# EXTREMUM SEEKING CONTROL FOR A CLASS OF NON HAMMERSTEIN-WIENNER NONLINEAR PLANTS

ALESSANDRO JACOUD PEIXOTO*, GUIDO JOSÉ VALENZANO TOCAIMASA†, DIEGO PEREIRA-DIAS‡, ALESSANDRO ROSA LOPES ZACHI†

*Universidade Federal do Rio de Janeiro (UFRJ)
Departamento de Engenharia Eletrônica e de Computação (DEL/Poli/UFRJ)
Rio de Janeiro, Rio de Janeiro, Brasil

†Centro Federal de Educação Tecnológica Celso Suckow da Fonseca (CEFET/RJ)
Programa de Pós-graduação em Engenharia Elétrica (PPEEL)
Rio de Janeiro, Rio de Janeiro, Brasil

‡Universidade Federal do Rio de Janeiro (UFRJ)
Laboratório de Ensaios Não Destrutivos, Corrosão e Soldagem (LNDC)
Rio de Janeiro, Rio de Janeiro, Brasil

Emails: jacoud@poli.ufrj.br, guidovalenzano@ieee.org, diego.dias@lndc.com.br, alessandro.zachi@cefet-rj.br

**Abstract—** This work addresses an extension to the classical design of a perturbation-based extremum seeking control (ESC) method applied to a class of non Hammerstein-Wienner (HW) plants. The original ESC scheme, when applied to a non HW system, requires a very low perturbation frequency. This results in excessively larger settling times, rendering the ESC method impractical for certain applications. This work addresses this problem by presenting two approaches for ESC applications: a gradient-reconstruction ESC (GR-ESC) and a pre-compensated ESC (PC-ESC). The first technique is used when some parameters of the plant are known *a priori* or can be estimated by probing the system, while the second one allows to deal with uncertain higher order plants. The proposed ESC techniques were validated considering a class of plants that captures the essential dynamic of the Eikrem's model for gas lifted oil wells. The ESC techniques applied in this context, indicate that it is possible to maintain the oil production around the optimum point of the well-performance curve (WPC) in gas lifted oil wells without reducing the frequency of the perturbation to forbidden practical values. The closed loop control performance is evaluated via numerical simulations.

**Keywords—** Extremum seeking control, non-Hammerstein-Wiener models, oil production, gas lifted wells.

**Resumo—** Este trabalho apresenta uma extensão do projeto clássico de controle baseado no método de busca extremal (ESC) a uma classe de plantas do tipo não Hammerstein-Wiener (HW). A formulação original do ESC quando aplicado a um sistema não HW requer uma forte redução na frequência de perturbação na qual pode tornar o ESC impraticável. Duas abordagens para aplicação do ESC são apresentadas: uma baseada diretamente na reconstrução do gradiente (GR-ESC) e uma que utiliza uma pré-compensação para reduzir o sistema para o tipo HW (PC-ESC). A primeira técnica é usada quando alguns parâmetros da planta são conhecidos *a priori* ou podem ser estimados, enquanto que o segundo método permite lidar com plantas incertas e de ordem superior. Baseado em resultados anteriores, a classe de plantas considerada captura a dinâmica essencial do modelo de Eikrem para um poço de petróleo que utiliza o método de elevação artificial. Neste sentido, este trabalho aborda a aplicação do ESC para a otimização da produção de petróleo. A técnica de ESC aplicada neste contexto mostra que é possível manter a produção de petróleo em torno de um ponto ótimo da curva de performance de poço de petróleo com elevação artificial sem reduzir a frequência da perturbação para valores impraticáveis. O desempenho da malha fechada de controle é avaliada por simulação numérica.

**Palavras-chave—** Controle por busca extremal, modelos do tipo não Hammerstein-Wiener, produção de petróleo, poços operando por elevação artificial.

## 1 Introduction

Extremum-seeking control (ESC) is an alternative approach for online optimization that deals with uncertain when the plant model and/or the cost function to optimize are not available to the designer. Using only input-output signals, the goal is to design a controller that dynamically searches for the optimizing inputs. This paper considers the applicability of ESC to face the optimization problem of oil production in gas lifted oil wells.

The modified version of Eikrem's model by [8] for gas lifted oil wells was evaluated via numerical simulations in [7] where no phase difference was observed between the input and output of the *well-performance curve* (WPC) mapping, when a high frequency periodic perturbation was used. An analogous simulation was performed with a very low perturbation frequency, yielding to a clearer phase difference at the expense of larger settling time (with duration of several years). This impairs the direct applicability of the ESC method. Seeking to circumvent this problem, in [7], the modified Eikrem's model is analyzed and a suitable model is proposed as a surrogate for the Eikrem's model: a simple stable first order linear system followed by a nonlinearity containing a product of the plant input and state. This model encompass a class of non Hammerstein-Wienner (HW) plants and, since the plant output is formed by a product of the plant input and the output of the nonlinearity, then a non-HW plant results. This model allows to capture the main features of the Eikrem' model [2] and clarify the main reason for the difficult to detect the phase difference between input-output [7].

