Thus, each minimization step k, the control action u_k^* and the optimal state x_{k+1}^* are given depending on the state x_k, which is available for hypothesis. Then consider the quadratic optimal control problem rephrased as follows Determine a optimal sequence $\{(x_{k+1}^*, u_k^*)\}_{k=0}^{N-1}$ considering

$$\min_{x_{k+1}, u_k, k=0,\ldots,N-1} \mathbb{E}\{\mathscr{J}(\theta, x, u)|\mathscr{O}_0\}, \tag{6}$$

subject to $x_{k+1} = F_{\theta_k,k}x_k + G_{\theta_k,k}u_k, k = 0, \ldots, N-1$. With the aid of the classical technique of dynamic programming, the optimal solution of the problem (6) is described in (Cerri et al., 2010), (Cerri, 2013) and (Terra et al., 2014). A quadratic programming problem in the mold of a least squares problem subject to a constraint of equality is obtained for each step. In this paper we use only the solution in the form a the following algorithm. Considering the problem (6), an optimal solution $\{(x_{k+1}^*, u_k^*)\}_{k=0}^{N-1}$ is given by

$$\begin{bmatrix} x_{k+1}^* \\ u_k^* \\ \mathscr{V}_k(x_k^*, \theta_k) \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & x_k^{*T} \end{bmatrix} \begin{bmatrix} L_{\theta_k,k} \\ K_{\theta_k,k} \\ P_{\theta_k,k} \end{bmatrix} x_k^*, k = 0, \ldots, N-1 \tag{7}$$

with $\theta_k \in \Theta$ and $L_{i,k}$,$K_{i,k}$ and $P_{i,k}$ are given by

$$\Psi_{i,k+1} = \left(\sum_{j=1}^{s} P_{j,k+1} p_{ij} \right), \tag{8}$$

$$\begin{bmatrix} L_{\theta_k,k} \\ K_{\theta_k,k} \\ P_{\theta_k,k} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -I \\ 0 & 0 & F_{i,k} \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}^T \cdot$$

$$\cdot \begin{bmatrix} \Psi_{i,k+1}^{-1} & 0 & 0 & 0 & I & 0 \\ 0 & R_{i,k}^{-1} & 0 & 0 & 0 & I \\ 0 & 0 & Q_{i,k}^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & -G_{i,k} \\ I & 0 & 0 & I & 0 & 0 \\ 0 & I & 0 & -G_{i,k}^T & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ -I \\ F_{i,k} \\ 0 \\ 0 \end{bmatrix} \tag{9}$$

for all $k = N-1, \ldots, 0$. The total optimal cost is $\mathscr{J}^*(\theta_0, x_0, u^*) = x_0^T P_{\theta_0,0} x_0$.

2.1 *Determining the Transition Probability Matrix*

2.1.1 Introductory example

We consider a plant with two states $\Theta = \{1, 2\}$, and performs jumps during $t = (1, \ldots, 10)$. For each instance of time, the plant performs a jump from the actual state into the other or stays in its actual state. Also we consider that, if the plant reaches the ultimate time instance, the plant jumps back into its first state. Thus we can map a Markovian chain as shown in Figure (1). Now we can verify by counting, how often the system

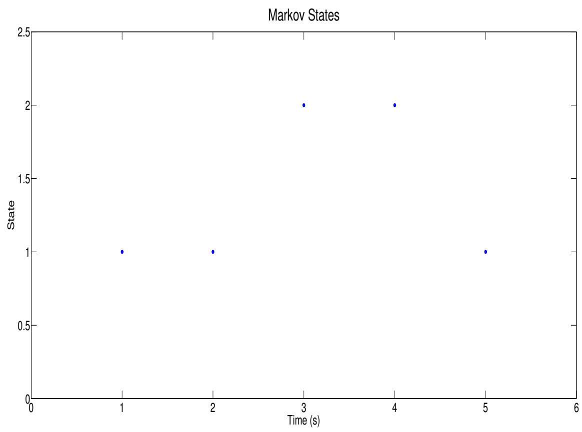

Figure 1: Example for a Markovian chain of a simple plant with tho possible Markovian states.