In [1], the perturbation-based ESC method was generalized for a class of dynamic plants stabilizable via state feedback. The general idea was to generate a closed-loop system with sufficiently fast dynamics in order to behave approximately as a static plant. A more general class of nonlinear plants were treated in [6] by assuming again that the system (in closed loop via state feedback) can behave approximately as a static one or by assuming that the period of the periodic perturbation is large when compared to the time constant of the system (low excitation frequency). For the class of HW systems, compensators can be added to the ESC scheme so that the period of the periodic perturbation can be reduced, leading to faster transients to reach the vicinity of the maximizer [1], [5].

It must be highlighted that, in all cases [6], [1] and [5], the ESC method is deeply dependent on a good phase difference detection between input and output for average values of the input below and above the maximizer. A clear phase difference appears only for very low frequencies of the periodic perturbation, which impairs the directly applicability of the ESC method.

This paper focus on the non-HW model described in [7]. Two methods are proposed that allow the use of perturbation-based ESC via output feedback for the class of uncertain nonlinear systems considered here, without reducing the frequency of operation. Two approaches for ESC application are presented: a gradient-reconstruction ESC (GR-ESC) and a pre-compensated ESC (PC-ESC). The pre-compensation is proposed in order to approximate the nonlinear system to a HW system, allowing to reduce the period of the periodic perturbation. As a result, the optimization of oil production via ESC for practical perturbation frequencies is shown possible by using the proposed techniques.

**Remark 1** *(Notation and Terminology) The symbol "$s$" represents either the Laplace variable or the differential operator "$d/dt$", according to the context. As in [3], $H(s)u$ denotes the output of a LTI system with transfer function $H(s)$ and input $u$. Pure convolutions $h(t) * u(t)$, with $h(t)$ being the impulse response from $H(s)$, will be eventually written as $H(s) * u$. Classes of $\mathscr{K}, \mathscr{K}_\infty, \mathscr{L}$ functions are defined according to [4, p. 144], in particular, a function $\beta : \mathbb{R}^+ \to \mathbb{R}^+$ belongs to class $\mathscr{L}$ if it is continuous, strictly decreasing and $\lim_{t\to\infty} \beta(t) = 0$. A function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ belongs to class $\mathscr{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$.*

## 2 Problem Statement

Consider the following class of nonlinear uncertain SISO plants

$$x = G(s)u\,, \tag{1}$$

$$y = h(x,u) = \beta(x)u\,. \tag{2}$$

where $G(s)$ is general *strictly stable* transfer function, $u \in \mathbb{R}$ is the control signal, $x \in \mathbb{R}$ is the linear plant output (not available), $y \in \mathbb{R}$ is measured output, and $\beta : \mathbb{R} \to \mathbb{R}$ is a class $\mathscr{L}$ ($\beta \in \mathscr{L}$) function. Let $G_0 := G(0) \neq 0$ be the **finite** DC gain and $G_1 := |G(j\omega)|$ be the frequency response gain evaluated at the ESC operating frequency $\omega$. In order to assure existence and forward uniqueness of solutions, it is assumed that the nonlinear function $\beta$ is sufficiently smooth in $x$ (all required derivatives are continuous). For each solution of the nonlinear system (1) and (2) there exists a maximal time interval of definition given by $[0, t_M)$, where $t_M$ may be finite or infinite[1]. It is clear that the control law $u = \theta_u$ assures that $x = G_0\theta_u$ is an equilibrium point globally exponential stable, where $\theta_u \in \mathbb{R}$ is a parameter. The corresponding plant output is given $\theta_y = \Phi(G_0\theta_u)/G_0$, where $\Phi : \mathbb{R} \to \mathbb{R}$ is the *static input-output mapping*:

$$\Phi(\theta) := \beta(\theta)\theta\,, \quad \theta \in \mathbb{R}\,. \tag{3}$$

Of course, for $G(s) = 1$, we have a static input-output mapping given by $y = \Phi(u)$. Note that, due to the presence of the control signal $u$ in (2), the system does not belong to the class of HW type systems[2], which is the most important class of nonlinear systems covered by ESC schemes founded in the literature, in particular, the perturbation-based ESC schemes [1].

Here, we address the global real-time optimization control problem – problem of extremum seeking – considering only the maximum seeking problem, without loss of generality. We deal with the maximization of the plant output (2) subjected to (1). We assume that the static mapping $\Phi : \mathbb{R} \to \mathbb{R}$ (3) has a unique maximizer $\theta^* \in \mathbb{R}$ such that $\Phi(\theta^*)$ is the maximum, i.e., $\forall \theta \in \mathbb{R}$ one has $\Phi(\theta) \leq \Phi(\theta^*)$. Moreover, we consider that $\theta^*$, $\Phi(\cdot)$ and its gradient are unknown for the control designer and that all the uncertain parameters belong to some compact set. It must be highlighted that this simple linear model approximates the Eikher's model for oil wells according to [7].

### 2.1 Preliminaries

Consider the perturbation-based extremum seeking control method described in [6]. A brief description is given in Appendix A. For plants with $G(s) = 1$ in (1), or with sufficiently fast dynamics when compared with the period of the disturbance, the following input-output static relationship results: $y = \Phi(u)$.

In the ESC method the control signal is composed by an estimated $\hat{\theta}$ for maximizer $\theta^*$ added to a sinusoidal perturbation $v = asin(\omega t)$, i.e., $u = \hat{\theta} + v$. The estimate $\hat{\theta}$ is given by:

$$\dot{\hat{\theta}} = \frac{k}{a}\xi\,, \quad \hat{\theta}(0) = \hat{\theta}_0\,, \tag{4}$$

[1]Note that the control signal $u$ is a function of the plant output, so that the closed loop system is a nonlinear system even when (1) is a linear plant.

[2]Note that, however, by adding a stable filter at the system input results in a augmented HW system.

where $k > 0$ is a design constant and $\xi$ is an estimate for the gradient of the function $\Phi$ in (3) evaluated at $\hat{\theta}$, i.e., $\Phi^{'}(\hat{\theta})$. Note that, when $\xi \approx \Phi^{'}(\hat{\theta})$ the resulting dynamics $\dot{\hat{\theta}} = \frac{k}{a}\Phi^{'}(\hat{\theta})$ has an asymptotically stable equilibrium point given by the maximizer $\hat{\theta} = \theta^{*}$.

Now, for plants with $G(s) \neq 1$ in (1), the period of the periodic disturbance is limited from below due to the dynamics in (1). There exists a difficult to detect the phase difference between input-output and a simple linear compensation, as the one approached in [] where **HW plants** were considered, is preclude due to the product of state and input presented in the output function (2). In fact, the direct application of the ESC to this class of non-HW plants is not feasible when the ESC frequency increases.

The ESC is based on time scaling separation so that we assume that the signal $\hat{\theta}$ is in the passband of the transfer function $G(s)$, according to the following assumption:

**(H0)** The estimate $\hat{\theta}(t)$ is such that the signal

$$G(s) * \hat{\theta}(t) - G_0\hat{\theta}(t) = \varepsilon(t)\,,$$

converges exponentially to a residual set of order $\mathscr{O}(\varepsilon^{*})$, i.e., $|\varepsilon(t)| \to \mathscr{O}(\varepsilon^{*})$, as $t \to \infty$, where $\varepsilon^{*}$ is a sufficiently small constant.

It is clear that when the estimate $\hat{\theta}(t)$ varies slowly with respect to the input frequency $\omega$, one can consider that $G(s) * \hat{\theta}(t) \approx G_0\hat{\theta}(t)$.

## 3 Non-HW Plants: Gradient Reconstruction ESC

The original ESC, when applied to a non HW system, requires a strong reduction of the perturbation frequency which can make the ESC be impractical. The proposed Gradient-Reconstruction ESC (GR-ESC) is employed when some parameters of the plant are known *a priori* or can be estimated by probing the system.

By considering **(H0)** and $u(t) = \hat{\theta}(t) + a\sin(\omega t)$, one can write

$$x = z + aG_1\sin(\omega t + \phi) + \pi_1\,, \quad z := \hat{\theta}G_0 + \varepsilon(t)\,, \tag{5}$$

where $\varepsilon(t)$ comes from **(H0)**, $\pi_1$ is an exponentially decaying term depending on the plant (1) initial conditions and on the transient response due to the sinusoidal input term and $\phi := \angle G(j\omega)$ is the phase-shift introduced by $G(s)$ in (1). By applying the product rule to $\Phi(z) := \beta(z)z$, one can write

$$\Phi^{'}(z) = \beta(z) + \beta^{'}(z)z\,, \tag{6}$$

where, the signal $z(t)$ is not available for feedback.

Similarly to the case where $G(s) = 1$, in what follows, an estimate $\xi$ for the gradient is obtained. The main difference is that the gradient is now evaluated at $z$, where $z(t) \to G_0\hat{\theta}(t)$, as $t \to \infty$, according to **(H0)**.

Note that, when $\xi \approx \Phi^{'}(G_0\hat{\theta})$ the resulting dynamics $G_0\dot{\hat{\theta}} = \frac{k}{a}\Phi^{'}(G_0\hat{\theta})$ has an asymptotically stable equilibrium point given by the maximizer $\hat{\theta} = \theta^{*}/G_0$.