jumps from its state θ_1 into the same state θ_1 and define this event as p_{11} and vice versa as p_{22} or how often it jumps from θ_1 into θ_2 as p_{12} and vice versa as p_{21} and finally divide the actual jumps by the total appearances of each individual state. Therefore we find under consideration that the systems jumps back into its initial state after reaching the last time instance

$$p_{11} = \frac{3}{4} = 0.75 \qquad p_{12} = \frac{1}{4} = 0.25,$$
$$p_{21} = \frac{1}{2} = 0.5 \qquad p_{22} = \frac{1}{2} = 0.5,$$

and thus obtain the TPM $\mathbb{P}$ as follows

$$\mathbb{P} = \begin{bmatrix} 0.75 & 0.25 \\ 0.5 & 0.5 \end{bmatrix}. \qquad (10)$$

2.2 TPM Derivation

Applying the above described method under consideration of a more general case in the space of the simulation, different scenarios for an application can be created for which we can derive a specific set of gains calculated with the corresponding TPM. First we define each joints range as follows:

- The trunk sector Q_n as $\sum_{i=1}^{n} \frac{\min_Q \leq Q_{(i)} \leq \max_Q}{p}$,
- The femur sector F_n as $\sum_{i=1}^{n} \frac{\min_F \leq F_{(i)} \leq \max_F}{p}$,
- The tibia sector T_n as $\sum_{i=1}^{n} \frac{\min_T \leq T_{(i)} \leq \max_T}{p}$.

For more details see (Mitschka et al., 2015) and the references therein.
Further we have the chain of Markovian states Γ, the matrix of appearances of each Markovian state $M^{p \times p}$, and the number of operational points p which are the points where the system is linearized in but are non equilibrium points of the type $q_0 = 0$. And define

$$\Gamma_{i|N} = \sum_{i=1}^{N} \{(Q_i - 1) + (F_i - 1) \cdot p + (T_i - 1) \cdot p^2\}, \quad (11)$$

which describes the appearances of each combination of linearization points and their transitions over the whole trajectory.

$$M_{\Gamma_i|\Gamma_{i+1}} = \sum_{i=1}^{N} \left[\left((M_{\Gamma_i}, M_{\Gamma_{i+1}} \right) + 1 \right], \qquad (12)$$

describes the matrix of transitions of the Markovian states from one state to the next one based on the chain Γ.

$$\mathbb{P}_{(i,j)} = \sum_{j=1}^{p^2} \left(\frac{M_{(i,j)}}{\sum_{i=1}^{N} M_{\Gamma_i|\Gamma_{i+1}}} \right), \qquad (13)$$

yields a TPM for a possible scenario.

3 Data measurement

3.1 Sensor Setup

To calculate the transition probabilities, a series of measurements with the MTw Development Kit from XSens was performed. The kit contains wireless motion trackers i.e. inertial measurement units (IMU's) which we use to record an accurate positioning of the joints. By gathering a high enough set of data, we then can apply our algorithm and thus identify possible Markovian states during locomotion. Such an IMU is shown in Figure (2). Data transfer is realized through the $Awinda^{TM}$ radio protocol. The Awinda protocol is based on the IEEE 802.15.4 PHY. Using this basis ensures that standard 2.4 GHz ISM chip-sets can be used. The Awinda protocol ensures time synchronization of up to 32 $MTw's$ across the wireless network to within 10 μs. XSens also provides a software development kit (SDK) which opens up the possibility for implementation of own filters or integration into other hardware applications.

The IMU's can be fixed on a persons joints, hands or head with a set of strips, each specific for the desired location where data needs to be collected as is shown in Figure (3).

Figure 2: MTw inertial motion tracker which was used to gather sufficient statistical data of a humans gait.

Figure 3: Sensors mounted on a human leg in the recording phase of data gathering.

4 Simulation results

For the simulations we used the same platform as introduced in (Mitschka et al., 2015) and the references therein. In those previous works, the performance of a Markovian controller which is solved by LMI techniques, was compared with other types of controllers and different treatment scenarios were created and thus we were able to find an optimal solution for this kind of solving method, which are now used to be compared with the rLQR subject to MJLS. Further, conditions were created which could demonstrate the characteristics of each control approach, for that two kind of disturbances during the trajectory were defined in an interval of iterations $\triangle i$. At first a step from $5 < i < 25$, and then a lowering from $100 < i < 120$ and else the simulations were ran under similar conditions.