### 3.1 Gradient Reconstruction

Since the output $y$ is measured then $\beta(x)$ can be obtained as $\beta(x) = \frac{y}{u}$, ($u \neq 0$), and satisfies $\beta(x) = \beta(z) + \beta^{'}(z)\left[aG_1\sin(\omega t + \phi) + \pi_1\right] + H.O.T\,,$ where *H.O.T.* denotes *high order terms* and $\beta^{'}(z) := \frac{d\beta(z)}{dz}$. Thus, since $\pi_1$ decays exponentially to zero, there exists $T_1 > 0$ such that $|\pi_1(t)| < a$, $\forall t \geq T_1$. Hence, by considering the first-order Taylor series approximation around $z$, one can write:

$$\beta(x) \approx \beta(z) + \beta^{'}(z)aG_1\sin(\omega t + \phi)\,, \quad \forall t \geq T_1\,. \tag{7}$$

Let the amplitude of the sine wave be denoted by *AC* while the bias be denoted by *DC*, i.e.,

$$AC := \beta^{'}(z)aG_1\,, \quad DC := \beta(z)\,. \tag{8}$$

Then, the gradient evaluated at $z$ is given by:

$$\Phi^{'}(z) = \beta(z) + \beta^{'}(z)z = DC + \frac{AC}{a}g\hat{\theta} + \varepsilon_{\phi}\,, \tag{9}$$

where $\varepsilon_{\phi} := \frac{AC}{aG_1}\varepsilon$ and $g := \frac{G_0}{G_1}$.

The DC component $\beta(z)$ can be estimated as the output of a low-pass filter driven by the available signal $\beta(x(t)), \forall t \geq T_1$. Analogously, the sinusoidal component $\beta'(z)\,aG_1\sin(\omega t + \phi)$ can be retrieved using a band-pass filter centered at $\omega$. Higher order terms may be obtained in a similar manner, but are neglected in this analysis since for $a < 1$, it becomes inherently smaller as the power of $a$ raises. From (8), the estimate $\widehat{DC}$ for the signal $DC = \beta(z)$ is defined as:

$$\beta(z) \approx \widehat{DC} := \frac{1}{\tau_{dc}s + 1}\beta(x)\,, \tag{10}$$

where $\tau_{dc} > 0$ is a constant designed to filtered the oscillatory component appearing in the approximation of the **available** signal $\beta(x) = y/u$ ($u \neq 0$). Estimating the amplitude of a sinusoidal signal is a well-known problem in engineering, and several solutions are available using continuous-time and discrete-time approaches. In particular, the signal *AC*, given in (8), can be estimated by

$$\widehat{AC} = \frac{2}{\tau_{ac}s + 1}\left[\sin(\omega t)\nu(t)\right]\,, \quad \nu = \frac{s}{s + w_{ac}}\beta(x)\,, \tag{11}$$

where $\tau_{ac}$ is a positive design constant and $\nu$ is the output of a high-pass filter, with $w_{ac}$ being a positive design constant.

Therefore, inspired in (9), the following estimate for the gradient $\Phi^{'}(z)$ in (6) is considered:

$$\Phi^{'}(z) \approx \xi := \widehat{DC} + \frac{\widehat{AC}}{a}\hat{g}\hat{\theta}\,, \tag{12}$$

where $\hat{g}$ is an estimate for the gain $g = G_0/G_1$ given in what follows. In fact, (12) with $\hat{g} = g$ is one possible estimate for the gradient when $g$ is a known gain. Moreover, it is clear that this estimate approximates the gradient as long as $\widehat{AC} \approx AC$, $\widehat{DC} \approx DC$, $\hat{g} \approx g$ and $\varepsilon_{\phi} \approx 0$.

ISSN 2525-8311

### *3.2 Estimating the Gain g: First Approach*

By applying a constant value $\bar{\theta}$, i.e. $\hat{\theta}(t) = \bar{\theta}$, one has that $z = G_0\bar{\theta} + \varepsilon$, with $\varepsilon \to 0$ being an exponential decaying term. Let $z_s := \bar{\theta}G_0$. Thus, one can write

$$x = \underbrace{\bar{\theta}G_0}_{z_s} + aG_1 \sin(\omega t + \phi) + \pi_2 \,, \qquad (13)$$

where $\pi_2 = \pi_1 + \varepsilon$ incorporates the exponential decaying term $\varepsilon$. Moreover, one has $\beta(x) \approx \beta(z_s) + \beta^{'}(z_s)aG_1 \sin(\omega t + \phi)$, $\forall t \geq T_2$, where $T_2$ is such that $|\pi_2(t)| < a$, $\forall t \geq T_2$. In this case, redefine the amplitude of the sine wave and the bias term by

$$AC := \beta^{'}(z_s)aG_1 \,, \quad DC := \beta(z_s) \,. \qquad (14)$$

In order to estimate the gain $g$, consider the following procedure:

1. Apply a constant signal $\hat{\theta}(t) = \bar{\theta}_1$ and $u = \bar{\theta}_1 + a\sin(\omega t)$. Let $T_3$ be sufficiently large to assure that the vanishing exponential term $\pi_2$ can be neglected for $t \geq T_3$.
2. Apply a constant signal $\hat{\theta}(t) = \bar{\theta}_2 = \bar{\theta}_1 + \delta$ and $u = \bar{\theta}_2 + a\sin(\omega t)$, with $\delta$ being an arbitrarily small design constant. Let $T_4$ be sufficiently large to assure that the vanishing exponential term $\pi_2$ can be neglected for $t \geq T_4$.

Recalling that

$$\beta^{'}(G_0\bar{\theta}_1) = \lim_{\bar{\theta}_1 \to \bar{\theta}_2} \frac{\beta(G_0\bar{\theta}_2) - \beta(G_0\bar{\theta}_1)}{G_0(\bar{\theta}_2 - \bar{\theta}_1)} \,, \qquad (15)$$

an estimate $\hat{\beta}^{'}(G_0\bar{\theta}_1)$ for $\beta^{'}(G_0\bar{\theta}_1)$ can be defined as

$$\beta^{'}(G_0\bar{\theta}_1) \approx \hat{\beta}^{'}(G_0\bar{\theta}_1) := \frac{\widehat{DC}_2 - \widehat{DC}_1}{G_0\delta} \,, \qquad (16)$$

when $G_0$ is a **known** constant, where $\widehat{DC}_1$ ($\widehat{DC}_2$) is the estimate for $DC_1 := \beta(G_0\bar{\theta}_1)$ ($DC_2 := \beta(G_0\bar{\theta}_2)$) obtained as in (10).

In addition, from (14) one can define $AC_1 := \beta^{'}(G_0\bar{\theta}_1)aG_1$. Therefore, from (16) one can define an estimate $\hat{g}$ for the **unknown** gain $g$ as:

$$\hat{g} := \frac{a}{\widehat{AC}_1} \frac{(\widehat{DC}_2 - \widehat{DC}_1)}{\delta} \,,$$

where $\widehat{AC}_1$ is the corresponding estimate of $AC_1$, as in (11). This procedure is feasible because $g$ is a constant gain, regardless the value of $\bar{\theta}$. It must be stressed that the finite difference method (16) can fail due to measurement noise, so an alternative approach based on a filtered version ("dirty derivative") is considered in what follows.

### *3.3 Estimating the Gain g: Second Approach*

Now set $u(t) = \hat{\theta}(t) + a\sin(\omega t)$ with $\hat{\theta}$ being the type ramp function given by

$$\dot{\hat{\theta}} = k_r \,, \quad \hat{\theta}(0) = \hat{\theta}_0 \,,$$

with $k_r, \hat{\theta}_0$ being constants.

In this case, one has that $z = G_0\hat{\theta} + \varepsilon$, with $\varepsilon = \varepsilon_r + \pi_r$, where $\varepsilon_r$ is a constant and $\pi_r$ is an exponential decaying term corresponding to the transient response due to the ramp input term. Let $z_r := \hat{\theta}G_0 + \varepsilon_r$. Thus, the output $x$ can be written as

$$x = \underbrace{\hat{\theta}G_0 + \varepsilon_r}_{z_r} + aG_1 \sin(\omega t + \phi) + \pi_3 \,, \qquad (17)$$

where $\pi_3 = \pi_1 + \pi_r$ is an exponentially vanishing term.

Similarly as in the first approach, one has $\beta(x) \approx \beta(z_r) + \beta^{'}(z_r)aG_1 \sin(\omega t + \phi)$, $\forall t \geq T_5$, where $T_5$ is such that $|\pi_3(t)| < a$, $\forall t \geq T_5$. In this case, redefine the amplitude of the sine wave and the bias term by

$$AC := \beta^{'}(z_r)aG_1 \,, \quad DC := \beta(z_r) \,. \qquad (18)$$

In addition, consider the auxiliary unavailable signal $\mu := \ln[\beta(z_r)]$, with time derivative given by

$$\dot{\mu} = \frac{d\mu}{dt} = \frac{d\mu}{dz_r}\frac{dz_r}{dt} = \frac{\beta^{'}(z_r)}{\beta(z_r)}\dot{z}_r \,, \qquad (19)$$

one has that $\frac{AC}{DC} = \frac{aG_1\dot{\mu}}{\dot{z}_r} = \frac{aG_1\dot{\mu}}{G_0\dot{\hat{\theta}}}$, since $z_r = \hat{\theta}G_0 + \varepsilon_r$. Hence, the gain $g = \frac{G_0}{G_1}$ can be expressed as