The desired trajectory used for the simulation was generated based on the model introduced by Eric Westervelt and Jessy Grizzle in (Westervelt and Grizzle, n.d.), based on (Spong and Vidyasagar, 2008) (obs.: 1989 version was the outgoing source not the new edition which is subject of the citation), a simulation was created which allows to implement any kind of control. The simulation results are shown

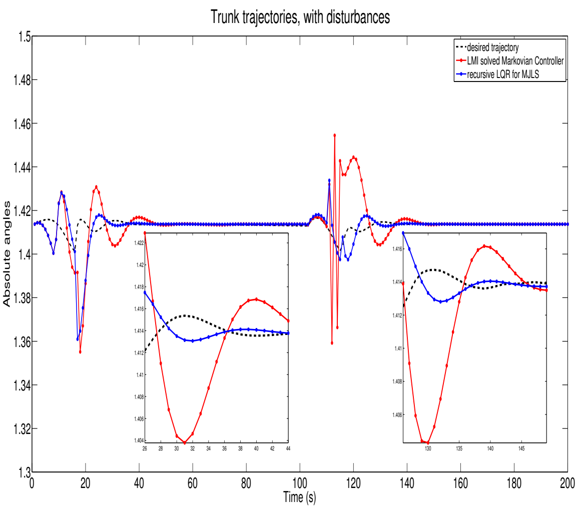

Figure 4: Performance of both controller types at the trunk; Black is the desired trajectory, red represents the LMI solved controller and blue the rLQR for MJLS. During the first iterations both controller have to adapt to the disturbances, but we can clearly observe the faster convergence of the rLQR for MJLS.

in the Figures (4), (5) and (6). The Markovian states in which the controllers enter are shown in Figure (7). The common segments of the trajectory consist of the same joint positions and thus each set of gains includes the same sector configurations.

5 Conclusion and future works

This paper compares two control strategies subjected to Markovian jumps, first the Markovian controller introduced in (Costa et al., 2005) by solving the H_∞ problem using LMI techniques, and second the in (Cerri et al., 2010) developed rLQR subject to MJLS for on-line applications. Both methods demand the determination of a TPM, such that the transition probabilities are well known and can be identified at any instance k of the process.

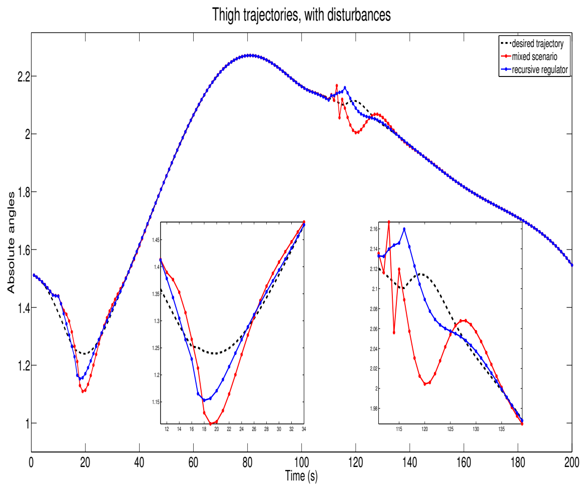

Figure 5: Performance of both controller types at the thigh; Black is the desired trajectory, red represents the LMI solved controller and blue the rLQR for MJLS. During the first iterations both controller have to adapt to the disturbances, but we can clearly observe the faster convergence of the rLQR for MJLS. The advantage of the rLQR for MJLS crystallizes more significantly at the thigh.

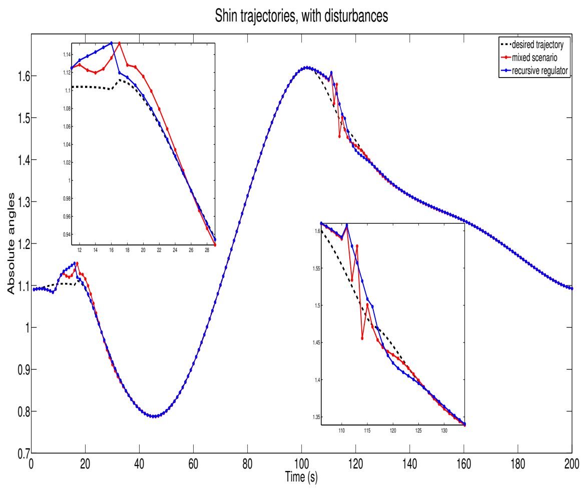

Figure 6: Performance of both controller types at the shin; Black is the desired trajectory, red represents the LMI solved controller and blue the rLQR for MJLS. During the first iterations both controller have to adapt to the disturbances, but we can clearly observe the faster convergence of the rLQR for MJLS.