$$g = \frac{DC}{AC}\frac{a\dot{\mu}}{\dot{\hat{\theta}}} \,. \qquad (20)$$

In what follows an estimate for the gain $g$ is developed. First, the estimates $\widehat{DC}$ for the signal $DC$ and $\widehat{AC}$ for the signal $AC$ are given as in the first approach. By noting that the unavailable signal $\mu = \ln[\beta(z_r)] = \ln[DC]$ can be estimated by $\hat{\mu} = \ln\left[\widehat{DC}\right]$ then the time derivative $\dot{\mu}$ can be approximated by $\widehat{\dot{\mu}} := \frac{s}{\tau_\mu s + 1}\hat{\mu}$, where $\tau_\mu > 0$ is an appropriate design constant. The estimate $\hat{g}$ for the gain $g$ in (20) is given by:

$$\hat{g} := \frac{\widehat{DC}}{\widehat{AC}}\frac{a\widehat{\dot{\mu}}}{k_r} \,, \qquad (21)$$

since $\dot{\hat{\theta}} = k_r$.

## 4 Non-HW Plants: Pre-Compensated ESC

The main idea is to reduce the system to a HW system which has the same input-output static mapping and for which the ESC frequency can be increased.

Here we use the following available auxiliary output signal

$$\mathscr{Y} := \frac{y}{u}\mathscr{X} \,, \quad u \neq 0 \,,$$

to generate the estimate $\xi$ for the gradient, by applying the traditional ESC to the resulting plant

$$x = G(s)u \,, \qquad (22)$$

$$\mathscr{Y} = \beta(x)\mathscr{X} \,. \qquad (23)$$

Note that, by choosing $\mathscr{X}$ such that $\mathscr{X} \propto x$, the system (22)–(23) has an input-output static mapping with the same maximizer as the original system (1)–(2). The advantage is that now (22)–(23) is (approximately) a HW system. This class of plants has been extensively addressed in the ESC literature. Recalling that a clear phase difference between input-output appears only for very low frequencies of the periodic perturbation when the non-HW plant is considered, now it is possible to detect phase difference between $\mathscr{Y}$ and $u$ for higher ESC operating frequencies.

### One Possible Choice for the Signal $\mathscr{X}$

Considering **(H0)**, the signal $x$ in (5) can be approximate by

$$x = \left[\hat{\theta} + ga\sin(\omega t + \phi)\right] G_0 \,, \tag{24}$$

*modulo* exponential decaying terms. Based on this approximation, the pre-compensation signal $\mathscr{X}$ is defined as:

$$\mathscr{X} := \left[\hat{\theta} + \hat{g}a\sin(\omega t + \hat{\phi})\right] . \tag{25}$$

where $\hat{g}$ can be obtained from (21). We can estimate the phase shift $\phi = \angle G(j\omega)$ by $\hat{\phi}$ using a phase lock loop estimator (PLL) applied to the available signal $y/u = \beta(x)$, when $u \neq 0$. Indeed, since $\beta \in \mathscr{L}$ the signal $\beta(x)$ has, approximately, the same phase shift ($\angle G(j\omega)$) as $x$ w.r.t. to the input $u$ (in steady state).

### Proposed Pre-Compensation

The initialization procedure is given by:

1. Apply a constant signal $\hat{\theta}(t) = \bar{\theta}$. Let $T_6$ be sufficiently large to assure that the vanishing exponential terms can be neglected for $t \geq T_6$ and the PLL reaches the steady state.

2. Apply a ramp type signal $\hat{\theta}(t)$ satisfying $\dot{\hat{\theta}} = k_\theta$, with $\hat{\theta}(0) = \hat{\theta}_0$. Let $T_7$ be sufficiently large to assure that the vanishing exponential terms can be neglected for $t \geq T_7$ and the gain estimator (21) is in steady state.

The signal $x$ in (24) can be approximated by

$$x \approx G_0 \mathscr{X} \,, \quad \forall t \geq T_8 \,, \tag{26}$$

where $T_8 > \max\{T_6, T_7\}$.

Note that, by applying the compensation signal $\mathscr{X} \approx x/G_0$, the system (22)–(23) with pre-compensation has the input-output static mapping $\Phi(x)/G_0 = \beta(x)x/G_0$ from $u$ to $\mathscr{Y} = (y/u)\mathscr{X}$ and now ($t > T_8$) behaves as an HW system. The uncertain parameter $G_0$ is irrelevant since the ESC gain $k$ can be adjusted to guarantee the transient behavior of the closed loop control system. The time instants $T_6, T_7$ and $T_8$ are unknown, however, a rough estimate can be obtained *a priori* observing the available signal $y/u = \beta(x)$. Moreover, the transition between applying a constant value and a ramp function can be accomplished on-line by starting with a very small ESC gain $k$ and increasing it smoothly.

## 5 Numerical Simulations

Consider the plant (1)–(2), with $G(s) = 1/(\tau s + 1)$, time constant $\tau = 1250$ and nonlinear output such that $\beta(x) = \frac{P(x)}{Q(x)}$, with $P(x) = 0.3x^4 - 7.8x^3 + 83.2x^2 - 475.2x + 1851.5$ and $Q(x) = -1.3x^4 + 34.3x^3 - 373.8x^2 + 3009.6x + 3033.3$. The corresponding cost function $\Phi(x) = \beta(x)x$ has a maximizer at $x = x^* = 5.31$. The perturbation parameters of the ESC scheme are: $a = 0.1$ and $\omega = 0.0031$. The pre-compensation ESC is implemented with the PLL and integral gain $k = 0.1$. The others ESC parameters appearing in Appendix A are the high-pass cut-off frequency $\omega_h = \omega/4$ and the low-pass cut-off frequency $\omega_L = 1/\tau_1 = \omega/4$. On the other hand, the ESC based on direct gradient estimation is implemented with integral gain $k = 8.3333 \times 10^{-5}$, a third-order low-pass Butterworth filter with cut-off frequency $\omega_L = \omega/2$ and a third-order band-pass Butterworth filter with cut-off frequencies $\omega_{PL} = \omega/2$ and $\omega_{PH} = 1.075\omega$. Additionally, an initial characterization of $\hat{G}_1 = 0.2474$ was made with $\theta_1 = 3$ and $\theta_2 = \theta_1 + 1 \times 10^{-5}$.

The output of both ESC schemes are shown in Fig. 1. It can be seen straightforwardly that both ESC schemes converge asymptotically to a vicinity of the maximizer. On the one hand, it would seem that the ESC based on direct gradient estimation converges to the maximizer, while the pre-compensated ESC presents steady state error. The pre-compensated ESC is deeply dependent on the accuracy of the estimates $\hat{g}$ and $\hat{\phi}$, revealing a high sensitivity. On the other hand, the pre-compensated ESC has a greater integral gain, which allows a faster convergence.

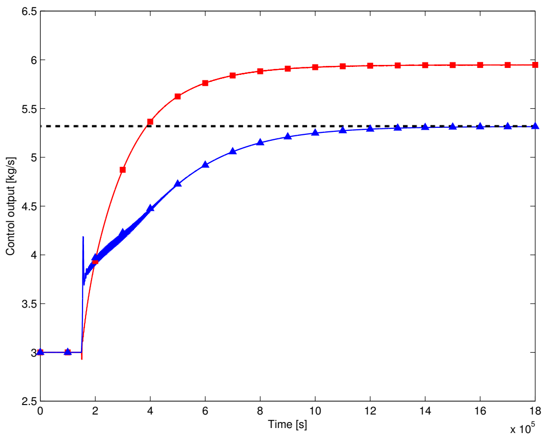


**Figure 1:** Simulation of the proposed ESC schemes. The signal $\hat{\theta}$ time behavior by using the pre-compensation ESC (red line), the ESC based on direct gradient estimation (blue line) and the true maximizer value (dashed line).

Now, consider the second order linear plant $G(s) = \frac{1}{(0.25s+1)(0.1s+1)}$, with the same nonlinearity as before. The pre-compensated ESC performance is satisfactory by using the parameters: $a = 0.01$, $\omega_h = 2\pi$, $\omega = 2\pi$ and $k = 5$. The results are illustrated in Figure 2.

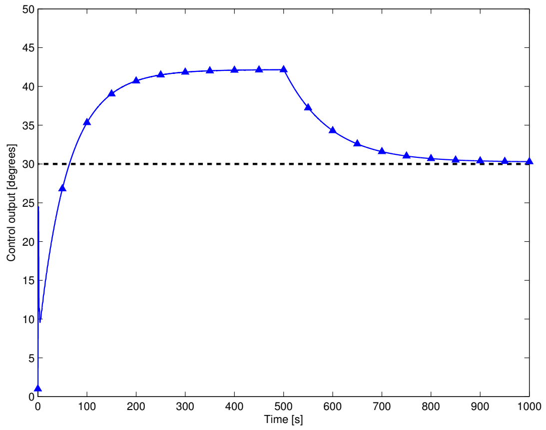


**Figure 2:** Numerical simulations for second order plant. Time history of the variable $\hat{\theta}$ (ESC output) with the maximizer $z^* = 30$. The estimate $\hat{\phi}$ switching from a rough initial value to the PLL estimate, after $t = 500\,$s.

The application faster sinusoids was only possible because the system was rebuilt as HW model, see Figure 2, where the estimate $\hat{\phi}$ varies from a rough initial value to the PLL estimate. Further research may be carried on by combining the strengths of both ESC schemes, encouraging to develop a hybrid method.

## 6 Conclusion

In this work two schemes for output-feedback extremum-seeking control (ESC) based on periodic perturbation were developed for a class of uncertain non Hammerstein-Wienner (HW) plants. When certain parameters of the plant are known *a priori* or can be estimated by probing the system, the gradient-reconstruction extremum-seeking control (GR-ESC) has proven to be effective in estimating the maximizer. The GR-ESC features a low-pass filter, a band-pass filter and an amplitude detector that allows to reconstruct the gradient of the output, thus detecting the optimal point. For more general plants with a higher order uncertain dynamics, the pre-compensation (PC-ESC) has shown a fast-response convergence to the maximizer. The PC-ESC approximates the system to an HW plant by using an estimate provided by a phase-locked loop (PLL).