During the implementation steps for both controllers, the effort for the rLQR was less work expensive simply due to the off-line computation cost which were to overcome with the the LMI technique to calculate the controllers gains. The on-line calculated gains for the rLQR and the resulting performance are more promising, especially in rehabilitation applications, due to its capacity to adapt to any kind of unexpected disturbances without being restricted by previ-

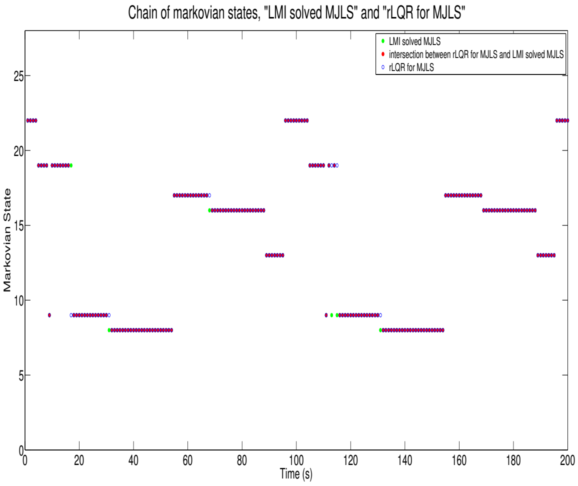

Figure 7: Markovian states for the plane scenario compared with the mixed scenario.

ously determined solutions of the gains.

As for the future works, the rLQR subject to Markovian jumps can be modeled with either uncertainties in the system matrices or the TPM.

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References

Cerri, J. P. (2013). *Control and Filtering for uncertain discrete linear systems subject to Markovian jumps*, PhD thesis, Eletrical Engineering Department - EESC.

Cerri, J. P., Terra, M. H. and Ishihara, J. Y. (2010). Recursive robust regulator for discrete-time Markovian jump linear systems via penalty game approach, *49th IEEE Conference on Decision and Control (CDC)* (2): 597–602.

Costa, E. F. and Val, J. B. R. D. (2002). Weak detectability and the linear-quadratic control problem of discrete-time Markov jump linear systems, *International Journal of Control* **75**(16-17): 1282–1292.

Costa, O., Fragoso, M. and Marques, R. (2005). *Probability and its Applications*, Springer-Verlag London.

do Val, J. B. R., Geromel, J. C. and Costa, O. L. V. (1999). Solutions for the linear quadratic control problem of Markov jump linear systems, *American Control Conference, 1999. Proceedings of the 1999*, Vol. 4, pp. 2793–2797 vol.4.

Matei, I., Martins, N. C. and Baras, J. S. (2008). Optimal Linear Quadratic Regulator for Markovian Jump Linear Systems, in the Presence of One Time-St, *IFAC World Congress*, Vol. 17, pp. 8056–8061.

Mitschka, C. M., Terra, M. H. and Siqueira, A. A. G. (2015). Derivation of a Markovian Controller for an exoskeleton by overcome the benchmarks of a single and double inverted pendulum, *2015 54th IEEE Conference on Decision and Control (CDC)*, pp. 5061–5066.

Spong, M. W. and Vidyasagar, M. (2008). *Robot dynamics and control*, John Wiley & Sons.

Terra, M. H., Cerri, J. P. and Ishihara, J. Y. (2014). Optimal Robust Linear Quadratic Regulator for Systems Subject to Uncertainties, *IEEE Transactions on Automatic Control* **59**(9): 2586–2591.

Westervelt, E. and Grizzle, J. (n.d.). Design of asymptotically stable walking for a 5-link planar biped walker via optimization, *Proceedings 2002 IEEE International Conference on Robotics and Automation (Cat. No.02CH37292)* **3**: 3117–3122.