The performance of the two ESC approaches were evaluated via numerical simulations. The behavior of a gas-lifted oil well was modeled using a simple nonlinear dynamic model which captures the essential dynamic of a modified version of the Eikrem's model, in both transient and steady state regimes. Based on this model, the closed-loop simulations generated a more accurate estimation of the maximizer of the plant when using the GR-ESC, whereas the use of PC-ESC presented a faster convergence despite the sensitivity to the PLL estimates.

## Referências

[1] Ariyur, K. B., and Krstic, M. *Real time optimization by extremum seeking control*. Wiley Interscience, 2003.

[2] Eikrem, G. O. *Stabilization of gas-lift wells by feedback control*. PhD thesis, Department of Engineering Cybernetics, Norwegian University of Science and Technology (NTNU), Trondheim, Norway, May 2006.

[3] Ioannou, P., and Sun, K. *Robust Adaptive Control*. Prentice Hall, 1996.

[4] Khalil, H. K. *Nonlinear Systems*. Prentice Hall, Upper Saddle River, New Jersey, 2002.

[5] Krstíc, M. Performance improvement and limitations in extremum seeking control. *Systems & Control Letters 39* (2000), 313–326.

[6] Krstić, M., and Wang, H.-H. Stability of extremum seeking feedback for general nonlinear dynamic systems. *Automatica 36*, 7 (April 2000), 595–601.

[7] Peixoto, A. J., Pereira-Dias, D., Xaud, A. F., and Secchi, A. R. Modelling and extremum seeking control of gas lifted oil wells. *IFAC-PapersOnLine 48*, 6 (2015), 21 – 26. 2nd {IFAC} Workshop on Automatic Control in Offshore Oil and Gas Production {OOGP} 2015Florianópolis, Brazil, 27–29 May 2015.

[8] Ribeiro, C. Controle preditivo multivariável com requisitos de qualidade em plataformas de produção de petróleo. Master's thesis, COPPE, Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil, Sept. 2012.

## A ESC for Static Plants

By using the first two terms of the Taylor series to approximate the input-output relationship $\Phi(u)$ with $u = \hat{\theta} + asin(\omega t)$, it follows that $y \approx \Phi(\hat{\theta}) + \Phi^{'}(\hat{\theta})asin(\omega t)$. In general, the output $y$ can be consider to present the form $y_{ss}(t) = \theta_1(t)\sin(\omega t) + \theta_2(t)$ in steady state[3]. Therefore, $\theta_1$ and $\theta_2$ can be estimated as follows. The amplitude $\theta_1$ can be estimated by $\hat{\theta}_1 = \frac{2}{\tau_1 s+1}[\sin(\omega t)\eta(t)]$, where $\tau_1$ is a positive design constant and $\eta$ is the output of a high-pass filter, i.e., $\eta = \frac{s}{s+w_h}y = y - \frac{w_h}{s+w_h}y$. Indeed, the filter $\frac{w_h}{s+w_h}$, designed properly, attenuate the term $\theta_1(t)\sin(\omega t)$ in $y = \theta_1(t)\sin(\omega t) + \theta_2(t)$. Then, $\eta \approx \theta_1(t)\sin(\omega t)$. The signal is then demodulated, multiplying by $\sin(\omega t)$, resulting in $\theta_1(t)\sin^2(\omega t)$. Reminding that $2\sin^2(\omega t) = 1 - \cos(2\omega t)$, only the DC component is not filtered by $\frac{2}{\tau_1 s+1}$. Hence, $\hat{\theta}_1 \approx \theta_1$. The average value (DC component) is estimated directly by $\hat{\theta}_2 = \frac{1}{\tau_2 s+1}y$, where $\tau_2$ is a positive design constant. Note that, $\hat{\theta}_2 \approx \theta_2$. Thus, $\hat{\theta}_1 \approx \Phi^{'}(\hat{\theta})a$, $\hat{\theta}_2 \approx \Phi(\hat{\theta})$ and an approximation for the gradient of the function $\Phi$ evaluated at $\hat{\theta}$ is given by $\xi = \hat{\theta}_1/a$. For the correct operation of the algorithm the cutoff frequencies $w_h$ and $1/\tau$ must be less than $\omega$. The amplitude $a$ of the sinusoidal perturbation defines the size of the oscillations around the optimal point, while the integrator gain $k$ defines how fast the output will reach this neighborhood. These two constants should be small enough to assure the stability of the algorithm [6].

[3] Note that, even for the case $G(s) \neq 1$, the dynamic system is stable and it still reasonable to consider that the output $y$ presents this form in steady state.